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On the least square prenucleolus for games with externalities

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Abstract
In this paper we address the problem of determining the least square prenucleolus for games with externalities. Specifically, this solution is based on the minimization of the variance of the excess vectors that can be associated to any allocation vector over the set of all embedded coalitions using a least square criterion. From a theoretical point of view, an axiomatization of this solution is provided. Besides, we analyse its relations with other proposals of solution in the game-theoretical literature.

Keywords Games with externalities · Least square prenucleolus · Optimization problem · Excess

1 The least square family for games with externalities

In this section, we briefly remind some notions on games with externalities. The model of *games with externalities* (or *games in partition function form*) is introduced in [14] to describe those situations in which the worth of a coalition substantially depends on how the remaining agents (outsiders to the coalition) are organized. In this framework the basic organization of agents is called an *embedded coalition*, which is a pair whose first component is an element of a partition and whose second component contains the remaining elements of the partition (sometimes, the whole partition).

Let $N = \{1, \dots, n\}$ be a set of agents and $\Pi(N)$ be the family of partitions of N. A coalition $S \subseteq N$ is a subset of s players of N. The number of partitions of the set N is given by the *Bell number* of n, denoted by B_n. Formally, an *embedded coalition* is given by a pair $(S; P)$ with $S \subseteq N$ and $P \in \Pi(N \setminus S)$. Although the empty set

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always belongs to any partition $P \in \Pi(N)$, it is omitted when defining embedded coalitions, with the exception of the case $(N;\emptyset)$. The family of those embedded coalitions is denoted by

$$\mathcal{EC}(N) = \left\{(S;P) : \; P \in \Pi(N \setminus S) \text{ and } S \subseteq N\right\}. \tag{1}$$

For notational convenience, we also introduce the set of those embedded coalitions $(S;P) \in \mathcal{EC}(N)$ with non-empty coalition S. We will denote this subset of $\mathcal{EC}(N)$ by

$$\mathcal{EC}^*(N) = \left\{(S;P) : \; P \in \Pi(N \setminus S) \text{ and } \emptyset \neq S \subseteq N\right\}. \tag{2}$$

Thus, the number of non-empty embedded coalitions can be determined as

$$\sum_{k=1}^{n} kS_{n,k}$$

with $S_{n,k}$ being the Stirling number of the second kind that quantifies the total amount of partitions with k groups that can be formed in a set of n players. That is,

$$S_{n,k} = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n, \quad \text{for every } 0 \leq k \leq n,$$

satisfying that $S_{0,0} = 1$ and $S_{n,0} = 0$. Then, notice that the Bell number of n can be directly given by $B_n = \sum_{k=0}^{n} S_{n,k}$.

A *game in partition function form* or a *game with externalities* is formally defined by a function $v : \mathcal{EC}(N) \longrightarrow \mathbb{R}$ such that $v(\emptyset;P) = 0$ for every $P \in \Pi(N)$. We will refer to S as the active coalition. We denote by $\mathcal{G}(N)$ the family of all games in partition function form with player set N.

Games with externalities deal with coalitions and allocations, and they consider groups of agents willing to allocate the joint benefits derived from their cooperation. Let $v \in \mathcal{G}(N)$ be a partition function form game. An *allocation* of v is an element $x \in \mathbb{R}^n$. We say that x satisfies *efficiency* if $\sum_{i=1}^{n} x_i = v(N;\emptyset)$. A *solution concept* (or a *value*) for games with externalities is a map φ that associates to every game $v \in \mathcal{G}(N)$ a vector $\varphi(v) \in \mathbb{R}^n$.

Criteria considered for the definition of values for TU games can be used, either directly or by redefining their meaning in this new context, when introducing values for games with externalities. For instance, minimizing the degree of dissatisfaction of players can be naturally considered, also under the presence of externalities and similarly to the interpretation that justifies the nucleolus [12] and the prenucleolus [13] for TU games. In such context, both solutions minimize, accordingly to the lexicographical order, the excess vector among all possible allocations and among all those allocations satisfying the property of individual rationality for a TU game, respectively. The determination of such allocation vector is based on the principles of fairness and stability, that requires the management of the excesses.

To extend these ideas, we firstly introduce the notion of *excess* of an allocation under the presence of externalities. Let x be an allocation of v and $(S;P) \in \mathcal{G}(N)$ such that $S \neq \emptyset$. The *excess of* x to $(S; P)$ is given by

$$e((S;P),x) = v(S;P) - x(S), \quad \text{with} \quad x(S) = \sum_{i=1}^{s} x_i.$$

Notice that $e((S;P),x)$ can be now interpreted as a measure of the dissatisfaction of the embedded coalition $(S;P)$ if x is suggested as final allocation, in the sense that as greater $e((S;P),x)$, worse-treated would feel agents in $(S;P)$.

