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Merton portfolio allocation under stochastic dividends

Lorenzo Reus[1]

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Abstract

Current methodologies for finding portfolio rules under the Merton framework employ hard-to-implement numerical techniques. This work presents a methodology that can derive an allocation à la Merton in a spreadsheet, under an incomplete market with a time-varying dividend yield and long-only constraints. The first step of the method uses the martingale approach to obtain a portfolio rule in a complete artificial market. The second step derives a closed-form optimal solution satisfying the long-only constraints, from the unconstrained solution of the first step. This is done by determining closed-form Lagrangian dual processes satisfying the primal-dual optimality conditions between the true and artificial markets. The last step estimates the parameters defined in the artificial market, to then obtain analytical approximations for the hedging demand component within the optimal portfolio rule of the previous step. The methodology is tested with real market data from 16 US stocks from the Dow Jones. The results show that the proposed solution delivers higher financial wealth than the myopic solution, which does not consider the time-varying nature of the dividend yield. The sensitivity analysis carried out on the closed-form solution reveals that the difference with respect to the myopic solution increases when the price of the risky asset is more sensitive to the dividend yield, and when the dividend yield presents a higher probability of diverging from the current yield. The proposed solution also outperforms a known Merton-type solution that derives the Lagrangian dual processes in another way.

Keywords Portfolio selection · Incomplete markets · Merton solution · Martingale method · Dividend yield

✉ Lorenzo Reus
lorenzo.reus@uai.cl

[1] Escuela de Negocios, Universidad Adolfo Ibáñez, Santiago, Chile

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1 Introduction

Continuous and time-varying allocations under the Merton [9] framework are still ignored by practitioners. As mentioned by Cochrane [4], "Merton's theory has almost no impact on portfolio practice" and solving Merton models "remains a productive and challenging enterprise". Evidently, ignoring the dynamic nature of markets in the portfolio allocation, as myopic solutions do, can lead to considerable welfare losses in the long term (see Larsen and Munk [8] and Castañeda and Reus [2] for examples).

To shed light on this matter, this work proposes an approximation method for a portfolio problem with one or two risky assets and a mean-reverting dividend yield process, to capture the well-known relationship between stock returns and the dividend-price ratio (see Cochrane [3]). The methodology is developed under an incomplete market, where the risk emerging from the dividend yield dynamics cannot be fully hedged. A closed-form solution is provided, using the "artificial markets"technique, developed by Karatzas et al. [7] and Cvitanić and Karatzas [5].

The method has two novel features, which are interesting for both academics and practitioners. The first is that it is very easy to implement, since the portfolio rule can be computed in a spreadsheet. Current à la Merton solutions involve implementing complex numerical methods which cannot be found in open-source programming languages. For example, Munk and Sørensen[10] use a finite-difference backwards iterative solution of the Hamilton–Jacobi–Bellman (HJB) equation to find the optimal investment in a life cycle problem with two assets and mean-reverting interest rates. Bick et al. [1] also employ the "artificial markets"technique to derive a portfolio rule for the same setting as in Munk and Sørensen [10]. However, the last step of their methodology includes a procedure for determining the near-optimal solution which is left unspecified.[1] More recently,Kamma and Pelsser [6] enhance the method from Bick et al.[1] to solve the a life cycle problem with more general return structures, general trading and liquidity constraints, and state-dependent utility functions.

The second is related to the way of determining the long-only allocations after deriving the unconstrained optimal allocation in the complete artificial market. Current methodologies prune (e.g Bick et al.[1]) or project (e.g Kamma and Pelsser [6]) the unconstrained solution to satisfy the portfolio constraints, which might lead to suboptimal solutions in the two-asset case. Instead, the method proposed derives a closed-form expression for the Lagrangian dual processes emerging from the long-only constraints. These dual processes, along with the constrained allocation, satisfy the primal-dual optimality conditions defined in Cvitanić and Karatzas[5]. It is important to remark that the formula can be applied in any portfolio problem involving two assets and long-only constraints, since the closed-form values of the dual processes are a function of nothing more than the unconstrained optimal solution and the asset's covariance matrix.

[1] Sect. 4 of Bick et al.[1] says "searching over different Ψ, we find the best of the feasible strategies $(c(\Psi^*), \pi(\Psi^*))$...", where Ψ is the set of parameters characterizing the processes from the artificial market.

2 Methodology

2.1 Market with one risky asset

Consider a market composed of constant risk-free rate r and a risky asset (stock) of price S_t, with the following dynamics

$$\mathrm{d}S_t/S_t = \mu_{S,t}\mathrm{d}t + \sigma_S \mathrm{d}W_{S,t}$$

where σ_S is the instantaneous volatility of the stock's return, $(W_t)_{0\le t\le T}$ is a standard Brownian Motion (BM) process. The instantaneous expected return $\mu_{S,t} = r + \alpha d_t$, where $\alpha > 0$ is a constant parameter and d_t is a mean-reverting process aimed at capturing the relationship between stock returns and the dividend-price ratio. The process d_t follows

$$\mathrm{d}d_t = \kappa(d_\infty - d_t)\mathrm{d}t + \sigma_d\left(\rho_{Sd}\mathrm{d}W_{S,t} + \sqrt{1-\rho_{Sd}^2}\mathrm{d}W_{d,t}\right), \tag{1}$$

where $(\kappa, d_\infty, \sigma_d, \rho_{Sd})$ are constant parameters, and W_d is an independent standard BM process. The investor's preferences are represented by the constant relative risk aversion (CRRA) utility

$$U_0(X_T) = \mathbf{E}_0\left[X_T^{1-\gamma}/(1-\gamma)\right],$$

where T is the planning horizon, $\gamma > 0$ is the Arrow-Pratt's coefficient of relative risk aversion. X_T is the terminal value of the financial wealth process, $X_t \ge 0\ \forall t \in [0, T]$, which follows

$$\mathrm{d}X_t = X_t\left(1-\pi_{S,t}\right)r\mathrm{d}t + X_t\pi_{S,t}\left(\mu_{S,t}\mathrm{d}t + \sigma_S\mathrm{d}W_{S,t}\right), X_0 \text{ given}, \tag{2}$$

where $\pi_{S,t} \in [a, b]$ is the fraction invested (of financial wealth, X_t) in the risky asset. It is assumed that the remains are invested in a risk-free asset (with return r).

Previous market is incomplete because it is not possible to fully hedge the risk emerging from process W_d in d_t. Thus, the "artificial markets"technique is used to complete the market with a fictitious second risky asset following $\mathrm{d}F_t/F_t = (r + \phi_{F,t})\mathrm{d}t + \mathrm{d}W_{d,t}$, where $\phi_{F,t}$ represents the market premium of the fictitious asset. Using the martingale approach, the investment problem can be written as

$$\max_{X_T} U_0(X_T), \text{ s.t. } \mathbf{E}_0\left[\xi_T X_T\right] \le X_0, \tag{3}$$

where ξ_t is the stochastic discount factor (SDF) in the artificial financial market. In this case, the SDF satisfies $\mathrm{d}\xi_t/\xi_t = -r\mathrm{d}t - \tilde{\theta}_t\mathrm{d}\tilde{W}_t$, with $\tilde{\theta}_t = (\frac{\alpha}{\sigma_S}d_t, \phi_{F,t})' \in \mathbb{R}^2$, $\tilde{W}_t = (W_{S,t}, W_{d,t})' \in \mathbb{R}^2$. Let $\pi_t' = (\pi_{S,t}, \pi_{F,t})' \in \mathbb{R}^2$ be the fraction of financial wealth invested in the real and fictitious assets, and let $\mu_t' = (\mu_{S,t}, \mu_{F,t})' \in \mathbb{R}^2$. The financial wealth X_t in this complete market follows

$$\mathrm{d}X_t = X_t\left(1-\pi_t'\mathbf{1}\right)r_t\mathrm{d}t + X_t\pi_t'\left(\mu_t\mathrm{d}t + \tilde{\sigma}\mathrm{d}\tilde{W}_t\right), \tag{4}$$

with $\tilde{\sigma} = \begin{pmatrix}\sigma_S & 0\\ 0 & 1\end{pmatrix}$. To find the unconstrained optimal solution, let ψ be the Lagrange multiplier of the budget constraint. KKT conditions in (3) deliver $\partial U(X_T)/\partial X_T = \psi\xi_T$. Thus, the optimal solution satisfies $X_T^* = (\psi\xi_T)^{-1/\gamma}$. Note that budget constraint is active in the optimal solution. If not, $\psi = 0$, which contradicts KKT. Defining $\xi_{t,T} = \xi_T/\xi_t$, then

$$X_t^* = \mathbf{E}_t\left[\xi_{t,T}X_T^*\right] = \mathbf{E}_t\left[\xi_{t,T}(\psi\xi_T)^{-\frac{1}{\gamma}}\right] = (\psi\xi_t)^{-\frac{1}{\gamma}}\mathbf{E}_t\left[\xi_{t,T}^R\right], \tag{5}$$

where $R = 1 - \frac{1}{\gamma}$. Define function $g_t(d_t)$ as

$$\begin{aligned}
g_t(d_t) &:= \mathbf{E}_t\left[\xi_{t,T}^R\right] = \mathbf{E}_t\left[\exp\left(-R\int_t^T (r + 0.5\tilde{\theta}_u'\tilde{\theta}_u)\mathrm{d}u - R\int_t^T \tilde{\theta}_u'\mathrm{d}W_u\right)\right]\\
&= \mathbf{E}_t\left[\exp\left(-Rr(T-t) - \frac{R}{2}\int_t^T\left(\frac{\alpha^2}{\sigma_S^2}d_u^2 + \phi_{F,u}^2\right)\mathrm{d}u - R\int_t^T\frac{\alpha d_u}{\sigma_S}\mathrm{d}W_{S,u} - R\int_t^T\phi_{F,u}\mathrm{d}W_{d,u}\right)\right]\\
&= \exp\left(-rR(T-t)\right)\\
&\quad \mathbf{E}_t\left[\exp\left(\int_t^T -\frac{R}{2}\left(\frac{\alpha^2}{\sigma_S^2}d_u^2 + \phi_{F,u}^2\right)\mathrm{d}u - \frac{R\alpha}{\sigma_S}\int_t^T d_u\mathrm{d}W_{S,u} - R\int_t^T\phi_{F,u}\mathrm{d}W_{d,u}\right)\right]
\end{aligned}$$

