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Design of Retarded Fractional Delay Differential Systems Using the Method of Inequalities

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Abstract: Methods based on numerical optimization are useful and effective in the design of control systems. This paper describes the design of retarded fractional delay differential systems (RFDDSs) by the method of inequalities, in which the design problem is formulated so that it is suitable for solution by numerical methods. Zakian′s original formulation, which was first proposed in connection with rational systems, is extended to the case of RFDDSs. In making the use of this formulation possible for RFDDSs, the associated stability problems are resolved by using the stability test and the numerical algorithm for computing the abscissa of stability recently developed by the authors. During the design process, the time responses are obtained by a known method for the numerical inversion of Laplace transforms. Two numerical examples are given, where fractional controllers are designed for a time-delay and a heat-conduction plants.

Keywords: Fractional systems, systems with time-delays, control systems design, method of inequalities, design formulation, parameter optimization.

1 Introduction

Recently, much research effort has been given to fractional differential systems. Many physical processes have their mathematical models described by fractional delay differential equations (see, for example, [1] and the references therein). Moreover, it is demonstrated[2] that if appropriately designed, feedback systems with fractional order controllers can yield better performances than those with integer order controllers.

Many investigators have been prompted to develop methods for designing fractional differential systems in order to enhance the system′s performances and robustness (see, for example, [2–4] and also the references therein). These methods are suitable for handling simple design problems with some specific design specifications. It is evident that the design problems become much more complicated when there are a number of design objectives to fulfill simultaneously. Therefore, it is desirable to have a systematic method that can solve the design problems for fractional differential systems efficaciously.

Methods based on numerical optimization have proved useful and effective in the design of control systems. The method of inequalities[5−8] (MoI) is a general multiobjective optimization method that requires the formulation of design problems as a set of inequalities. The method facilitates a realistic formulation of the design problem by allowing the designer to express the constraints and the performance specifications directly in terms of inequalities, whereas all tedious computations are carried out by efficient numerical algorithms. The method has been successfully applied to many difficult design problems (see, for example, [9–14]) and also many references cited in [8]). So far, no one has considered the design of fractional differential systems using the MoI.

The objective of this paper is to describe the design of retarded fractional delay differential systems (RFDDSs) by the MoI, where the transfer functions of the systems are of the form

$$\frac{q_0(s) + \sum_{k=1}^{n_2} q_k(s)\mathrm{e}^{-\beta_k s} + \sum_{k=1}^{\tilde{n}_2} \tilde{q}_k(s)\mathrm{e}^{-v_k(s)}}{p_0(s) + \sum_{k=1}^{n_1} p_k(s)\mathrm{e}^{-\gamma_k s} + \sum_{k=1}^{\tilde{n}_1} \tilde{p}_k(s)\mathrm{e}^{-u_k(s)}} \tag{1}$$

the delays γ_k and β_k are such that $0 < \gamma_1 < \cdots < \gamma_{n_1}$ and $0 < \beta_1 < \cdots < \beta_{n_2}$, the polynomials p_k, q_k, $\tilde{p}_k$, and $\tilde{q}_k$ are of the form $\sum_{j=0}^{l_k} a_j s^{\alpha_j}$ with all $\alpha_j \geqslant 0$, $\deg(p_0) > \deg(p_k)$ for $k = 1, 2, \cdots, n_1$, $\deg(p_0) \geqslant \deg(q_0)$ and $\deg(p_0) > \deg(q_k)$ for $k = 1, 2, \cdots, n_2$, u_k, and v_k are polynomials of the form $\sum_{j=1}^{m_k} b_j s^{\delta_j}$ with $0 < \delta_j \leqslant 1$ and $b_j \geqslant 0$, and none of u_k and v_k assumes the form αs.

Obviously, the class of systems (1) is very general and includes rational systems (RSs) and retarded delay differential systems (RDDSs) as special cases. This is illustrated in Fig. 1.