Formally, for any $v \in \mathcal{G}(N)$, the average of all excesses of v for a given allocation x over the set of all embedded coalitions is obtained as

$$\overline{e}(v,x) = \frac{1}{\sum_{k=1}^{n} kS_{n,k}} \sum_{(S;P)\in\mathcal{EC}(N)} e((S;P),x).$$

Notice that the average excess at any efficient allocation x is always the same, that is, $\overline{e}(v,x)$ does not depend on x. Hence, $\overline{e}(v)$ will denote $\overline{e}(v,x)$ in order to simplify the notation in the remainder of the paper. Lemma 1 proves this statement.

Lemma 1 *For any game with externalities $v \in \mathcal{G}(N)$, the sum of the excesses of all embedded coalitions is constant for all efficient allocation vectors.*

Proof Take $v \in \mathcal{G}(N)$ and take x an allocation for v satisfying efficiency, i.e. $x(N) = v(N;\emptyset)$. Thus, we immediately have that

$$\begin{aligned}
\sum_{(S;P)\in\mathcal{EC}(N)} e((S;P),x) &= \sum_{(S;P)\in\mathcal{EC}(N)} (v(S;P) - x(S)) \\
&= \sum_{P\in\Pi(N)} \sum_{S\in P} (v(S;P_{-S}) - x(S)) \\
&= \sum_{P\in\Pi(N)} \Big(\sum_{S\in P} v(S;P_{-S}) - x(N) \Big) \\
&= \sum_{P\in\Pi(N)} \sum_{S\in P} v(S;P_{-S}) - B_n \cdot v(N;\emptyset),
\end{aligned}$$

where $P_{-S} \in \Pi(N \setminus S)$ denotes the partition induced by any partition $P \in \Pi(N)$ with $S \in P$. Then, it holds that

$$\begin{aligned}
\overline{e}(v,x) &= \frac{1}{\sum_{k=1}^{n} kS_{n,k}} \sum_{(S;P)\in\mathcal{EC}(N)} e((S;P),x) \\
&= \frac{1}{\sum_{k=1}^{n} kS_{n,k}} \sum_{P\in\Pi(N)} \sum_{S\in P} v(S;P_{-S}) - \frac{B_n}{\sum_{k=1}^{n} kS_{n,k}} v(N;\emptyset).
\end{aligned}$$

Hence, we check that $\overline{e}(v,x)$ does not vary with the allocation x for v considered. □

As [10] and [11] do for the case of TU games, we now analyse the problem of of determining that allocation for v that minimizes the variance of the excesses over the set of all embedded coalitions using a least square criterion. For this purpose, we formalize the following problem for any game with externalities $v \in \mathcal{G}(N)$, being $\overline{e}(v)$ the average excess over the set of all non-empty embedded coalitions.

(Problem 1)

$$\begin{aligned} \min \quad & \sum_{(S;P)\in\mathcal{EC}(N)} \left(e((S;P),x) - \overline{e}(v)\right)^2 \\ s.t. \quad & \textstyle\sum_{i=1}^{n} x_i = v(N;\emptyset). \end{aligned} \tag{3}$$

Then, the optimal solution of Problem 1 in (3) determines that allocation vector that minimizes the variance of the vector of excesses.

From Lemma 1 and after some basic algebra, we straightforwardly have that the objective function of Problem 1 also satisfies for every allocation x that

$$\sum_{(S;P)\in\mathcal{EC}(N)} \left(e((S;P),x) - \overline{e}(v)\right)^2 = \sum_{(S;P)\in\mathcal{EC}(N)} e((S;P),x)^2 - \Big(\sum_{k=1}^{n} kS_{n,k}\Big)\overline{e}(v)^2.$$

The fact of that $\overline{e}(v)$ remains fixed over the feasible region of allocations ensures that the optimal solution of Problem 1 in (3) also coincides with the optimal solution of its alternative formulation, described below and denoted by Problem 1^a.

(Problem 1^a)

$$\begin{aligned} \min \quad & \sum_{(S;P)\in\mathcal{EC}(N)} \left(e((S;P),x)\right)^2 \\ s.t. \quad & \textstyle\sum_{i=1}^{n} x_i = v(N;\emptyset). \end{aligned} \tag{4}$$

2 The least square prenucleolus for games with externalities

In this section we analyse the problem of determining the allocation that minimizes the objective function of Problem 1 for any game with externalities $v \in \mathcal{G}(N)$. According to the final comments of the previous section, from now on we will consider directly its alternative formulation Problem 1^a in (4) for the sake of simplicity.

Definition 1 The *least square prenucleolus* for games with externalities (hereafter, LS^E-prenucleolus) is an optimal solution of Problem 1^a.

Now we can state the following result that determines the exact expression of the LS^E-prenucleolus for any game in partition form $v \in \mathcal{G}(N)$.

Theorem 1 *Let $v \in \mathcal{G}(N)$ be a game with externalities. Then, the LS^E-prenucleolus is the unique optimal solution of Problem 1^a that is given by*

$$\hat{x}_k = \frac{v(N;\emptyset)}{n} + \frac{1}{n\alpha} \sum_{i\in N} \sum_{P\in\Pi(N)} \left(v(P_{(k)};P_{-P_{(k)}}) - v(P_{(i)};P_{-P_{(i)}}) \right), \tag{5}$$

for all $k \in N$, *where* $P_{(i)} \in P$ *such that* $i \in P_{(i)}$, *for all* $i \in N$, *and* $\alpha = \sum_{s=1}^{n-1} \binom{n-2}{s-1} B_{n-s}$.