From (5) and the fact that $\mathbf{E}_0\left[\xi_T X_T\right] = X_0$, is not hard to find that $\psi = \frac{X_0}{E_0(\xi_T^R)}^{-\gamma}$, and thus the indirect utility function in this artificial market (which depends of process ϕ_F) is

$$J_0(X_0, \phi_F) := \max_\pi U_0(X_T) = \frac{X_0^{1-\gamma}}{1-\gamma}g_0(d_0)^\gamma \tag{6}$$

By applying Itô's Lemma on $(\psi\xi_t)^{-\frac{1}{\gamma}}$, its stochastic component equals $-\frac{(\psi\xi_t)^{-\frac{1}{\gamma}}}{\gamma}\tilde{\theta}_t\mathrm{d}\tilde{W}_t$. Analogously, the stochastic component of $g_t(d_t)$ equals $\partial_d g_t(d_t)(\sigma_d\rho_S, \sigma_d\sqrt{1-\rho_{Sd}^2})\mathrm{d}\tilde{W}_t$, where $\partial_d g_t(d_t)$ denotes the derivative with respect to d_t. The stochastic component of X_t^* on (5) equals $X_t^*\frac{\tilde{\theta}_t}{\gamma}\mathrm{d}\tilde{W}_t + X_t^*\frac{\partial_d g_t(d_t)}{g_t(d_t)}(\sigma_d\rho_S, \sigma_d\sqrt{1-\rho_{Sd}^2})\mathrm{d}\tilde{W}_t$. By matching the stochastic component of this SDE with the stochastic component defining X_t in (4), the following holds

$$X_t^*\tilde{\sigma}'\pi_t^* = \left[\frac{\tilde{\theta}_t}{\gamma} + \frac{\partial_d g_t(d_t)}{g_t(d_t)}\left(\sigma_d\rho_S, \sigma_d\sqrt{1-\rho_{Sd}^2}\right)'\right]X_t^*$$

where $\pi_t^* = (\pi_{S,t}^*, \pi_{F,t}^*)' \in \mathbb{R}^2$ are the unconstrained optimal allocations in the risky and fictitious assets respectively. Thus

$$\pi_{S,t}^* = \frac{\alpha d_t}{\gamma\sigma_S^2} + \frac{\sigma_d\rho_{Sd}}{\sigma_S}\frac{\partial_d g_t(d_t)}{g_t(d_t)} = \frac{\mu_{S,t} - r}{\gamma\sigma_S^2} + \frac{\sigma_d\rho_{Sd}}{\sigma_S}\frac{\partial_d g_t(d_t)}{g_t(d_t)} \tag{7}$$

$$\pi^*_{F,t} = \frac{\phi_{F,t}}{\gamma} + \sigma_d\sqrt{1-\rho^2_{Sd}}\frac{\partial_d g_t(d_t)}{g_t(d_t)}, \tag{8}$$

The first component of $\pi^*_{S,t}$ is the solution that would be obtained if $\mu_{S,t}$ is assumed to remain unchanged after time t. This would be the myopic solution. The second term is the *hedging demand* component (HD hereafter), which takes into account the time-varying nature of d_t after time t.

Since $\pi_{S,t} \in [a,b]$, then the artificial market proposed in Cvitanić and Karatzas [5] is used. This unconstrained market has a Lagrange multiplier process $\lambda_{S,t} \subseteq \mathbb{R}$, with a risk-free rate of $r + \delta_{[a,b]}(\lambda_{S,t})$, where $\delta_{[a,b]}(\lambda_{S,t}) := \sup_{x\in[a,b]}(-\lambda_{S,t} \cdot x)$. Therefore, $\tilde{\theta}_{t,1} = \alpha d_t/\sigma_S$ changes to $\alpha d_t + \lambda_{S,t}/\sigma_S$. Replacing last change into (7), the new optimal solution equals to $\tilde{\pi}_{S,t} = \pi^*_{S,t} + \frac{1}{\gamma\sigma_S^2}\lambda_{S,t}$. Finally, Proposition 8.3 of Cvitanić and Karatzas [5] is applied; if there is a $\lambda_{S,t}$ satisfying

$$\tilde{\pi}_{S,t} = \pi^*_{S,t} + \frac{1}{\gamma\sigma_S^2}\lambda_{S,t} \in [a,b]$$

$$\pi^*_{S,t}\lambda_{S,t} + \frac{1}{\gamma\sigma_S^2}\lambda^2_{S,t} + \delta_{[a,b]}(\lambda_{S,t}) = 0,$$

then $\tilde{\pi}_{S,t}$ is the optimal solution in set $[a, b]$. Defining $\tilde{\lambda}_{S,t} = \lambda_{S,t}/\gamma$, latter conditions can be written as

$$\tilde{\pi}_{S,t} = \pi^*_{S,t} + \frac{1}{\sigma_S^2}\tilde{\lambda}_{S,t} \in [a,b] \tag{9}$$

$$\pi^*_{S,t}\tilde{\lambda}_{S,t} + \frac{1}{\sigma_S^2}\tilde{\lambda}^2_{S,t} + \delta_{[a,b]}(\tilde{\lambda}_{S,t}) = 0, \tag{10}$$

Is not hard to see that conditions (9) and (10) are satisfied with

$$\begin{aligned} \tilde{\lambda}_{S,t} &= \sigma_S^2[\max\{a-\pi^*_{S,t},0\} - \max\{\pi^*_{S,t}-b,0\}] \\ \tilde{\pi}_{S,t} &= \pi^*_{S,t} + \max\{a-\pi^*_{S,t},0\} - \max\{\pi^*_{S,t}-b,0\} \end{aligned} \tag{11}$$

Proof

- Case $\pi^*_{S,t} < a$: Solution $\tilde{\pi}_{S,t} = a \Rightarrow \tilde{\lambda}_{S,t} = \sigma_S^2(a-\pi^*_{S,t})$. Since $\tilde{\lambda}_{S,t} > 0$ then $\delta_{[a,b]}(\tilde{\lambda}_{S,t}) = -a\tilde{\lambda}_{S,t}$, and $\pi^*_{S,t}\tilde{\lambda}_{S,t} + \frac{1}{\sigma_S^2}\tilde{\lambda}^2_{S,t} - a\tilde{\lambda}_{S,t} = 0$.
- Case $\pi^*_{S,t} \in [a,b]$: Solution $\tilde{\pi}_{S,t} = \pi^*_{S,t}$, $\Rightarrow \tilde{\lambda}_{S,t} = 0$. Thus $\delta_{[a,b]}(\tilde{\lambda}_{S,t}) = 0$, and $\pi^*_{S,t}\tilde{\lambda}_{S,t} + \frac{1}{\sigma_S^2}\tilde{\lambda}^2_{S,t} = 0$.
- Case $\pi^*_{S,t} > b$: Solution $\tilde{\pi}_{S,t} = b \Rightarrow \tilde{\lambda}_{S,t} = \sigma_S^2(b-\pi^*_{S,t})$. Since $\tilde{\lambda}_{S,t} < 0$, then $\delta_{[a,b]} = -b\tilde{\lambda}_S$, and $\pi^*_{S,t}\tilde{\lambda}_{S,t} + \frac{1}{\sigma_S^2}\tilde{\lambda}^2_{S,t} - b\tilde{\lambda}_{S,t} = 0$.

□

The final step is to derive a closed-form approximation for $\partial_d g_t(d_t)/g_t(d_t)$. In such way the method can provide a closed-form approximation for $\pi^*_{S,t}$ in (7), which allows to obtain $\tilde{\lambda}_{S,t}$ and $\tilde{\pi}_{S,t}$ using (11). To do so, $\phi_{F,t}$ must be estimated first. As explained in [1], the indirect utility in the artificial market is an upper bound for the indirect utility function of the true market. It is an upper bound for any values of $\phi_{F,t}$. Thus, the best bound is the process $\phi_{F,t}$ minimizing the indirect utility function of the artificial market. As an approximation, later function is minimized over the set $\mathcal{A}$ composed of the non-stochastic processes $\phi_{F,t}$. In such case, $\int_t^T \phi_{F,u} \mathrm{d}W_{S,u} \sim \mathcal{N}(0, \int_t^T \phi^2_{F,u} \mathrm{d}u)$. Thus

$$\begin{aligned}\min_{\phi_F \in \mathcal{A}} J_t(d_t, \phi_F) &:= \min_{\phi_F \in \mathcal{A}} \mathbf{E}_t \left[X_T^{1-\gamma}/(1-\gamma) \right] = \min_{\phi_F \in \mathcal{A}} \frac{X_t^{1-\gamma}}{1-\gamma} g_t(d_t)^\gamma \\ &= \min_{\phi_F \in \mathcal{A}} \frac{1}{1-\gamma} \exp\left(-\gamma \frac{R(1-R)}{2} \int_t^T \phi^2_{F,u} \mathrm{d}u \right) \\ &= \min_{\phi_F \in \mathcal{A}} \frac{1}{1-\gamma} \exp\left(\frac{1}{2} \left(\frac{1-\gamma}{\gamma} \right) \int_t^T \phi^2_{F,u} \mathrm{d}u \right)\end{aligned}$$

which is minimized when $\phi_{F,u} = 0 \; \forall u \in [t, T]$. The next step is to assume that d_u is not stochastic (only for the approximation of $g_t(d_t)$). For example, d_u can be equal to the expected value of such process, that is, $d_u = d_t \exp(-\kappa(u-t)) + d_\infty [1 - \exp(-\kappa(u-t))]$. In that case, $\int_t^T d_u \mathrm{d}W_{S,u} \sim \mathcal{N}(0, \int_t^T d_u^2 \mathrm{d}u)$ and thus

$$\hat{g}_t(d_t) = \exp\left[-R \left(r(T-t) + \frac{\alpha^2(1-R)}{2\sigma_S^2} \int_t^T d_u^2 \mathrm{d}u \right) \right]. \tag{12}$$