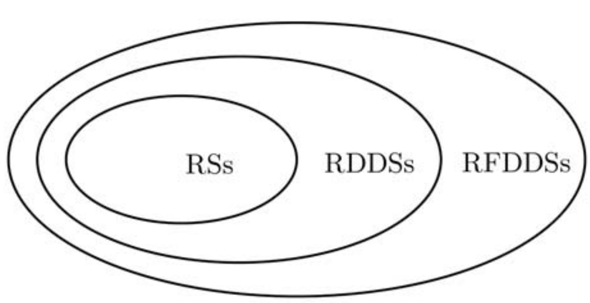

Fig. 1 Class of RFDDSs includes RSs and RDDSs

The formulation of the design problem considered here

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was first proposed by Zakian and Al-Naib[5] in connection with rational systems. In order to make the use of this formulation possible for the more general case of systems described by transfer functions of the form (1), the associated stability problems are resolved by using the stability test and the numerical method for computing the abscissa of stability that have been recently developed by the authors (see Section 3.1 for details). During the design process, the system′s time responses are obtained by using a known method for numerical inversion of Laplace transforms (see Section 3.2 for the details).

The organization of the paper is as follows. Section 2 describes the MoI and the general principle of the design formulation that was considered in [5]. In Section 3, such a formulation is extended to the case of RFDDSs so that the design problem is suitable for solution by numerical methods. In addition, the numerical methods involved are briefly described. In Section 4, two illustrative numerical examples are given, where fractional controllers are designed, respectively, for a time-delay plant and a heat-conduction process. Conclusions and discussion are provided in Section 5.

2 Design by the MoI

The MoI requires that a design problem be expressed as a set of inequalities

$$\phi_i(p) \leqslant C_i, \quad i = 1, 2, \cdots, m \tag{2}$$

where $p \in \mathbf{R}^n$ is the design parameter vector, $\phi_i : \mathbf{R}^n \rightarrow \mathbf{R} \cup \{\infty\}$ represent performance measures or physical properties of the system, and the bounds C_i are the supremal values of $\phi_i(p)$ that can be tolerated. Any point p satisfying (2) is an acceptable design solution.

In practice, numerical methods are usually employed to compute a solution of (2). For this reason, it is important that the design problem (2) be formulated in such a way that it is suitable for solution by numerical methods.

It is noted in [5, 6, 8, 15] that the process of solving inequalities (2) by numerical methods involves two phases of computation as follows:

- Phase I. Determine a point p such that $\phi_i(p) < \infty$ for every i.
- Phase II. Determine a solution of inequalities (2) by starting at the point p obtained in Phase I.

Each phase gives rise to distinct computational problems.

2.1 Phase I

Define a stability point as a point p satisfying

$$\phi_i(p) < \infty, \quad \forall i \tag{3}$$

and define the stability region Σ as the set of all stability points. The main problem in Phase I is how to locate a stability point by starting from any arbitrary point in $\mathbf{R}^n$, as shown in Fig. 2.

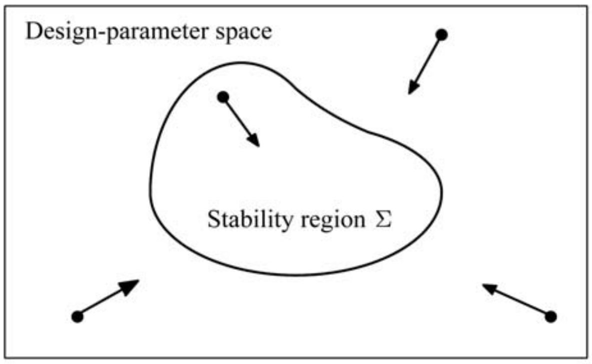

Fig. 2 In Phase I, a stability point is located from an arbitrary point in the design parameter space $\mathbf{R}^n$

In general, inequalities (3) are not soluble by numerical methods. That is, a stability point cannot be generated using only the functions ϕ_i and a descent method. This is because a gradient or similar property of ϕ_i cannot be defined outside the stability region Σ.

Zakian[5, 8, 15] advocates that a possible method for obtaining a stability point by numerical methods is to replace condition (3) by an equivalent inequality

$$\alpha(p) < 0 \tag{4}$$

such that 1) $\alpha(p) < \infty$ for all $p \in \mathbf{R}^n$, and 2) α can be computed economically in practice.