Proof Take $v \in \mathcal{G}(N)$ and consider Problem 1^a in (4). It is easy to check that the objective function is strictly convex and the feasible set is also convex. Immediately, there exists at most one optimal solution that can be obtained through the Lagrangian conditions.

The Lagrangian function of Problem 1^a is given, for any $x \in \mathbb{R}^n$ and for any $\lambda \in \mathbb{R}$, by

$$L(\lambda, x) = \sum_{(S;P) \in \mathcal{EC}(N)} (e((S;P), x))^2 + \lambda \left(\sum_{i=1}^{n} x_i - v(N;\emptyset) \right).$$

Besides, the partial derivates of $L(\lambda, x)$ are given by

$$\begin{aligned} \frac{\partial L}{\partial \lambda} &= \sum_{i=1}^{n} x_i - v(N;\emptyset) \\ \frac{\partial L}{\partial x_i} &= -2 \sum_{\substack{(S;P) \in \mathcal{EC}(N) \\ i \in S}} e((S;P), x) + \lambda, \qquad \text{for all } i \in N. \end{aligned}$$

In order to obtain a stationary point, we need to solve the following system:

$$\begin{aligned} 0 &= \sum_{i=1}^{n} x_i - v(N;\emptyset) \\ 0 &= -2 \sum_{\substack{(S;P) \in \mathcal{EC}(N) \\ i \in S}} e((S;P), x) + \lambda, \qquad \text{for all } i \in N. \end{aligned} \tag{6}$$

Below, we determine that point satisfying such conditions. Fix $i \in N$ and rewrite Eq. (6) using the definition of excess as follows:

$$\begin{aligned} 0 &= -2 \sum_{\substack{(S;P) \in \mathcal{EC}(N) \\ i \in S}} \big(v(S;P) - x(S) \big) + \lambda \\ 0 &= -2 \sum_{\substack{(S;P) \in \mathcal{EC}(N) \\ i \in S}} \big(v(S;P) - x_i - x(S \setminus i) \big) + \lambda. \end{aligned} \tag{7}$$

By the efficiency of x, i.e. $x(N) = \sum_{i \in N} x_i = v(N;\emptyset)$, we have now that

$$\sum_{\substack{(S;P)\in\mathcal{EC}(N)\\ i\in S}} \big(v(S;P)-x_i-x(S\setminus i)\big) = \sum_{\substack{(S;P)\in\mathcal{EC}(N)\\ S\neq N,\ i\in S}} \big(v(S;P)-x_i-x(S\setminus i)\big)$$
$$= \sum_{\substack{(S;P)\in\mathcal{EC}(N)\\ S\neq N,\ i\in S}} v(S;P) - \sum_{\substack{S\subsetneq N\\ i\in S}} \sum_{P\in\Pi(N\setminus S)} x_i - \sum_{\substack{S\subsetneq N\\ i\in S}} \sum_{P\in\Pi(N\setminus S)} x(S\setminus i).$$

Besides, by analysing each addend of the above expression separately, we have that

$$\sum_{\substack{S\subsetneq N\\ i\in S}} \sum_{P\in\Pi(N\setminus S)} x_i = x_i \sum_{s=1}^{n-1} \binom{n-1}{s-1} B_{n-s},$$

and

$$\sum_{\substack{S\subsetneq N\\ i\in S}} \sum_{P\in\Pi(N\setminus S)} x(S\setminus i) = \sum_{j\in N\setminus i} x_j \sum_{s=2}^{n-1} \binom{n-2}{s-2} B_{n-s},$$

for the third one. Then, rewriting Eq. (7), we alternatively have that

$$0 = -2\Bigg(\sum_{\substack{(S;P)\in\mathcal{EC}(N)\\ S\neq N,\ i\in S}} v(S;P) - x_i \sum_{s=1}^{n-1}\binom{n-1}{s-1}B_{n-s}\Bigg) - 2\Bigg(-\sum_{j\in N\setminus i} x_j \sum_{s=2}^{n-1}\binom{n-2}{s-2}B_{n-s}\Bigg) + \lambda. \tag{8}$$

For every $k\in N\setminus i$, Eq. (8) can be also rewritten as

$$0 = -2\Bigg(\sum_{\substack{(S;P)\in\mathcal{EC}(N)\\ S\neq N,\ k\in S}} v(S;P) - x_k \sum_{s=1}^{n-1}\binom{n-1}{s-1}B_{n-s}\Bigg) - 2\Bigg(-\sum_{j\in N\setminus k} x_j \sum_{s=2}^{n-1}\binom{n-2}{s-2}B_{n-s}\Bigg) + \lambda. \tag{9}$$