Hence

$$\frac{\partial_d \hat{g}_t(d_t)}{\hat{g}_t(d_t)} = -\frac{R}{\gamma} \frac{\alpha^2}{\sigma_S^2} \frac{1}{2} \partial_d \int_t^T d_u^2 \mathrm{d}_u = -\frac{R}{\gamma} \frac{\alpha^2}{\sigma_S^2} D^\kappa_{t,T},$$

with

$$\begin{aligned}D^\kappa_{t,T} &:= \frac{1}{2} \partial_d \int_t^T d_u^2 \mathrm{d}u = \int_t^T d_u \partial_d d_u \mathrm{d}u = \frac{1 - \exp(-2\kappa(T-t))}{2\kappa} (d_t - d_\infty) \\ &+ \frac{1 - \exp(-\kappa(T-t))}{\kappa} d_\infty.\end{aligned}$$

Therefore, the closed-form unconstrained allocation for the risky asset equals

$$\hat{\pi}_{S,t} = \frac{\alpha d_t}{\gamma \sigma_S^2} - \frac{\sigma_d \rho_{Sd}}{\sigma_S} \frac{R}{\gamma} \frac{\alpha^2}{\sigma_S^2} D^\kappa_{t,T} = \frac{\mu_{S,t} - r}{\gamma \sigma_S^2} - \frac{\sigma_d \rho_{Sd}}{\sigma_S} \left(\frac{\alpha}{\sigma_S} \right)^2 D^\kappa_{t,T} \frac{1}{\gamma} \left(1 - \frac{1}{\gamma} \right) \tag{13}$$

The HD component in (13) increases in magnitude with (i) an increase in $\frac{\sigma_d \rho_{Sd}}{\sigma_S}$, which represents the sensitivity of the dividend yield with respect to the asset return, (ii) an increase in the α/σ_S ratio, (iii) increases in the current and long-run dividend yields, since $D^{\kappa}_{t,T}$ increases with increases in d_t and d_∞, (iv) a decrease in κ, since $D^{\kappa}_{t,T}$ decreases with an increase in κ, (v) an increase in the time to horizon, since $D^{\kappa}_{t,T}$ increases with an increase in $T-t$. The previous sensitivity results are as expected. The HD component increases in magnitude either when the dividend yield diverges more frequently from d_t (e.g., smaller κ, longer horizon), or when the price of the risky asset is more sensitive to the dividend yield (e.g., higher d_t, α, or $\frac{\sigma_d \rho_{Sd}}{\sigma_S}$).

2.2 Market with two risky assets

The main complexity of this extension comes in determining the Lagrangian processes emerging from the long only-constraints. Now there are two possibly correlated risky asset with prices $S_t = (S_{1,t}, S_{2,t})$ following

$$\frac{\mathrm{d}S_t}{S_t} = \mu_{S,t}\mathrm{d}t + \sigma_S \mathrm{d}W_{S,t}$$

where $\sigma_S \in \mathbb{R}^{2\times 2}$ is a volatility matrix and $W_{S,t} \in \mathbb{R}^2$. Let $\mu_{S,t} = r + \tilde{d}_t$, with $\tilde{d}_t = (\alpha_1 d_{1,t}, \alpha_1 d_{2,t})' \in \mathbb{R}^2$, and d_t following

$$\mathrm{d}d_t = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} (d_\infty - d_t)\mathrm{d}t + \sigma_d (\mathrm{d}W_{S,t}, \mathrm{d}W_{d,t})' \tag{14}$$

where $\sigma_d = \begin{pmatrix} \sigma_{d,1}\rho_{Sd,1} & 0 & \sigma_{d,1}\sqrt{1-\rho^2_{Sd,1}} & 0 \\ 0 & \sigma_{d,2}\rho_{Sd,2} & 0 & \sigma_{d,2}\sqrt{1-\rho^2_{Sd,2}} \end{pmatrix}$. Equation (2) is adapted to

$$\mathrm{d}X_t = X_t\big(1 - \mathbf{1}'\pi_{S,t}\big)r\mathrm{d}t + X_t \pi'_{S,t}\big(\mu_{S,t}\mathrm{d}t + \sigma_S \mathrm{d}W_{S,t}\big), \tag{15}$$

where $\pi_{S,t} \in \mathcal{K} = \{(\pi_{S_1,t}, \pi_{S_2,t}) \in \mathbb{R}^2 : (\pi_{S_1,t}, \pi_{S_2,t}) \geq 0, \pi_{S_1,t} + \pi_{S_2,t} \leq 1\}$.

To complete the market, two fictitious assets are added $F_t = (F_{1,t}, F_{2,t})$ following $\mathrm{d}F_t/F_t = (r + \phi_{F,t})\mathrm{d}t + \mathrm{d}W_{d,t}$, where $\phi_{F,t} \in \mathbb{R}^2$ represents the market premium of both assets. The SDF then satisfies $\mathrm{d}\xi_t/\xi_t = -r\mathrm{d}t - \tilde{\theta}_t \mathrm{d}\tilde{W}_t$, with $\tilde{\theta}_t = (\sigma_S^{-1}\tilde{d}_{S,t}, \phi_{F,t})' \in \mathbb{R}^4$, $\tilde{W}_t = (W_{S,t}, W_{d,t})' \in \mathbb{R}^4$. Following the same martingale approach with one risky asset, it is straightforward to show that the optimal unconstrained solution in this complete market equals

$$\pi^*_{S,t} = \frac{1}{\gamma}(\sigma_S\sigma_S')^{-1}\tilde{d}_t + (\sigma_S')^{-1}\begin{pmatrix}\sigma_{d,1}\rho_{Sd,1} & 0\\ 0 & \sigma_{d,2}\rho_{Sd,2}\end{pmatrix}\frac{\partial_d g(d_t)}{g(d_t)}$$
$$= \frac{1}{\gamma}(\sigma_S\sigma_S')^{-1}(\mu_{S,t}-r) + (\sigma_S')^{-1}\begin{pmatrix}\sigma_{d,1}\rho_{Sd,1} & 0\\ 0 & \sigma_{d,2}\rho_{Sd,2}\end{pmatrix}\frac{\partial_d g(d_t)}{g(d_t)} \tag{16}$$
$$\pi^*_{F,t} = \frac{\phi_{F,t}}{\gamma} + \begin{pmatrix}\sigma_{d,1}\sqrt{1-\rho^2_{Sd,1}} & 0\\ 0 & \sigma_{d,2}\sqrt{1-\rho^2_{Sd,2}}\end{pmatrix}\frac{\partial_d g(d_t)}{g(d_t)}$$

with $g_t(d_t)$ as

$$g_t(d_t) := \mathbf{E}_t\left[\exp\left(-R\int_t^T (r+0.5\tilde{\theta}_u'\tilde{\theta}_u)\mathrm{d}u - R\int_t^T \tilde{\theta}_u'\mathrm{d}\tilde{W}_u\right)\right]$$
$$= \exp(-rR(T-t))\mathbf{E}_t\left[\exp\left(-\frac{R}{2}\int_t^T\left(\tilde{d}_u'(\sigma_S\sigma_S')^{-1}\tilde{d}_u + \phi_{F,u}'\phi_{F,u}\right)\mathrm{d}u\right)\right.$$
$$\left.\exp\left(-R\int_t^T \tilde{d}_u'(\sigma_S')^{-1}\mathrm{d}W_{S,u} - R\int_t^T \phi_{F,u}'\mathrm{d}W_{d,u}\right)\right]$$

Since $\pi_{S,t} \in \mathcal{K}$, then the artificial market proposed in Cvitanić and Karatzas [5] is used. This unconstrained market has a Lagrange multiplier process $\lambda_{S,t} \subseteq \mathbb{R}^2$, with a risk-free rate of $r + \delta_{\mathcal{K}}(\lambda_{S,t})$, where $\delta_{\mathcal{K}}(\lambda_{S,t}) := \sup_{x\in\mathcal{K}}(-\lambda_{S,t}'x)$. Therefore, $\tilde{\theta}_{t,1} = \sigma_S^{-1}\tilde{d}_{S,t}$ changes to $\sigma_S^{-1}\tilde{d}_{S,t} + \sigma_S^{-1}\lambda_{S,t}$. Replacing last change into (16), the new optimal solution equals to $\tilde{\pi}_{S,t} = \pi^*_{S,t} + \frac{1}{\gamma}(\sigma_S\sigma_S')^{-1}\lambda_{S,t}$. Finally, Proposition 8.3 of Cvitanić and Karatzas [5] is applied exactly as with the one-asset case; if there is a $\tilde{\lambda}_{S,t} \subseteq \mathbb{R}^2$ satisfying

$$\tilde{\pi}_{S,t} = \pi^*_{S,t} + (\sigma_S\sigma_S')^{-1}\tilde{\lambda}_{S,t} \in \mathcal{K} \tag{17}$$

$$\pi^*_{S,t}\tilde{\lambda}_{S,t} + \tilde{\lambda}_{S,t}'(\sigma_S\sigma_S')^{-1}\tilde{\lambda}_{S,t} + \delta_{\mathcal{K}}(\tilde{\lambda}_{S,t}) = 0, \tag{18}$$