Once there exists such a function α, condition (4) becomes soluble by numerical methods. Consequently, iterative numerical methods can be used to locate a stability point by starting from any arbitrary point in $\mathbf{R}^n$.

For rational systems, the function α was chosen as the abscissa of stability of the characteristic polynomial in [5]. In this connection, an economical algorithm for computing the abscissa of stability was given in [16], which avoids calculating all the characteristic roots. This useful approach was then extended to the case of retarded delay differential systems in [17, 18] and to the case of RFDDSs in [19]. See also Section 3 for further details.

Hence, for RFDDSs (including rational and retarded delay differential systems), condition (3) is replaced by

$$\alpha(p) < 0, \quad \alpha \triangleq \sup\{\mathrm{Re}\,(s) : f(s) = 0\} \tag{5}$$

where α is called the abscissa of stability of $f(s)$, and $f(s)$ denotes the characteristic function of the system. Usually, the inequality (5) is replaced by a practical sufficient condition

$$\alpha(p) \leqslant -\varepsilon \quad (\varepsilon > 0) \tag{6}$$

where the bound $-\varepsilon$ is introduced so as to ensure that the system is stable as long as the magnitude of error in the computed value of $\alpha(p)$ is less than ε.

2.2 Phase II

By starting from a stability point, a search method locates a solution of (2) within the stability region Σ, as shown in Fig. 3. To avoid any risk in stepping outside the stability region, one checks the stability of the system at every point generated by the search method. In doing this, one needs to determine the sign of $\alpha(p)$, which is less demanding than to compute the value of $\alpha(p)$. Therefore, for the purpose of maintaining the system′s stability during the search in Phase II, only a stability test is required.

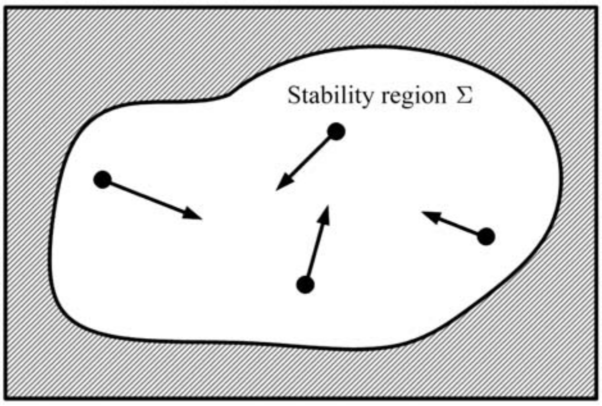

Fig. 3 In Phase II, a design solution is sought within the stability region Σ

Once the system is found to be stable, the numerical search algorithm generates a long sequence of points $p \in \Sigma$. In doing this, $\phi_i(p)$ are to be computed repeatedly. It is evident that in contrast to Phase I, the main problem in Phase II is how to compute $\phi_i(p)$ economically.

Usually, the most difficult functions ϕ_i to compute are those defined in terms of the time responses of the system. Once the time responses are obtained, such functions ϕ_i can be computed using known numerical methods (see Section 4 for examples of these). To evaluate these ϕ_i is computationally expensive. For this reason, the efficiency of the design procedure mainly depends on that of computing the time responses. Hence, an efficient and reliable algorithm for computing the time responses is needed.

Note, in passing, that the functions ϕ_i defined in terms of the frequency responses (for example, bandwidth and phase margin) are readily computable. Consequently, when required, these ϕ_i can be easily incorporated into the design.

3 Design formulation for RFDDS

Following the general principle given in Section 2, this section describes a practical formulation for the design of RFDDSs by the MoI, which is suitable for solution by numerical methods. In addition, the computational methods involved are briefly explained.