Thus, if we substract Eqs. (8) and (9), we immediately obtain that

$$\sum_{s=1}^{n-1}\binom{n-1}{s-1}B_{n-s}(x_i-x_k)+(x_k-x_i)\sum_{s=2}^{n-1}\binom{n-2}{s-2}B_{n-s} - \sum_{P\in\Pi(N)}\left(v(P_{(i)};P_{-P_{(i)}})-v(P_{(k)};P_{-P_{(k)}})\right)=0$$

and

$$(x_i-x_k)\left(\binom{n-1}{0}B_{n-1}+\sum_{s=2}^{n-1}\left(\binom{n-1}{s-1}-\binom{n-2}{s-2}\right)B_{n-s}\right) - \sum_{P\in\Pi(N)}\left(v(P_{(i)};P_{-P_{(i)}})-v(P_{(k)};P_{-P_{(k)}})\right)=0.$$

To simplify the previous equation, we should use that

$$\binom{n-1}{r}=\binom{n-2}{r}+\binom{n-2}{r-1},\ \text{for every } r=0,\dots,n-2,\ \text{with} \binom{n-2}{-1}=0,$$

and that $\binom{n-1}{0}=\binom{n-2}{0}$. Thus, we have that

$$(x_i-x_k)\sum_{s=1}^{n-1}\binom{n-2}{s-1}B_{n-s}-\sum_{P\in\Pi(N)}\left(v(P_{(i)};P_{-P_{(i)}})-v(P_{(k)};P_{-P_{(k)}})\right)=0.$$

From the previous equality, it holds that

$$x_i=x_k+\frac{1}{\sum_{s=1}^{n-1}\binom{n-2}{s-1}B_{n-s}}\sum_{P\in\Pi(N)}\left(v(P_{(i)};P_{-P_{(i)}})-v(P_{(k)};P_{-P_{(k)}})\right),$$

for all $k\in N\setminus\{i\}$. Using the efficiency of the allocation x, we have that

$$v(N;\emptyset)=nx_k+\frac{1}{\sum_{s=1}^{n-1}\binom{n-2}{s-1}B_{n-s}}\sum_{i\in N}\sum_{P\in\Pi(N)}\left(v(P_{(i)};P_{-P_{(i)}})-v(P_{(k)};P_{-P_{(k)}})\right)$$

which directly implies that

$$\hat{x}_k=\frac{v(N;\emptyset)}{n}+\frac{1}{n\sum_{s=1}^{n-1}\binom{n-2}{s-1}B_{n-s}}\sum_{i\in N}\sum_{P\in\Pi(N)}\left(v(P_{(k)};P_{-P_{(k)}})-v(P_{(i)};P_{-P_{(i)}})\right)$$

is the unique optimal solution of Problem 1^a. □

First, we provide some alternative formulations for the LS^E-prenucleolus of a game with externalities $v \in \mathcal{G}(N)$ in Expression (5), that can be useful in the analysis of their theoretical properties.

Remark 1 The LS^E-prenucleolus of a game with externalities $v \in \mathcal{G}(N)$ in Expression (5) can be also rewritten as

$$\hat{x}_k = \frac{v(N;\emptyset)}{n} + \frac{1}{n\sum_{s=1}^{n-1}\binom{n-2}{s-1}B_{n-s}}\left(n\sum_{\substack{(S;P)\in\mathcal{EC}(N)\\ k\in S}} v(S;P) - \sum_{i\in N}\left(\sum_{\substack{(S;P)\in\mathcal{EC}(N)\\ i\in S}} v(S;P)\right)\right).$$

Also from Expression (5), some aspects related to the interpretation of the least square prenucleolus of a game with externalities $v \in \mathcal{G}(N)$ can be mentioned.

Remark 2 The least square prenucleolus for games with externalities in Expression (5) enables the following alternative formulation:

$$\hat{x}_k = \hat{y}_k + \frac{1}{n}\left(v(N;\emptyset) - \frac{1}{\sum_{s=1}^{n-1}\binom{n-2}{s-1}B_{n-s}}\sum_{i\in N}\sum_{P\in\Pi(N)} v(P_{(i)};P_{-P_{(i)}})\right), \tag{10}$$

for all $k \in N$, being $\hat{y}_k$ such that

$$\begin{aligned}\hat{y}_k &= \frac{1}{\sum_{s=1}^{n-1}\binom{n-2}{s-1}B_{n-s}}\sum_{P\in\Pi(N)} v(P_{(k)};P_{-P_{(k)}})\\ &= \frac{1}{\sum_{s=1}^{n-1}\binom{n-2}{s-1}B_{n-s}}\sum_{S\subseteq N\setminus k}\ \sum_{P\in\Pi(N\setminus(S\cup k))} v(S\cup k;P).\end{aligned}$$

In practice, each coordinate k of vector $\hat{y}$, with $k \in N$, represents the average value of that function defining the game with externalities for those embedded coalitions in which k belongs to the active coalition.

Since the quantity $\sum_{s=1}^{n-1}\binom{n-2}{s-1}B_{n-s}$ represents the total amount of embedded coalitions of the form $(S; P)$ to which player k belongs to S, the first addend of Expression (10) can be interpreted as the average worth of those embedded coalitions in which player k is a member of S. So, this implies that the alternative expression of the least square prenucleolus can be naturally introduced as an additive normalization of the allocation given by $(\hat{y}_k)_{k\in N}$.