then $\tilde{\pi}_{S,t}$ is the optimal solution in $\mathcal{K}$. Is not hard to see that $\delta_{\mathcal{K}}(\tilde{\lambda}_S) := \sup_{x\in\mathcal{K}}(-x'\tilde{\lambda}_{S,t}) = \max\{-\tilde{\lambda}_{S_1,t}, -\tilde{\lambda}_{S_2,t}, 0\}$. Let $\sigma_S\sigma_S' = \begin{pmatrix}\sigma_1^2 & \sigma_1\sigma_2\rho_{12}\\ \sigma_1\sigma_2\rho_{12} & \sigma_2^2,\end{pmatrix}$, representing the covariance matrix of the two assets and denote $(\pi_1, \pi_2) = (\pi_{S_1,t}, \pi_{S_2,t})$. Appendix 1 shows that the solution of Eqs. (17) and (18) is given by

$$\tilde{\lambda}_S = \begin{cases} (0,0) & \text{if } (\pi_1^*,\pi_2^*) \in \mathcal{K} \\ -(\sigma_1^2\pi_1^*(1-\rho_{12}^2),0) & \text{if } \{0 \le A_2 \le \sigma_2^2, \pi_1^* < 0\} \\ -(0,\sigma_2^2\pi_2^*(1-\rho_{12}^2)) & \text{if } \{0 \le A_1 \le \sigma_1^2, \pi_2^* < 0\} \\ -(A_1,A_2) & \text{if } \{A_1<0, A_2<0, (\pi_1^*<0 \cup \pi_2^*<0)\} \\ -(A_1-\sigma_1^2, & \text{if } \{A_1 > \sigma_1^2, \pi_2^* < 0\} \cup \{A_1>\sigma_1^2, \\ \quad A_2-\sigma_1\sigma_2\rho_{12}) & \quad A_2 \le A_1-(\sigma_1^2-\sigma_1\sigma_2\rho_{12}), \pi_1^*\ge 0, \pi_2^*\ge 0, \pi_1^*+\pi_2^*>1\} \\ -(A_1-\sigma_1\sigma_2\rho_{12}, & \text{if } \{A_2>\sigma_2^2, \pi_1^*<0\} \cup \{A_2>\sigma_2^2, \\ \quad A_2-\sigma_2^2) & \quad A_2 \ge A_1+(\sigma_2^2-\sigma_1\sigma_2\rho_{12}), \pi_1^*\ge 0, \pi_2^*\ge 0, \pi_1^*+\pi_2^*>1\} \\ -\frac{\sigma_1^2\sigma_2^2(\pi_1^*+\pi_2^*-1)}{\sigma_2^2-2\sigma_1\sigma_2\rho_{12}+\sigma_1^2}(1,1) & \text{otherwise} \end{cases} \tag{19}$$

where $A_1 = \pi_1^*\sigma_1^2+\pi_2^*\sigma_1\sigma_2\rho_{12}$ and $A_2 = \pi_2^*\sigma_2^2+\pi_1^*\sigma_1\sigma_2\rho_{12}$.

An interesting remark of the closed-form solution in (19) is that it can be applied on other investment opportunity sets, since it only depends on the covariance matrix of both assets and the unconstrained portfolio rule on such market setting.

The last step step is to derive a closed-form approximation for $\partial_d g_t(d_t)/g_t(d_t)$. To derive $\phi_{F,t}$, the same methodology as in the one-asset case is used. Similarly, $J_0(X_0,\phi_F)$ is minimized when $\phi_{F,u} = 0$, thus:

$$\hat{g}_t(d_t) = \exp\left[-R\left(r(T-t)+\frac{(1-R)}{2}\int_t^T \tilde{d}_u'(\sigma_S\sigma_S')^{-1}\tilde{d}_u \mathrm{d}u\right)\right]$$

Let $B = diag\begin{pmatrix}\exp(-\kappa_1(u-t))\\ \exp(-\kappa_2(u-t))\end{pmatrix}$. To derive $g_t(d_t)$, d_u can be approximated with $d_u = Bd_t+d_\infty[I-B]$. Then

$$\frac{\partial_d \hat{g}_t(d_t)}{\hat{g}_t(d_t)} = -\frac{R}{\gamma}\begin{pmatrix}\alpha_1^2 & 0\\ 0 & \alpha_2^2\end{pmatrix}\frac{1}{2}\partial_d\int_t^T d_u'(\sigma_S\sigma_S')^{-1}d_u \mathrm{d}u = -\frac{R}{\gamma}\begin{pmatrix}\alpha_1^2 & 0\\ 0 & \alpha_2^2\end{pmatrix}D_{t,T}^{\kappa_1,\kappa_2}$$

with

$$(D_{t,T}^{\kappa_1,\kappa_2})_1 := (\sigma_S\sigma_S')^{-1}_{1,*}\left[diag\begin{pmatrix}\frac{1-\exp(-2\kappa_1(T-t))}{2\kappa_1}\\ \frac{1-\exp(-(\kappa_1+\kappa_2)(T-t))}{\kappa_1+\kappa_2}\end{pmatrix}(d_t-d_\infty)+\frac{1-\exp(-\kappa_1(T-t))}{\kappa_1}d_\infty\right]$$
$$(D_{t,T}^{\kappa_1,\kappa_2})_2 := (\sigma_S\sigma_S')^{-1}_{2,*}\left[diag\begin{pmatrix}\frac{1-\exp(-(\kappa_1+\kappa_2)(T-t))}{\kappa_1+\kappa_2}\\ \frac{1-\exp(-2\kappa_2(T-t))}{2\kappa_2}\end{pmatrix}(d_t-d_\infty)+\frac{1-\exp(-\kappa_2(T-t))}{\kappa_2}d_\infty\right]$$

Therefore, the closed-form unconstrained allocation for the risky assets equals

$$\hat{\pi}_{S,t} = \frac{(\sigma_S\sigma_S')^{-1}}{\gamma}\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix} d_t - (\sigma_S')^{-1}\begin{pmatrix}\sigma_{d,1}\rho_{Sd,1} & 0\\ 0 & \sigma_{d,2}\rho_{Sd,2}\end{pmatrix}\begin{pmatrix}\alpha_1^2 & 0\\ 0 & \alpha_2^2\end{pmatrix} D_{t,T}^{\kappa_1,\kappa_2}\frac{1}{\gamma}\left(1-\frac{1}{\gamma}\right) \tag{20}$$

2.2.1 Pruning suboptimality example

Note also that the solution proposed can be different from the one obtained by *pruning* the unconstrained optimal solution to the *nearest* solution meeting the long-only constraints.[2] For illustration purposes, suppose that the dividend yield is constant, in which case the unconstrained optimal solution does not change over time and equals the myopic component. Under this assumption and by applying Itô's Lemma to (15), it is not hard to see that the indirect utility function is

$$J_0(X_0) = \frac{X_0^{1-\gamma}}{1-\gamma}\exp\left((r + (\mu_S - r)'\pi_S - \frac{\gamma}{2}\pi_S'\sigma_S\sigma_S'\pi_S)T(1-\gamma)\right)$$

Instead of using $J_0(X_0)$ to measure the performance of a portfolio rule, the certainty equivalent (CE) is used. The CE is the initial wealth value X_0^{CE} such that $U_0(X_0^{CE}) = J_0(X_0) \Leftrightarrow X_0^{CE}(\pi_S) = [(1-\gamma)J_0(X_0)]^{\frac{1}{1-\gamma}}$. Note that in the example

$$X_0^{CE}(\pi_S) = X_0 \exp\left((r + (\mu_S - r)'\pi_S - \frac{\gamma}{2}\pi_S'\sigma_S\sigma_S'\pi_S)T\right) \tag{21}$$

In the following setting $\sigma_1 = 5\%, \sigma_1 = 20\%, \mu_S = (0\%, 10\%)', \rho_{12} = 0.7, r = 2\%, \gamma = 4$, the unconstrained optimal solution $\pi_S^* = \frac{1}{\gamma}(\sigma_S\sigma_S')^{-1}(\mu_S - r) = (-20/3, 7/3)'$. From (19), $\tilde{\lambda}_S = (0.0085, 0)'$, which implies $\tilde{\pi}_S = (0, 0.5)'$. For $X_0 = 1, T = 50$, then $X_T^{CE}(\tilde{\pi}_S) = 7.39$ (or annual return of 4%). The nearest feasible solution from π_S^* is $\pi_S^{near} = (0, 1)'$, with $X_T^{CE}(\pi_S^{near}) = 2.71$ (or annual return of 2%).

3 Results

To test the methodology with data from the US equity market, a sample of monthly dividend yields and prices were taken from the stocks of the Dow Jones index. For calibration purposes, the stocks chosen were those for which data were available from Jan-1990 to Feb-2024. Stocks that did not pay dividends for long periods of time (e.g. APPL) were not considered. The final sample consisted of 16 stocks out of the 30 stocks composing the index. Figure 4 in Appendix 3 depicts the dividend yields and prices of these 16 stocks.

The next step was to calibrate the model for each of the 16 stocks. For illustration purposes, Table 1 shows the calibration results for the eight stocks that have the highest hedging demands, relative to the unconstrained solution of Eq. (13), when

[2] With one risky asset, Eq. (11) shows that the solution proposed coincides with the *nearest* solution meeting the long-only constraints.