3.1 Stability test and stabilization

It is known[20] that a system characterized by the transfer function (1) is bounded-input bounded-output (BIBO) stable if and only if the characteristic function

$$f(s) = p_0(s) + \sum_{k=1}^{n_1} p_k(s)\mathrm{e}^{-\gamma_k s} + \sum_{k=1}^{\tilde{n}_1} \tilde{p}_k(s)\mathrm{e}^{-u_k(s)} \qquad (7)$$

has all zeros with negative real parts. Accordingly, an RFDDS is BIBO stable if and only if

$$\alpha < 0. \qquad (8)$$

Recently, both a computational stability test and a practical algorithm for computing the abscissa of stability for RFDDSs have been established in [19]. The algorithm makes repeated use of the stability test and thereby avoids computing of all zeros of the characteristic function $f(s)$.

For RFDDSs, once the abscissa of stability α can be efficiently computed, a stability point is readily obtainable by simply solving the inequality (6) by iterative numerical methods.

3.1.1 Stability test

Let $H(\rho)$ be a right half plane and described by

$$H(\rho) \triangleq \{s \in \mathbf{C} : \mathrm{Re}(s) \geqslant \rho\}$$

where $\rho \in \mathbf{R}$ is specified. An RFDDS is said to be $H(\rho)$-stable if none of its characteristic roots lies in $H(\rho)$.

By extending the stability test due to Hwang and Cheng[21], a numerical procedure for testing the $H(\rho)$-stability for any given number ρ, which is more general, is devised in [19]. The key idea of the $H(\rho)$-stability test is briefly summarized as follows.

The half plane $H(\rho)$ is represented by a semicircle with infinite radius and the well-known Cauchy′s residue theorem is applied to determine whether the characteristic function $f(s)$ has no zeros in $H(\rho)$.

Notice that, in general, $f(s)$ in (7) has a branch cut along the negative real axis. Therefore, when $\rho \leqslant 0$, the contour is indented to avoid crossing the branch cut along the negative real axis. In this case, the search for any roots of the characteristic function $f(s)$ on the interval $[\rho, 0]$ can be performed readily by using one-dimensional search methods.

In Phase II, neither should the search method step outside the stability region Σ, nor should it generate trial points very close to the boundary of Σ. This is because of the requirement (6) and because, if the design problem is properly formulated, a design solution usually lies well inside the region Σ. For these reasons, in Phase II, it is advisable to perform an $H(-\varepsilon)$-stability test with $\varepsilon > 0$ sufficiently small, instead of performing an $H(0)$-stability test.

Notice that when the system is $H(-\varepsilon)$-stable, condition (6) is always satisfied. Hence, the $H(-\varepsilon)$-stability test can be used in practice to ensure that a given point p lies inside the stability region Σ (in other words, the system is stable).

3.1.2 Computation of abscissa of stability

The numerical method for computing the abscissa of stability for RFDDSs is devised by modifying Zakian′s algorithm[16], which is a bisection method and makes repeated use of the $H(\rho)$-stability test.

In essence, $\alpha(p)$ is computed by the following bisection algorithm. First, determine two numbers a and b such that $\alpha(p) \in (a, b)$; that is, the system is both $H(a)$-unstable and $H(b)$-stable. Compute the midpoint $c = (a + b)/2$ and perform an $H(c)$-stability test so as to determine which of the two intervals (a, c) and (c, b) contains $\alpha(p)$. Repeat the bisection until the interval containing $\alpha(p)$ is sufficiently small.

The details of the algorithm can be found in [19].

3.2 Computation of time responses

There are methods available for computing the time responses of RFDDSs. One may, for example, obtain the time responses by solving a system of differential equations of fractional order (see, for example, [1]). However, since the Laplace transforms of the time responses are readily obtainable in our case, it is convenient to compute the time responses from their Laplace transforms.

In this paper, we compute the time responses of RFDDSs by employing I_{MN} approximants[22, 23], which result in a useful formula for numerically inverting Laplace transforms (see also [24, 25]).

3.2.1 Definition of I_{MN} approximants

Zakian[22, 23] defines the I_{MN} approximant of $x(t)$ for $t \geqslant 0$ by the improper integral

$$I_{MN} \triangleq \int_0^\infty x(\lambda t) \sum_{i=1}^{N} K_i \mathrm{e}^{-\alpha_i \lambda} \mathrm{d}\lambda \tag{9}$$

where (α_i, K_i) are defined constants, and the nonnegative integers M and N are, respectively, the orders of the numerator and the denominator of the Laplace transform of $\sum_{i=1}^{N} K_i \mathrm{e}^{-\alpha_i \lambda}$.