3 TU game-based analysis for the least square prenucleolus

As mentioned, the task of allocating the profit or the cost of the cooperation is also relevant under the presence of externalities. In literature, there exist some proposals that make use of games with transferable utility for this purpose that can be intuitively associated to any $v \in \mathcal{G}(N)$. Recall that a *game with transferable utility or TU game* is a pair (N, w), where N is a finite set of agents and w is a map from 2^N to $\mathbb{R}$ that satisfies that $w(\emptyset) = 0$ (cf. [6]). The class of TU games with a set of agents N is denoted by G^N.

Let $v \in \mathcal{G}(N)$ be a game with externalities. Among other proposals, [2] assigns to v, the TU game $(N, \bar{v}) \in G^N$ given by

$$\bar{v}(S) = \begin{cases} v(N;\emptyset) & \text{if } S = N, \\ \frac{1}{B_{n-s}} \sum_{P \in \Pi(N \setminus S)} v(S;P) & \text{otherwise,} \end{cases} \tag{11}$$

for every $S \subseteq N$. Each coalition $S \subseteq N$ obtains the expected worth of the cooperation of the members of S in v, that is, the average over the whole set of the embedded coalitions $(S; P)$ with $P \in \Pi(N \setminus S)$ when they are equally likely.

The study of the least square prenucleolus was already considered in [10] for the case of TU games when denoting by $\overline{e}(w, x) = \frac{1}{2^n - 1} \sum_{S \subseteq N} e(S, x)$ the average excess of a given allocation x for every $(N, w) \in G^N$, and being the excess defined by $e((S;P), x) = w(S) - x(S)$ for each $S \subseteq N$. Thus, the least square prenucleolus of a general TU game (N, w) (hereafter, LS-prenucleolus) is the optimal solution of

(Problem 1*)

$$\begin{aligned} \min \quad & \sum_{S \subseteq N} \left(e(S, x) - \overline{e}(w, x) \right)^2 \\ s.t. \quad & \sum_{i=1}^{n} x_i = w(N). \end{aligned} \tag{12}$$

[10] determine and characterize the least square prenucleolus of a given TU game (N, w), the LS-prenucleolus for w, that is given by

$$z_k = \frac{w(N)}{n} + \frac{1}{n 2^{n-2}} \left(n \sum_{\substack{S \subseteq N \\ k \in S}} w(S) - \sum_{j=1}^{n} \left(\sum_{\substack{S \subseteq N \\ j \in S}} w(S) \right) \right), \text{ for all } k \in N, \tag{13}$$

and that corresponds to the optimal solution of Problem 1* in (12).

Moreover, [11] extend the original problem by using a coalitional weight function $m : 2^{|N|} \setminus \emptyset \longrightarrow \mathbb{R}$, that assigns to every non-empty coalition S of N a real number $m(S)$. Specifically, the weight function m is assumed positive, with $m(S) > 0$ for all $S \subseteq N$, and symmetric, with $m(S) = m(T)$ for any pair of coalitions $S, T \subseteq N$ such that $s = t$. The interpretation of this function usually refers to the distribution of probability of any coalition S forming.

Directly, Problem 1$^{*\mathcal{W}}$ formalized a weighted version of Problem 1*.

(Problem $1^{*\mathcal{W}}$)

$$\begin{aligned} \min \quad & \sum_{S\subseteq N} m(S)\big(e(S,x)-\overline{e}(w,x)\big)^2 \\ s.t. \quad & \sum_{i=1}^{n} x_i = w(N). \end{aligned} \tag{14}$$

[11] determine and study the optimal solution of Problem $1^{*\mathcal{W}}$ in (14). Thus, the weighted least square prenucleolus for w, denoted by the weighted LS-prenucleolus for w, is

$$z_k^m = \frac{w(N)}{n} + \frac{1}{\alpha n}\left(n\Bigg(\sum_{\substack{S\subseteq N\\ k\in S}} m(S)w(S)\Bigg) - \sum_{j=1}^{n}\Bigg(\sum_{\substack{S\subseteq N\\ j\in S}} m(S)w(S)\Bigg)\right), \text{ for all } k\in N,$$

with $\alpha = \sum_{s=1}^{n-1} m(s)\binom{n-2}{s-1}$.

Using these notions, the LS^E-prenucleolus of a given game with externalities v can be interpreted as the weighted LS-prenucleolus (cf. [11]) of the TU game of Albizuri et al. in (11). Then, if we consider the coalitional weight function $m : 2^{|N|}\backslash\emptyset \longrightarrow \mathbb{R}$ defined by

$$m(S) = B_{n-s}, \text{ for all } S\subseteq N,$$

it holds that the LS^E-prenucleolus for v in (5) coincides with the weighted LS-prenucleolus for the TU game $\bar{v}$, given in (11).