Table 1 (Upper): Estimation results for eight stocks from the Dow Jones index

	HD	AXP	HON	WMT	MCD	JPM	JNJ	KO
α	6.3	5.8	4.8	4.1	4.4	2.9	3.1	2.4
σ_S	26%	30%	27%	22%	20%	32%	19%	20%
κ	0.10	0.19	0.17	0.08	0.12	0.40	0.19	0.14
d_∞	1.4%	2.0%	2.7%	1.2%	1.7%	3.7%	2.3%	2.3%
σ_d	.5%	.9%	.7%	.3%	.4%	2.5%	.4%	.4%
ρ_{Sd}	−0.70	−0.85	−0.89	−0.76	−0.65	−0.63	−0.91	−0.89
$\sigma_d \rho_{Sd}/\sigma_S$	−1.22%	−2.43%	−2.38%	−1.04%	−1.43%	−5.12%	−2.02%	−1.91%
$(\alpha/\sigma_S)^2$	570.5	359.9	303.3	357.0	466.1	82.7	281.0	151.4
$D^\kappa_{0,50}$	0.14	0.10	0.16	0.16	0.15	0.09	0.12	0.17
$HD^{\gamma=2}_{t=0,T=50}$	25%	22%	29%	15%	25%	10%	18%	12%
Myopic $\pi^{\gamma=2}_{t=0,T=50}$	64%	62%	86%	54%	92%	54%	104%	73%
$\hat{\pi}^{\gamma=2}_{t=0,T=50}$	89%	84%	115%	69%	117%	64%	122%	85%
$HD/\hat{\pi}$	28%	26%	25%	22%	21%	16%	14%	14%

The sample consists of monthly dividend yields and stock prices, from the period Jan-90 to Feb-24. The estimation was done with AR(1) regressions on the dividend yield, and regressions on the stock's return and the dividend yield. (Lower): Myopic and dynamic unconstrained allocations for $t = 0$, $T = 50$ and $\gamma = 2$, as in Eq. (13). The first rows show the HD decomposition, i.e. $HD^\gamma_{t,T} = \left[-\frac{\sigma_d \rho_{Sd}}{\sigma_S}(\frac{\alpha}{\sigma_S})^2 D^\kappa_{t,T}\right]\frac{1}{\gamma}(1-\frac{1}{\gamma})$. The risk-free rate r equals to the Dec-23 yield of the 10-year US treasury bond ($r = 3.8\%$). The yield at $t = 0$ is set to $d_0 = d_\infty$ for every stock

$t = 0$, $T = 50$ and $\gamma = 2$. A priori, these eight stocks are the ones presenting the highest potential for the methodology to outperform the myopic solution. Calibration results for the other eight stocks can be found in Table 2 (Appendix 3).

3.1 Market with one risky asset

Figure 1 shows the increase in the CE obtained by the dynamic solution with respect to the CE of the myopic solution, in the presence of long-only constraints, i.e. $\pi_{S,t} \in [0, 1]$. Note that the formula in (21) cannot be used to estimate the CE, because the dividend yield is dynamic (thus $\mu_{S,t}$ and $\tilde{\pi}_{S,t}$ are dynamic too). Hence, the CE is estimated through simulations. The CE is obtained by performing 200,000 simulations of the risky asset and the dividend yield process, with a discretization of $dt = 1/200$ (200 times in a year). These simulations were done in MATLAB R2021b on a personal computer, specifically a MacBook Pro with a Quad-Core Intel Core i5 and 16 GB RAM. For $T = 50$, a simulation is completed in approximately one minute.

As expected, the stocks with higher (lower) differences in CE coincide with the ones with higher (lower) HDs/$\hat{\pi}$ ratios from Table 1. This relationship is useful because the potential benefit given by the non-myopic solution for a stock can be measured before the simulation is run. Results for the other eight stocks can be

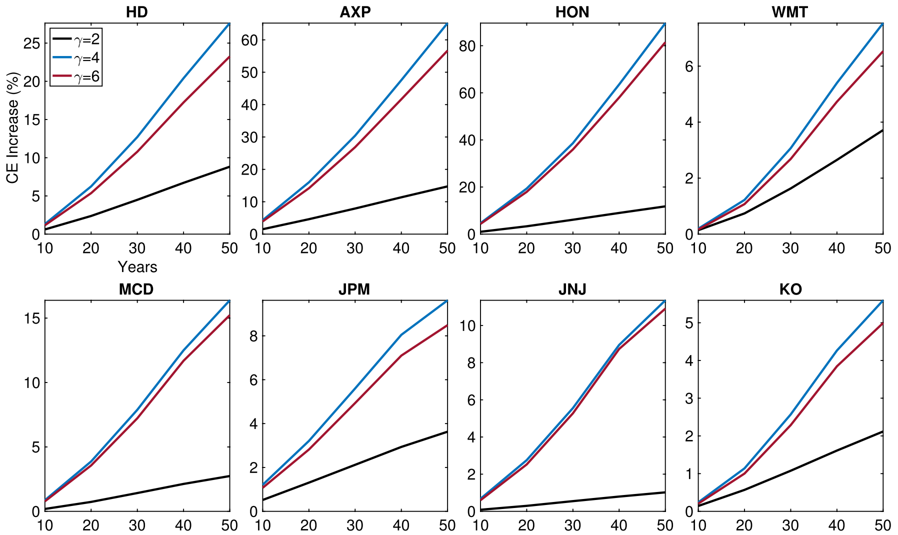

Fig. 1 Increase in CE (%) from the proposed solution with respect to the CE of the myopic solution, for different risk aversion levels γ and horizons T (years). The CE is obtained by doing 200,000 simulations of the risky asset and the dividend yield process, with a discretization of $dt = 1/200$ (200 times in a year). As a remark, the CE increase is equivalent to the loss defined in Larsen and Munk[8]

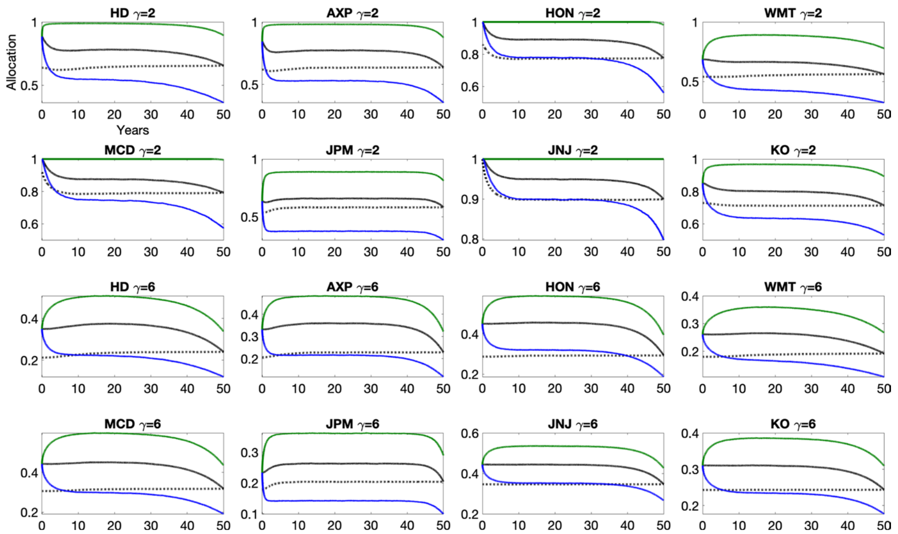

Fig. 2 Average allocations obtained by the proposed solution (black) and the myopic solution (dashed black) when horizon $T = 50$ and $\gamma = \{2, 6\}$. The green (blue) lines are the average allocations of the solution when the dividend yield is above (below) the long-term dividend yield d_∞ (color figure online)

found in Fig. 5 (Appendix 3). Figure 2 shows the average allocations to the risky asset for cases $\gamma = \{2, 6\}$. Since the HD is positive, it is expected that the dynamic solution will over-weight the allocation to the risky asset. This is because a negative shock to the price of the risky asset generally comes with a positive shock to the dividend yield ($\rho_{Sd} \ll 0$), which increases the drift of that asset (recall $\mu_{S,t} = r + \alpha d_t$). The figure also shows how allocations to the risky asset increase (decrease) in the scenarios where the dividend yield is above (below) the long-term dividend yield d_∞. As expected, the HD decreases when approaching the horizon and the allocation to the risky asset decreases when the risk aversion is higher.

Appendix 2 shows how to implement the method explained in Bick et al. [1] (BKM hereafter) for this dividend model. For the one-asset case, the BKM method produces almost the same results as the methodology proposed in this study. The reasons are the following: (i) the approximation of $g_t(d_t)$ to be made in BKM is similar to $\hat{g}_t(d_t)$ in (12) with these data, thus both methods deliver the same unconstrained solution, and (ii) the constrained solution found using (11) coincides with the pruned solution of BKM. Even if both solutions are similar, the implementation of the BKM method is more complex than that of the method proposed in this study. As explained in Appendix 2, the BKM method requires solving an optimization problem to derive the parameters included in the artificial market, which has to be done numerically. Function $g_t(d_t)$ and its derivative must be computed using numerical integration methods. Thus, the solution cannot be found on a spreadsheet and finding the CE through a simulation can take a considerable amount of time.

3.2 Market with two risky assets

With these calibration results, the unconstrained solution derived in Eq. (20) is similar to the unconstrained solution derived using the BKM method. However, there are differences when delivering a solution satisfying long-only constrains. Similar to the example in Sect. 2.2.1, BKM prunes the unconstrained solution instead of finding the Lagrangian solution in (19). The following link provides an spreadsheet that demonstrates how to find the constrained solution using the method proposed in this study.

Figure 3 shows the CE increase w.r.t. the myopic solution and w.r.t the BKM solution for four pairs of the eight stocks shown in Table 1. As expected, the CE increase is higher w.r.t. the myopic solution in most cases. However, the CE increase w.r.t the myopic solution is much higher than the CE increase w.r.t the BKM solution for $\gamma = 4$ and $\gamma = 6$. In those settings, this means that considering the HD is more important than using the Lagrange duals in (19). When risk aversion increases, allocation to both stocks are reduced. Hence, the unconstrained solution satisfies long-only constraints, which explains the similarities of the proposed solution to BKM. For $\gamma = 2$, the unconstrained allocations in both stocks can add more than one. Thus deriving the Lagrange duals in (19), instead of pruning the unconstrained solution, produces more differences w.r.t. the BKM solution. In terms of computational time, it took around 25 min to derive the CE for $T = 50$, by performing 200,000 simulations of each risky asset and each dividend yield process, with a discretization of $dt = 1/200$.