In this work, we restrict our attention only to the full grade I_{MN} approximants[23], whose constants (α_i, K_i) are defined by

$$\sum_{i=1}^{N} \frac{K_i}{z+\alpha_i} = \mathrm{e}_{MN}^{-z} \quad \text{and} \quad \mathrm{Re}(\alpha_i) > 0, \ \ \forall i \tag{10}$$

where e_{MN}^{-z} denotes the $[M/N]$ Padé approximate to e^{-z}.

The full grade I_{MN} approximants have many remarkably useful properties (see [23, 25] for details on this) and have been successfully applied to many practical problems (see, for example, [5, 24, 26] and the references cited in [25]).

3.2.2 Inversion formula

Let $X(s)$ denote the Laplace transform of $x(t)$, evaluated at s. That is,

$$X(s) \triangleq L[x(t)] = \int_0^\infty x(t)\mathrm{e}^{-st}\mathrm{d}t \tag{11}$$

where s is a complex number such that the integral converges to a finite limit. From (9), it can be readily verified that[22]

$$I_{MN}(x,t) = \frac{1}{t}\sum_{i=1}^{N} K_i X\Big(\frac{\alpha_i}{t}\Big), \quad t > 0. \tag{12}$$

When N is even, all the constants α_i and K_i occur in complex conjugate pairs and hence computational economy can be obtained by replacing (12) with

$$I_{MN}(x,t) = \frac{2}{t}\sum_{i=1}^{N/2} \mathrm{Re}\left[K_i X\Big(\frac{\alpha_i}{t}\Big)\right], \quad t > 0.$$

Evidently, (12) provides a useful formula for the numerical inversion of Laplace transforms. For a given value of t, $x(t)$ is obtained by evaluating its Laplace transforms $X(s)$ at certain points $s = (\alpha_i/t)$ on the complex plane, which can be done very fast.

In this paper, we normally use $M = 11$ and $N = 18$ with double precision arithmetic operations. However, whenever there is a doubt in the accuracy of the obtained results, we recompute by using $M = 30$ and $N = 40$ with quad-precision arithmetic operations. The details of how to choose appropriate values of M and N for the inversion formula (12) can be found in [24].

4 Numerical examples

This section demonstrates how to design RFDDSs by the MoI using the formulation described in Section 2 and the algorithms mentioned in Section 3. For clarity, we focus our attention only on the design of single-input single-output (SISO) systems. However, it is important to note that the systematic method developed in the paper can readily be applied to multiple input multiple output (MIMO) systems of fractional order.

4.1 Design formulation and specifications

Consider an SISO control system shown in Fig. 4, where $G(s)$ is the plant transfer function, $K(s,p)$ is the controller transfer function with design parameter p. Suppose that the reference r is the unit step function.

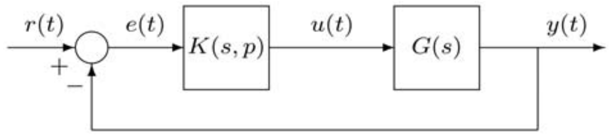

Fig. 4 A unity feedback control system

In the following examples, the design objectives are twofold. First, to achieve a good step response, that is to say, the response with small maximum overshoot, settling time, and rise time. Second, to avoid the control saturation by requiring that the control signal be not too large. Accordingly, the design parameter p is to be determined so that it satisfies the following design specifications.