Considering all the comments above, the analysis of the properties that LS^E-prenucleolus satisfy readily follows. For this purpose, we largely use the ideas in [7] about properties to satisfy those solutions for externalities games that are obtained from solutions for TU games using an "average approach".

Let $v \in \mathcal{G}(N)$ be a game with externalities and take φ a value for v. Below, we provide an initial list of those more basic properties on φ.

(EFF) **Efficiency**. φ satisfies *efficiency* if it provides an efficient allocation for v, i.e. $\sum_{i\in N}\varphi_i(v) = v(N;\emptyset)$.

(L) **Linearity**. φ satisfies *linearity* property if for any pair of games with externalities $v, v' \in \mathcal{G}(N)$, $\varphi(v+v') = \varphi(v)+\varphi(v')$, and for any scalar $\lambda\in\mathbb{R}$, $\varphi(\lambda v) = \lambda\varphi(v)$.

(C) **Continuity**. φ satisfies *continuity* property if it is a continuous function from the set of games with externalities to $\mathbb{R}^n$.

(IN) **Inessential game**. φ satisfies *inessential game* property if $\varphi_i(v) = v(i;\lfloor N\backslash i\rfloor)$ whenever v is additive, and being $\lfloor N\backslash i\rfloor = \{\{j\} : j \in N\backslash\{i\}\}$.[1] Notice that $v \in \mathcal{G}(N)$ is said to be additive if $v(S;P) = \sum_{i\in S} v(i;\lfloor N \setminus i\rfloor)$.

(SE) **Strategic equivalence**. φ satisfies *strategic equivalence* property if, for a given $\rho > 0$ and for a given collection of real numbers $\gamma_1, \dots, \gamma_n$ such that $v_2(S;P) = \rho v_1(S;P) + \sum_{i\in S} \gamma_i$, for any $(S;P) \in \mathcal{EC}(N)$ and for any pair of games with externalities $v_1, v_2 \in \mathcal{G}(N)$, it holds that

$$\varphi_i(v_2) = \rho\varphi_i(v_1) + \gamma_i, \text{ for all } i \in N.$$

In this case, v_1 and v_2 are said to be games with externalities strategically equivalent.

(S) **Standard for two person games with externalities**. φ satisfies this property if, for any two person game with externalities $v \in \mathcal{G}(N)$, it holds that

$$\varphi_i(v) = v(i;\{j\}) + \frac{1}{2}\big(v(N;\emptyset) - v(i;\{j\}) - v(j;\{i\})\big).$$

Before going on to introduce those properties that specifically characterize a value φ built under the least square criterion for $v \in \mathcal{G}(N)$, we introduce some notation. For a given coalition $S \subseteq N$ and a given partition $P \in \Pi(N\backslash S)$, $\sigma_{S,P}P$ denotes a new partition of $N \setminus S$ resulting from a permutation of $N \setminus S$ with as many groups as P and with the same distribution of group sizes as P. Besides, we define the game $\sigma_{S,P}v \in \mathcal{G}(N)$ as $(\sigma_{S,P}v)(S;P) = v(S;\sigma_{S,P}P)$, $(\sigma_{S,P}v)(S;\sigma_{S,P}P) = v(S;P)$, and $(\sigma_{S,P}v)(R;Q) = v(R;Q)$ for all $(R;Q) \in \mathcal{EC}(N)\backslash\{(S;P), (S;\sigma_{S,P}P)\}$.

Next, we complete the previous list by adding the properties of strong symmetry and of coalitional monotonicity on φ, that are originally introduced in [7].

(SSYM) **Strong symmetry**. φ satisfies the *strong symmetry* property if for any permutation σ of N, $\varphi(\sigma v) = \sigma\varphi(v)$, and for any $(S;P) \in \mathcal{EC}(N)$, and for any permutation $\sigma_{S,P}$, $\varphi(\sigma_{S,P}v) = \varphi(v)$.

(CM) **Coalitional monotonicity**. φ satisfies the *coalitional monotonicity* property if $\varphi_i(v_1) \geq \varphi_i(v_2)$ for all $i \in S$ whenever v_1 and $v_2 \in \mathcal{G}(N)$ are such that $v_1(S;P) > v_2(S;P)$ for some $P \in \Pi(N \setminus S)$ and $v_1(R;Q) = v_2(R;Q)$ for $(R;Q) \neq (S;P)$.

[1] Unlike the case of TU games, the worth of individually acting each player i under the presence of externalities, for every $i \in N$, is usually non-constant over the set of partitions for $N \setminus \{i\}$. In some specific contexts, the worth of $v(i;\lfloor N\backslash i\rfloor)$ is clearly tied to the sense of the externalities of v (see [1, 5]). This notion requires the introduction of an order relation between partitions. Specifically, given $P, Q \in \Pi(N)$, *P is finer than Q*, $P \preceq Q$, if for every $S \in P$ there is $T \in Q$ such that $S \subseteq T$. Thus, $v \in \mathcal{G}(N)$ has *negative externalities*, if for every $S \subseteq N$, $P, Q \in \Pi(N \setminus S)$ such that $P \preceq Q$, $v(S;P) \geq v(S;Q)$. This ensures the best-case scenario for i since that $v(i;\lfloor N\backslash i\rfloor) = \max_{P\in\Pi(N\backslash\{i\})} v(\{i\};P)$. Conversely, if $v \in PG(N)$ has *positive externalities*, for every $S \subseteq N$, $P, Q \in \Pi(N \setminus S)$ such that $P \preceq Q$, $v(S;P) \leq v(S;Q)$. This ensures the worst-case scenario for i among all the structures of N\i, that is, $v(i;\lfloor N\backslash i\rfloor) = \min_{P\in\Pi(N\backslash\{i\})} v(\{i\};P)$.