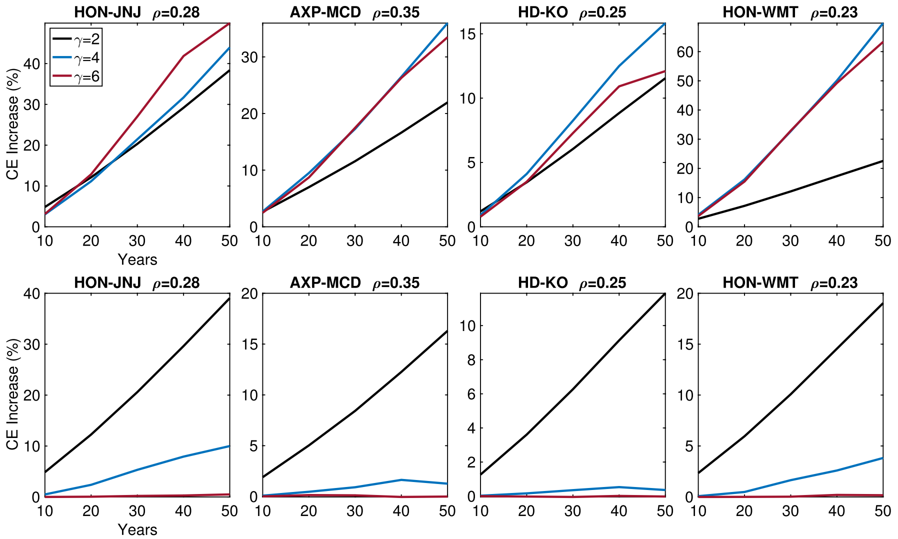

Fig. 3 (Upper): Increase in CE (%) from the proposed solution with respect to the CE of the myopic solution, for four pairs of stocks, risk aversion levels γ and horizons T (years). (Lower): Increase in CE (%) from the proposed solution with respect to the CE of the BKM solution

4 Conclusion

This work presents an example of how to solve a portfolio problem under the Merton framework with closed-form approximations instead of hard-to-implement numerical methods. The examples illustrate how important it is to include the time-varying nature of markets to prevent welfare losses, and also how to correctly use the non-constrained solution to derive the constrained solution. The methodology presented can be reduced to solving the following tasks: (i) derive the parameters of the artificial assets (e.g., $\phi_{F,t}$), (ii) find closed-form approximations of the HD (e.g., $g_t(d_t)$) and (iii) determine the Lagrange processes emerging from portfolio constraints (e.g., $\tilde{\lambda}_{S,t}$). Evidently, this work can be extended by changing the investment opportunity set. For example, include another time-varying driver besides the dividend yield, or develop a method to derive $\tilde{\lambda}_{S,t}$ for multiple assets and/or other portfolio constraints. Another possibility is to apply the method to a life cycle problem.

Appendix 1: Optimal Lagrange multiplier process

Let $(\sigma_S\sigma_S')^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho_{12}^2)}\begin{pmatrix}\sigma_2^2 & -\rho_{12}\sigma_1\sigma_2\\ -\rho_{12}\sigma_1\sigma_2 & \sigma_1^2\end{pmatrix}$. Conditions (17) and (18) reduce to

$$\tilde{\pi}_1 = \pi_1 + \frac{1}{\sigma_1^2(1-\rho_{12}^2)}\tilde{\lambda}_1 - \frac{\rho_{12}}{\sigma_1\sigma_2(1-\rho_{12}^2)}\tilde{\lambda}_2$$
$$\tilde{\pi}_2 = \pi_2 - \frac{\rho_{12}}{\sigma_1\sigma_2(1-\rho_{12}^2)}\tilde{\lambda}_1 + \frac{1}{\sigma_2^2(1-\rho_{12}^2)}\tilde{\lambda}_2$$
$$\tilde{\pi}_1\tilde{\lambda}_1 + \tilde{\pi}_2\tilde{\lambda}_2 + \max\{-\tilde{\lambda}_1, -\tilde{\lambda}_2, 0\} = 0$$

Define $A_1 = \pi_1\sigma_1^2 + \pi_2\sigma_1\sigma_2\rho_{12}$ and $A_2 = \pi_2\sigma_2^2 + \pi_1\sigma_1\sigma_2\rho_{12}$

1. Case $\pi_1 < 0, \pi_2 < 0$ It is not possible to have $A_1 > 0$ and $A_2 > 0$, otherwise term $\pi_1 A_1 + \pi_2 A_2 < 0 \Rightarrow \pi_1^2\sigma_1^2 + \pi_2^2\sigma_2^2 + 2\pi_2\pi_1\sigma_1\sigma_2\rho_{12} < 0$, which is a contradiction. Thus

   Solution 1: $\tilde{\pi} = (0, 0) \Rightarrow \tilde{\lambda}_1 = -\pi_1\sigma_1^2 - \pi_2\sigma_1\sigma_2\rho_{12} = -A_1$ and $\tilde{\lambda}_2 = -A_2$. To satisfy (18), then $\delta_{\mathcal{K}} = 0$ must hold. Thus, $\tilde{\lambda}_1 \geq 0$ and $\tilde{\lambda}_2 \geq 0$, that is, $A_1 \leq 0$ and $A_2 \leq 0$.

   Solution 2: $\tilde{\pi} = (\pi_1 + \frac{\rho_{12}\sigma_2}{\sigma_1}\pi_2, 0) = (\frac{A_1}{\sigma_1^2}, 0)$, where $\tilde{\lambda} = (0, -\pi_2(1-\rho_{12}^2)\sigma_2^2)$. Thus $\delta_{\mathcal{K}} = 0$ and (18) holds. To satisfy (17), $0 \leq A_1 \leq \sigma_1^2$ must hold. Since $0 \leq A_1$, then $A_2 \leq 0$.

   Solution 3: $\tilde{\pi} = (1, 0)$. In this case $\tilde{\lambda}_1 = (1-\pi_1)\sigma_1^2 - \pi_2\sigma_1\sigma_2\rho_{12} = \sigma_1^2 - A_1$ and $\tilde{\lambda}_2 = -\pi_2\sigma_2^2 + (1-\pi_1)\sigma_1\sigma_2\rho_{12} = -A_2 + \sigma_1\sigma_2\rho_{12}$. To satisfy (18), $\delta_{\mathcal{K}} = -\tilde{\lambda}_1$- must hold. Thus, $\tilde{\lambda}_1 \leq 0$ and $\tilde{\lambda}_1 \leq \tilde{\lambda}_2$, that is, $A_1 > \sigma_1^2$. Again, $A_2 \leq 0$. By symmetry, there are two more solutions. In summary

   - $\tilde{\pi} = (0, 0), \tilde{\lambda} = (-A_1, -A_2)$ if $A_1 \leq 0, A_2 \leq 0$
   - $\tilde{\pi} = (\pi_1 + \frac{\rho_{12}\sigma_2}{\sigma_1}\pi_2, 0), \tilde{\lambda} = (0, -\pi_2(1-\rho_{12}^2)\sigma_2^2)$ if $0 \leq A_1 \leq \sigma_1^2, A_2 \leq 0$
   - $\tilde{\pi} = (1, 0), \tilde{\lambda} = (\sigma_1^2 - A_1, -A_2 + \sigma_1\sigma_2\rho_{12})$, if $A_1 \geq \sigma_1^2, A_2 \leq 0$
   - $\tilde{\pi} = (0, \pi_2 + \frac{\rho_{12}\sigma_1}{\sigma_2}\pi_1), \tilde{\lambda} = (-\pi_1(1-\rho_{12}^2)\sigma_1^2, 0)$ if $0 \leq A_2 \leq \sigma_2^2, A_1 \leq 0$
   - $\tilde{\pi} = (0, 1), \tilde{\lambda} = (-A_1 + \sigma_1\sigma_2\rho_{12}, \sigma_2^2 - A_2)$, if $A_2 \geq \sigma_2^2, A_1 \leq 0$

2. Case $\pi_1 < 0, \pi_2 \in [0, 1]$

   Solution 1: $\tilde{\pi} = (0, 0), \tilde{\lambda} = (-A_1, -A_2)$ satisfy conditions if $A_1 \leq 0, A_2 \leq 0$. In this case though, $A_2 \leq 0$ implies $A_1 \leq 0$. Thus $A_2 \leq 0$ must hold.

   Solution 2: $\tilde{\pi} = (0, \pi_2 + \frac{\rho_{12}\sigma_1}{\sigma_2}\pi_1), \tilde{\lambda} = (-\pi_1(1-\rho_{12}^2)\sigma_1^2, 0)$ satisfy conditions if $0 \leq A_2 \leq \sigma_2^2$.

   Solution 3: $\tilde{\pi} = (0, 1), \tilde{\lambda} = (-A_1 + \sigma_1\sigma_2\rho_{12}, \sigma_2^2 - A_2)$ satisfy conditions if $A_2 \geq \sigma_2^2$.
3. Case $\pi_1 < 0, \pi_2 > 1$: Same solution as case 2.
4. Case $\pi_1 \in [0, 1], \pi_2 < 0$: By symmetry with case 2

   Solution 1: $\tilde{\pi} = (0, 0), \tilde{\lambda} = (-A_1, -A_2)$ satisfy conditions if $A_1 \leq 0, A_2 \leq 0$. In this case though, $A_1 \leq 0$ implies $A_2 \leq 0$. Thus $A_1 \leq 0$ must hold.