$$\phi_i(p) \leqslant C_i, \quad i = 1,2,3,4 \tag{13}$$

where ϕ_1 is the maximum overshoot, ϕ_2 the rise time, ϕ_3 the settling time, and ϕ_4 the peak of the control signal. That is to say,

$$\begin{aligned} &\phi_1 \triangleq \max\left\{\frac{y(t)-y_\infty}{y_\infty} : t \geqslant 0\right\} \\ &\phi_2 \triangleq \min\{t : y(t) = 0.9\, y_\infty\} \\ &\phi_3 \triangleq \min\{\tau : |y(t)-y_\infty| \leqslant 0.02\, y_\infty, \quad \forall\, t \geqslant \tau\} \\ &\phi_4 \triangleq \max\{|u(t)| : t \geqslant 0\} \end{aligned} \tag{14}$$

where y_∞ denotes the steady state value of the step response y, that is,

$$y_\infty \triangleq \lim_{t\to\infty} y(t).$$

In many practical applications, the actuator has a saturation characteristic. To avoid the control saturation during the operation of the system, the design requirement $\phi_4 \leqslant C_4$ in (13) needs to be taken into design consideration. This fundamental requirement makes the design problem become difficult to solve even for SISO systems. However, it should be noted that the MoI can solve such a design problem effectively in a systematic manner (see, for example, [9, 10]).

In addition to the design specifications in (13), if there are constraints on the design parameter p (or permissible ranges of p), they can be incorporated into the set of design inequalities (13) very easily. This still results in the inequalities of the form (2).

Throughout this work, the design inequalities (13) or (2) are solved by using a numerical search algorithm called the moving boundaries process (MBP)[5]. For the efficiency in the design process, tasks such as the $H(\rho)$-stability test, the computation of $\alpha(p)$ and the step responses are implemented in Fortran programming language.

4.2 Controller design for a time-delay plant

Let the plant transfer function $G(s)$ be given by

$$G(s) = \frac{2\mathrm{e}^{-2s}}{(s+1)(s+2)}. \tag{15}$$

Assume that a fractional proportional-integral (PI) controller is used and the controller transfer function $K(s, p)$ is

$$K(s,p) = \frac{p_1 + p_2 s^{p_3}}{s^{p_3}} \tag{16}$$

where $p = [p_1,\ p_2,\ p_3]^{\mathrm{T}}$ is the vector of design parameters that are constrained to be positive.

It is easy to verify that the transfer function of the closed-loop system takes the form (1) and is given by

$$H(s) = \frac{2(p_1 + p_2 s^{p_3})\mathrm{e}^{-2s}}{s^{p_3}(s+1)(s+2) + 2(p_1 + p_2 s^{p_3})\mathrm{e}^{-2s}}. \tag{17}$$

Suppose that the bounds C_i are specified as follows.

$$C_1 = 0.05,\ C_2 = 5.7,\ C_3 = 6.5,\ C_4 = 1.1. \tag{18}$$

By solving the inequality

$$\alpha(p_0) \leqslant -0.001$$

and using the MBP algorithm, a stability point p_0 is obtained. Then, after a number of iterations, the MBP algorithm locates a design solution

$$p = [0.225,\ 0.491,\ 1.043]^{\mathrm{T}} \tag{19}$$

where $\alpha(p) = -0.2714$ and the corresponding performance measures are

$$\phi_1 = 0.02,\ \phi_2 = 5.62,\ \phi_3 = 6.37,\ \phi_4 = 1.06.$$

The control and output signals of the system with the controller parameter p in (19) are shown in Fig. 5.

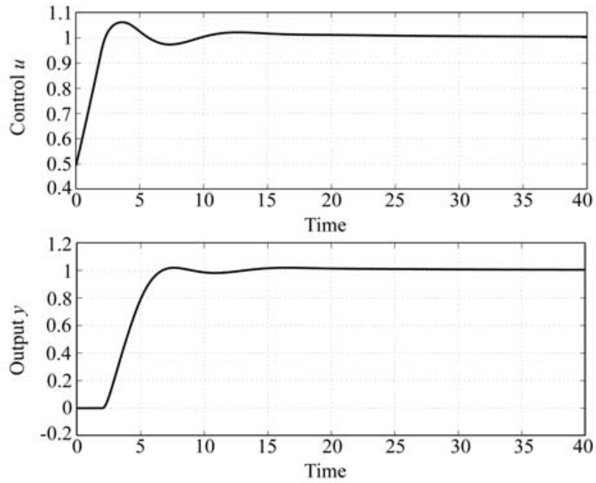

Fig. 5 Step responses of the time-delay system

4.3 Controller design for a heat-conduction process

Consider the plant whose transfer function $G(s)$ is given by

$$G(s) = \frac{1}{\sqrt{s}\,\sinh(\sqrt{s}\,)}. \tag{20}$$

The nonrational transfer function in (20) occurs when the plant is governed by a heat conduction (or diffusion) equation (see, for example, [26, 27]).