Immediately, the next result ensures that least square prenucleolus for games with externalities satisfies all of the above properties.

Proposition 1 *Let $v \in \mathcal{G}(N)$ be a game with externalities. Then, the LS^E-prenucleolus verifies the properties of efficiency (EFF), linearity (L), continuity (C), inessential game (IN), strategic equivalence (SE), strong symmetry (SSYM), coalitional monotonicity (CM), and it is standard for two person games with externalities (S).*

Proof Take $v \in \mathcal{G}(N)$ and take the least square prenucleolus for v given by Expression (5) in Theorem 1.

By construction, the efficiency (EFF) of the LS^E-prenucleolus for v is clearly satisfied. After some basic algebra, the properties of continuity (C) and of strategic equivalence (SE) can directly established, as well as the property of being standard for two person games with externalities (S). More specifically, the fulfilment of (S) can be immediately stated from the expression of the LS^E-prenucleolus in (5) for the case of only considering two players.

Since that the least square prenucleolus of a game with externalities is the weighted least square prenucleolus of the TU game of [2], the fulfilment of the remaining properties is plain from the properties that [11] prove for the latter. This is an immediate consequence of Theorem 10 in [7]. □

3.1 Surplus approach-based analysis for the least square prenucleolus

Below, we extend a solution considered in [10] in the analysis of the least square values for TU games for the case of games with externalities. Its definition is based on the ideas of surplus of a player against other that justify solution concepts for TU games as the kernel ([4]) and the prekernel ([8, 9]).

Below, we formalize the *average surplus of player i against j* at an efficient allocation vector x for any game with externalities $v \in \mathcal{G}(N)$ as

$$\sigma_{ij}(x,v) = \frac{1}{\sum_{s=1}^{n-1} \binom{n-2}{s-1} B_{n-s}} \sum_{S \subseteq N \setminus \{i,j\}} \sum_{P \in \Pi(N \setminus (S \cup i))} \big(v(S \cup i;P) - x(S \cup i)\big), \tag{15}$$

for each pair of players i and $j \in N$ such $i \neq j$.

From the solution given in (15), the following set of allocations can be directly established for any game with externalities $v \in \mathcal{G}(N)$.

Definition 2 Take $v \in \mathcal{G}(N)$ a game with externalities. The average prekernel for v is given by

$$pK^{av}(v) = \Big\{ x \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N;\emptyset), (\sigma_{ij}(x,v) - \sigma_{ji}(x,v)) = 0, \quad \text{for all } i,j \in N, \quad \text{with } i \neq j \}.$$

Next result study the average prekernel for a given game with externalities v. As a direct consequence of Theorem 1, for any game with externalities v, the sum of the excesses of all embedded coalitions containing a player at the LS^E-prenucleolus for v is the same for all players. That is, using Eq. (6) in the proof Theorem 1 yields that

$$\sum_{\substack{(S;P)\in\mathcal{EC}(N)\\ i\in S}} e((S;P),\hat{x}) = \sum_{\substack{(S;P)\in\mathcal{EC}(N)\\ j\in S}} e((S;P),\hat{x}),\ \text{for all } i,j\in N, \tag{16}$$

and being $\hat{x}$ the LS^E-prenucleolus for v.

Proposition 2 *Take $v \in \mathcal{G}(N)$ a game with externalities. Thus, the LS^E-prenucleolus for v is the unique point of the average prekernel.*

Proof The result can be directly proved from the fact of that $\sigma_{ij}(x,v) = \sigma_{ji}(x,v)$ for all $i,j \in N$ such that $i \neq j$ ensures that any allocation in $pK^{av}(v)$ satisfies that

$$\sum_{\substack{(S;P)\in\mathcal{EC}(N)\\ i\in S}} e((S;P),x) = \sum_{\substack{(S;P)\in\mathcal{EC}(N)\\ j\in S}} e((S;P),x).$$

Thus, from the condition in (16), the only allocation satisfying the previous equality is the LS^E-prenucleolus for v. □

4 An axiomatization of the least square prenucleolus

Although the condition imposed in (16) for LS^E-prenucleolus jointly with its efficiency (EFF) provides a first characterization of this solution, we establish in this section new results that make use of usual properties of solution concepts.