   Solution 2: $\tilde{\pi} = (\pi_1 + \frac{\rho_{12}\sigma_2}{\sigma_1}\pi_2, 0), \tilde{\lambda} = (0, -\pi_2(1-\rho_{12}^2)\sigma_2^2)$ satisfy conditions if $0 \leq A_1 \leq \sigma_1^2$

   Solution 3: $\tilde{\pi} = (1, 0), \tilde{\lambda} = (\sigma_1^2 - A_1, -A_2 + \sigma_1\sigma_2\rho_{12})$ satisfy conditions if $A_1 \geq \sigma_1^2$.

5. Case $\pi_1 > 1, \pi_2 < 0$: Same solution as case 4
6. Case $\pi_1 \in [0,1], \pi_2 \in [0,1], \pi_1 + \pi_2 \le 1$: The only solution is $\tilde{\pi} = (\pi_1, \pi_2), \tilde{\lambda} = (0,0)$
7. Case $\pi_1 \in [0,1], \pi_2 \in [0,1], \pi_1 + \pi_2 > 1$: The following cannot hold

(i) $A_1 < 0$, $A_2 < 0$: To see this, suppose otherwise. Term $\pi_1 A_1 + \pi_2 A_2 < 0 \Rightarrow \pi_1^2\sigma_1^2 + \pi_2^2\sigma_2^2 + 2\pi_2\pi_1\sigma_1\sigma_2\rho_{12} < 0$, which is a contradiction.
(ii) $A_2 < \sigma_2^2, A_2 > A_1 + (\sigma_2^2 - \sigma_1\sigma_2\rho_{12})$: $A_2 < \sigma_2^2$ implies $A_2 < A_1 + (\sigma_2^2 - \sigma_1\sigma_2\rho_{12})$.
(iii) $A_1 < \sigma_1^2, A_2 < A_1 - (\sigma_1^2 - \sigma_1\sigma_2\rho_{12})$: $A_1 < \sigma_1^2$ implies $A_2 > A_1 - (\sigma_1^2 - \sigma_1\sigma_2\rho_{12})$.
(iv) $A_1 < 0, A_2 > A_1 + (\sigma_2^2 - \sigma_1\sigma_2\rho_{12})$: Since $A_1 < 0$, then $\sigma_1\sigma_2\rho_{12} < 0$. This implies $(1-\pi_2)(\sigma_2^2 - \sigma_1\sigma_2\rho_{12}) + \pi_1(\sigma_1^2 - \sigma_1\sigma_2\rho_{12}) > 0$, that is $A_2 < A_1 + (\sigma_2^2 - \sigma_1\sigma_2\rho_{12})$.
(v) $A_2 < 0, A_2 < A_1 - (\sigma_1^2 - \sigma_1\sigma_2\rho_{12})$: Since $A_2 < 0$, then $\sigma_1\sigma_2\rho_{12} < 0$. This implies $(1-\pi_1)(\sigma_1^2 - \sigma_1\sigma_2\rho_{12}) + \pi_2(\sigma_2^2 - \sigma_1\sigma_2\rho_{12}) > 0$, that is $A_2 > A_1 - (\sigma_1^2 - \sigma_1\sigma_2\rho_{12})$.

Solution 1: $\tilde{\pi} = (1,0), \tilde{\lambda} = (\sigma_1^2 - A_1, -A_2 + \sigma_1\sigma_2\rho_{12})$. To satisfy (18), $\delta_{\mathcal{K}} = -\tilde{\lambda}_1$ must hold. Thus $A_1 \ge \sigma_1^2$ and $A_2 \le A_1 - (\sigma_1^2 - \sigma_1\sigma_2\rho_{12})$. Such conditions imply $0 \le A_2$.

Solution 2: $\tilde{\pi} = (0,1), \tilde{\lambda} = (-A_1 + \sigma_1\sigma_2\rho_{12}, \sigma_2^2 - A_2)$. To satisfy (18), $\delta_{\mathcal{K}} = -\tilde{\lambda}_2$ must hold. Thus $A_2 \ge \sigma_2^2$ and $A_2 \ge A_1 + (\sigma_2^2 - \sigma_1\sigma_2\rho_{12})$. Such conditions imply $0 \le A_1$.

Solution 3:

$$\tilde{\pi}_1 = \pi_1 - \frac{(\sigma_2^2 - \sigma_1\sigma_2\rho_{12})(\pi_1+\pi_2-1)}{\sigma_2^2 - 2\sigma_1\sigma_2\rho_{12} + \sigma_1^2}, \quad \tilde{\pi}_2 = \pi_2 - \frac{(\sigma_1^2 - \sigma_1\sigma_2\rho_{12})(\pi_1+\pi_2-1)}{\sigma_2^2 - 2\sigma_1\sigma_2\rho_{12} + \sigma_1^2},$$

which implies $\tilde{\lambda}_1 = \tilde{\lambda}_2 = -\frac{\sigma_1^2\sigma_2^2(\pi_1+\pi_2-1)}{\sigma_2^2 - 2\sigma_1\sigma_2\rho_{12} + \sigma_1^2}$. Note that $\tilde{\pi}_1 + \tilde{\pi}_2 = 1$ and $\tilde{\lambda}_1 \le 0$. Thus, (18) holds. To satisfy (17), $0 \le \tilde{\lambda}_1 \le 1$ must hold. This means that the following two inequalities must hold: $A_2 \ge A_1 - (\sigma_1^2 - \sigma_1\sigma_2\rho_{12})$ and $A_2 \le A_1 + (\sigma_2^2 - \sigma_1\sigma_2\rho_{12})$. Note that both inequalities imply $A_1 \le \sigma_1^2$ and $A_2 \le \sigma_2^2$.

8. Case $\pi_1 \in [0,1]$, $\pi_2 > 1$: Solutions and conditions are the same as in case 7. What changes with respect to case 7 is that 7.(iv) can hold.
9. Case $\pi_1 > 1$, $\pi_2 \in [0,1]$: Solutions and conditions are the same as in case 7. What changes with respect to case 7 is that 7.(v) can hold.
10. Case $\pi_1 > 1$, $\pi_2 > 1$: Solutions and conditions are the same as in case 7. What changes with respect to case 7 is that 7.(iv) and 7.(v) can hold.

Appendix 2: BKM implementation

BKM assumes that both $\phi_{F,t}$ and λ_t are affine, i.e. $\phi_{F,t} = \phi_0 + \phi_1 t$ and $\lambda_{S,t} = \lambda_0 + \lambda_1 t$. The first step is to estimate parameters $\mathcal{A} := \{\phi_0, \phi_1, \lambda_0, \lambda_1\}$ by minimizing the indirect utility function of the artificial market. For the one-asset case, this market has risk-free rate of $r + \delta_{[0,1]}(\lambda_{S,t}) = r + \max\{-\lambda_{S,t}, 0\}$, and $\tilde{\theta}_{t,1} = \alpha d_t + \lambda_{S,t}/\sigma_S$. Thus

$$\min_{\mathcal{A}} J_t(d_t, \phi_F, \lambda_S) := \min_{\mathcal{A}} \mathbf{E}_t\left[X_T^{1-\gamma}/(1-\gamma)\right] = \min_{\mathcal{A}} \frac{X_t^{1-\gamma}}{1-\gamma} g_t(d_t)^\gamma =$$
$$\min_{\mathcal{A}} \frac{1}{1-\gamma} \mathbf{E}_t\Bigg[\exp\Bigg(-\gamma R \int_t^T r + \max\{-\lambda_{S,u}, 0\} du - \frac{\gamma R}{2} \int_t^T \left((\frac{\alpha d_u}{\sigma_S} + \frac{\lambda_{S,u}}{\sigma_S})^2 + \phi_{F,u}^2 \right) \mathrm{d}u$$
$$-\gamma R \int_t^T (\frac{\alpha d_u}{\sigma_S} + \frac{\lambda_{S,u}}{\sigma_S}) \mathrm{d}W_{S,u} - \gamma R \int_t^T \phi_{F,u} \mathrm{d}W_{d,u}\Bigg)\Bigg]$$

Using the approximation $d_u = d_t \exp(-\kappa(u-t)) + d_\infty[1 - \exp(-\kappa(u-t))]$ (same as when deriving $\hat{g}_t(d_t)$ in (12)), then

$$\min_{\mathcal{A}} J_t(d_t, \phi_F, \lambda_S) = \min_{\mathcal{A}} \frac{1}{1-\gamma} \exp\Bigg(-\gamma R \int_t^T \max\{-\lambda_{S,u}, 0\} du$$
$$- \frac{\gamma R(1-R)}{2} \int_t^T \frac{(\alpha d_u + \lambda_{S,u})^2}{\sigma_S^2} \mathrm{d}u - \frac{\gamma R(1-R)}{2} \int_t^T \phi_{F,u}^2 \mathrm{d}u\Bigg)$$
$$= \min_{\mathcal{A}} \frac{1}{1-\gamma} \exp\Bigg((1-\gamma) \int_t^T \max\{-\lambda_{S,u}, 0\} du$$
$$- \frac{1-\gamma}{2\gamma} \int_t^T \frac{(\alpha d_u + \lambda_{S,u})^2}{\sigma_S^2} \mathrm{d}u - \frac{1-\gamma}{2\gamma} \int_t^T \phi_{F,u}^2 \mathrm{d}u\Bigg)$$