Assume that a fractional phase-lead controller is used and its transfer function is

$$K(s,p) = \frac{p_1(s^{p_4} + p_2)}{(s^{p_4} + p_3)} \tag{21}$$

where $p_3 > p_2 > 0$ and $p_4 > 0$. It is worth noting (see, for example, [28]) that the controller in (21) can be realized in practice.

The closed-loop transfer function takes the form (1) and is given by

$$H(s) = \frac{2p_1(s^{p_4} + p_2)\mathrm{e}^{-\sqrt{s}}}{\sqrt{s}(s^{p_4} + p_3)(1 - \mathrm{e}^{-2\sqrt{s}}) + 2p_1(s^{p_4} + p_2)\mathrm{e}^{-\sqrt{s}}}. \tag{22}$$

Suppose that the bounds C_i are specified as follows.

$$C_1 = 0.05,\ C_2 = 0.35,\ C_3 = 0.4,\ C_4 = 10.0. \tag{23}$$

By starting from a stability point p_0 such that

$$\alpha(p_0) \leqslant -0.001$$

the following design solution is found by the MBP algorithm.

$$p = [9.240,\ 7.513,\ 15.204,\ 1.101]^{\mathrm{T}} \tag{24}$$

where $\alpha(p) = -4.4383$ and the corresponding performance measures are

$$\phi_1 = 0.02,\ \phi_2 = 0.34,\ \phi_3 = 0.39,\ \phi_4 = 9.24.$$

The control and output signals of the system with the controller parameter p in (24) are shown in Fig. 6.

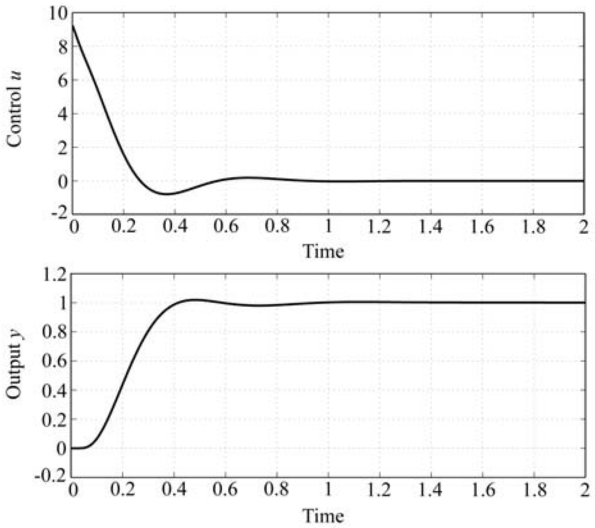

Fig. 6 Step responses of the heat-conduction system

5 Conclusions

This paper describes the design of RFDDSs by the MoI, in which the design problem is formulated so that it is suitable for solution by numerical methods. This is an extension of the design formulation which was first used in [5] in conjunction with rational systems and subsequently in [17, 18] in conjunction with retarded delay differential systems.

The use of this formulation is made possible for RFDDSs because the associated stability problems have been resolved by using the stability test and the algorithm for computing the abscissa of stability which have been recently developed in [19]. Moreover, in this formulation, the time responses of the system are obtained efficiently by using the Laplace transform inversion formula based on Zakian′s I_{MN} approximants. Once the time responses are obtained, the performances defined in terms of the responses (which are usually the most difficult ones to compute) are easily obtainable by known numerical algorithms.

The numerical results evidently show that by using the MoI, one can design RFDDSs effectively in a systematic way. Consequently, one can deal more easily with a sophisticated design problem and, provided that appropriate design criteria are used, can arrive at an accurate and realistic formulation of the design problem.

It is of interest to note that the design formulation for RFDDSs presented in this paper is not only useful for the MoI but also for other numerical optimization methods that search for a solution in a design-parameter space.

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