Under this structure, $\{\phi_0, \phi_1\}$ can be found independently from $\{\lambda_0, \lambda_1\}$. Similar to the results in Sect. 2.1, is not hard to see that the optimal is found with $\phi_0 = \phi_1 = 0$. Thus the problem reduces in finding

$$\min_{\lambda_0, \lambda_1} \frac{1}{1-\gamma} \exp\left((1-\gamma) \int_t^T \max\{-\lambda_{S,u}, 0\} du - \frac{1-\gamma}{2\gamma} \int_t^T \frac{(\alpha d_u + \lambda_{S,u})^2}{\sigma_S^2} \mathrm{d}u\right),$$

which can be obtained numerically. Let $\lambda_{S,t}^* = \lambda_0^* + \lambda_1^* t$ be the optimal found previously. Hence, function $g_t(d_t)$ can be estimated as

$$g_t^{BKM}(d_t) = \exp\left(-R \int_t^T \max\{-\lambda_{S,u}^*, 0\} du - \frac{R(1-R)}{2} \int_t^T \frac{(\alpha d_u + \lambda_{S,u}^*)^2}{\sigma_S^2} \mathrm{d}u\right)$$

Thus, the approximation of the optimal allocation in the risky asset is similar to Eq. (7), that is

$$\pi_{S,t}^{BKM} = \frac{\alpha d_t}{\gamma \sigma_S^2} + \frac{\sigma_d \rho_{Sd}}{\sigma_S} \frac{\partial_d g_t^{BKM}(d_t)}{g_t^{BKM}(d_t)}$$

Numerical methods must be used to estimate $\frac{\partial_d g_t^{BKM}(d_t)}{g_t^{BKM}(d_t)}$; first to estimate $g_t^{BKM}(d_t)$ and then to estimate the derivative with respect to d_t. The constrained allocation is

$$\tilde{\pi}_{S,t}^{BKM} = \max\{\min\{1, \pi_{S,t}^{BKM}\}, 0\}$$

Appendix 3: Supplementary data and results

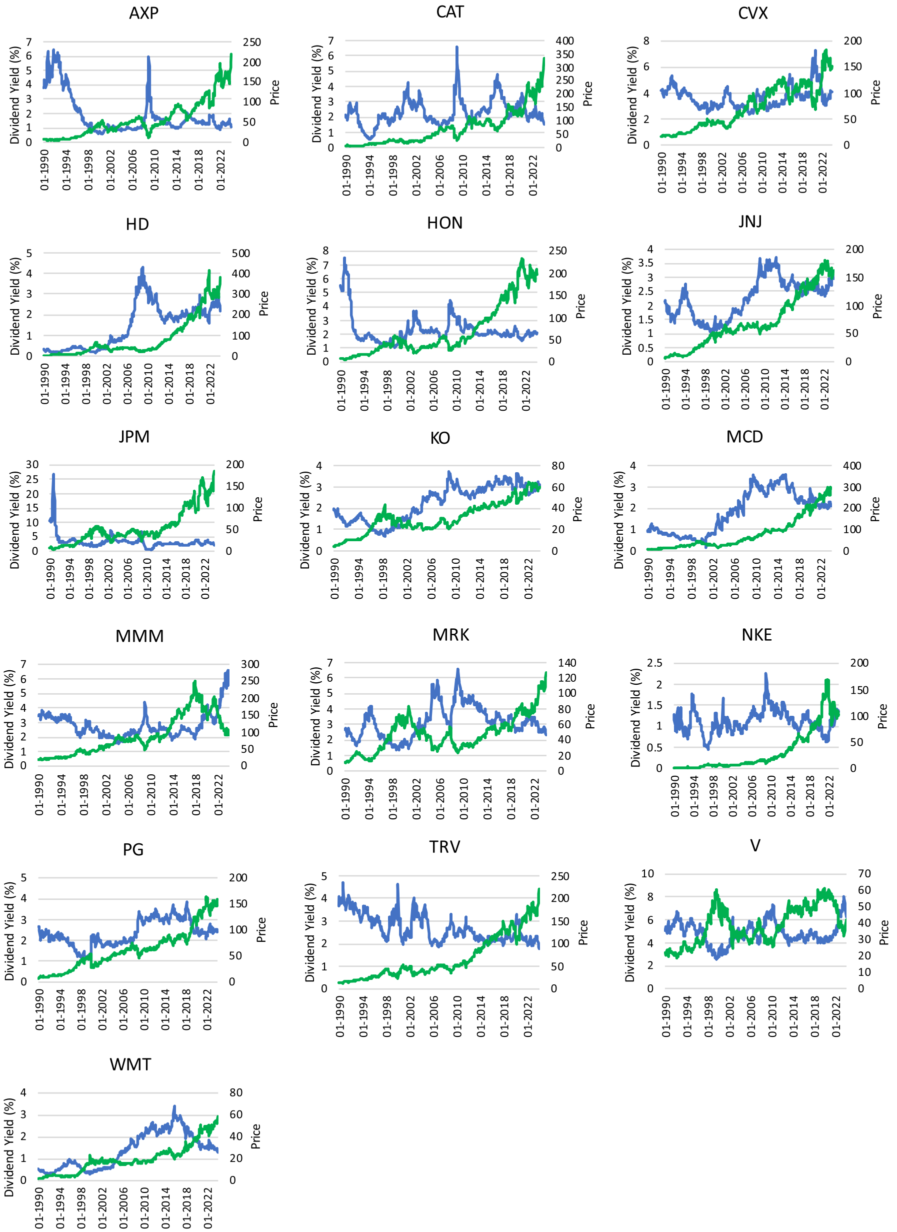

Fig. 4 Monthly dividend yields (blue) and prices (green) for 16 stocks of the Dow Jones index, from the period Jan-90 to Feb-24 (color figure online)

Table 2 (Upper): Estimation results for eight stocks from the Dow Jones index

	CAT	MMM	PG	MRK	NKE	TRV	CVX	V
α	6.4	1.1	3.1	2.2	16.3	3.7	2.0	0.5
σ_S	31%	21%	19%	24%	30%	24%	22%	22%
κ	0.61	0.12	0.27	0.33	0.90	0.69	0.69	0.54
d_∞	2.2%	3.3%	2.4%	3.1%	1.1%	2.7%	3.7%	4.9%
σ_d	0.9%	0.7%	0.4%	0.8%	0.4%	0.7%	0.9%	1.0%
ρ_{Sd}	−0.90	−0.93	−0.95	−0.92	−0.93	−0.94	−0.95	−0.96
$\sigma_d \rho_{Sd}/\sigma_S$	−2.68%	−3.02%	−2.16%	−3.17%	−1.14%	−2.89%	−3.9%	−4.62%
$(\alpha/\sigma_S)^2$	425.3	26.8	266.9	83.0	2957.3	235.8	81.0	4.8
$D^\kappa_{0,50}$	0.04	0.27	0.09	0.09	0.01	0.04	0.05	0.09
$HD^{\gamma=2}_{t=0,T=50}$	11%	6%	13%	6%	10%	7%	4%	0%
Myopic $\pi^{\gamma=2}_{t=0,T=50}$	75%	44%	102%	58%	102%	86%	75%	25%
$\hat{\pi}^{\gamma=2}_{t=0,T=50}$	86%	50%	115%	64%	112%	93%	79%	25%
$HD/\hat{\pi}$	13%	12%	11%	9%	9%	8%	5%	0%

The sample consists of monthly dividend yields and stock prices, from the period Jan-90 to Feb-24. (Lower): Myopic and dynamic unconstrained allocations for $t = 0$, $T = 50$ and $\gamma = 2$, as in Eq. (13). The first rows show the HD decomposition, i.e. $HD^\gamma_{t,T} = \left[-\frac{\sigma_d \rho_{Sd}}{\sigma_S}(\frac{\alpha}{\sigma_S})^2 D^\kappa_{t,T}\right]\frac{1}{\gamma}(1-\frac{1}{\gamma})$. The risk-free rate r equals to the Dec-23 yield of the 10-year US treasury bond ($r = 3.8\%$). The yield at $t = 0$ is set to $d_0 = d_\infty$ for every stock

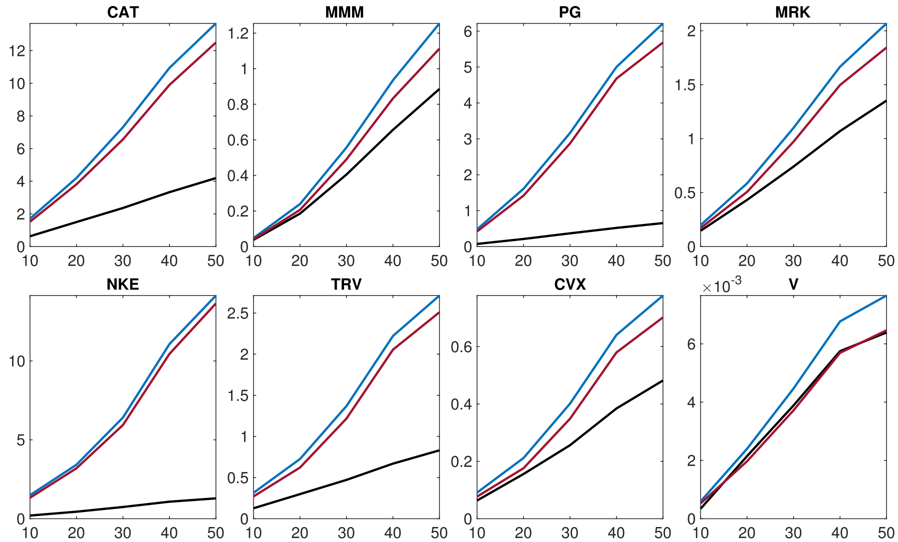

Fig. 5 Increase in CE (%) from the proposed solution with respect to the CE of the myopic solution, for different risk aversion levels γ and horizons T (years). The CE is obtained by doing 200,000 simulations of the risky asset and the dividend yield process, with a discretization of $dt = 1/200$ (200 times in a year)

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Data availibility statement The dataset used in the experiments is publicly available, since it corresponds to historical dividends and prices of public traded companies.

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