To do this, we extend the property of *Average Marginal Contribution Monotonicity* for solutions for TU games considered in [10] to characterize solutions for games with externalities. However, the notion of marginal contribution of a player that bases it for TU games can not be explicitly handled here. By using its alternative formulation, we rename it under the presence of externalities as monotonicity in the average worth. The property of *Monotonicity in the average worth* ensures that, given $v \in \mathcal{G}(N)$, φ a value for v and for any pair of players $i,j \in N$, if the average worth of those embedded coalitions which player i belongs to but not containing player j is larger than the average worth of those containing j but not i, i receives more than j when using φ. We formalize it below for any game with externalities $v \in \mathcal{G}(N)$ and any value φ for v.

(M) Monotonicity in the average worth. φ satisfies this property if

$$\sum_{\substack{(S;P)\in\mathcal{EC}(N)\\ i\in S,\ j\notin S}} v(S;P) \geq \sum_{\substack{(S;P)\in\mathcal{EC}(N)\\ j\in S,\ i\notin S}} v(S;P)$$

implies that $\varphi_i(v) \geq \varphi_j(v)$, for all $i,j \in N$ and for all $v \in \mathcal{G}(N)$.

Below, we characterize the least square prenucleolus for games with externalities but first we must state the following auxiliary result.

Lemma 2 *Take $v \in \mathcal{G}(N)$ a game with externalities. A value φ for v verifies efficiency (EFF), linearity (L) and monotonicity in the average worth (M), if and only if*

$$\varphi_k(v) = \frac{v(N;\emptyset)}{n} + \beta\left(n \sum_{\substack{(S;P)\in\mathcal{EC}(N)\\ k\in S}} v(S;P) - \sum_{i\in N}\left(\sum_{\substack{(S;P)\in\mathcal{EC}(N)\\ i\in S}} v(S;P)\right)\right), \tag{17}$$

for all $k \in N$.

Proof The scheme of the proof follows the one given for Lemma 14 in [10]. First, it is easy to check that the value for v in (17) satisfies the three required properties.

To prove the reverse, we now take a basis of the games with externalities defined over the subset of non-empty embedded coalitions of N given by $\mathcal{EC}^*(N)$ in (2). Thus, we will denote such basis by $(w^{(S;P)})_{\{(S;P)\in\mathcal{EC}^*(N)\}}$ and it will be specified by:

$$w^{(S;P)}(T;Q) = \begin{cases} 1, \text{ if } (S;P) = (T;Q), \\ 0, \text{ if } (S;P) \neq (T;Q), \end{cases}$$

for all $(S;P) \in \mathcal{EC}^*(N)$.

In this case, the properties of (EFF) and (M) ensures that $\varphi_i(w^{(N;\emptyset)}) = 1/n$, for all $i \in N$. For each embedded coalition $(S;P) \in \mathcal{EC}^*(N)$, with $S \neq N$, using again (EFF) and (M), there exists a family of values $\beta_{(S;P)} \geq 0$ satisfying that

$$\varphi_i(w^{(S;P)}) = \begin{cases} \frac{\beta_{(S;P)}}{s}, & \text{if } i \in S, \\ \frac{-\beta_{(S;P)}}{n-s}, & \text{if } i \notin S. \end{cases}$$

As any game with externalities $v \in \mathcal{G}(N)$ can be expressed in terms of the considered basis and using (L), we have that

$$\begin{aligned}\varphi_k(v) &= \varphi_k\left(\sum_{(S;P)\in\mathcal{EC}^*(N)} v(S;P)w^{(S;P)}\right) = \sum_{(S;P)\in\mathcal{EC}^*(N)} v(S;P)\varphi_k\left(w^{(S;P)}\right) \\ &= v(N;\emptyset)\varphi_k(w^{(N;\emptyset)}) \\ &\quad + \sum_{\substack{(S;P)\in\mathcal{EC}^*(N)\\ S\neq N,\ k\in S}} \frac{\beta_{(S;P)}}{s} v(S;P) - \sum_{\substack{(S;P)\in\mathcal{EC}^*(N)\\ S\neq N,\ k\notin S}} \frac{\beta_{(S;P)}}{n-s} v(S;P),\end{aligned} \tag{18}$$

by the linearity property that φ satisfies.

Similarly to the reasoning done in [10], by using (L) and (M) on φ, we can state that there exists a real value $\beta \geq 0$ such that

$$\beta = \frac{\beta_{(S;P)}}{s(n-s)}, \text{ for all } (S;P) \in \mathcal{EC}^*(N).$$

Using the condition on (18), we immediately have that

$$\varphi_k(v) = \frac{v(N;\emptyset)}{n} + \beta\left(n \sum_{\substack{(S;P) \in \mathcal{EC}(N)\\ k \in S}} v(S;P) - \sum_{i\in N}\left(\sum_{\substack{(S;P) \in \mathcal{EC}(N)\\ i \in S}} v(S;P)\right)\right),$$

concluding the proof. □

Now, we provide the following characterization of the least square prenucleolus for games with externalities.

Theorem 2 *Take $v \in \mathcal{G}(N)$ a game with externalities. The LS^E-prenucleolus for v is the unique value for v satisfying efficiency* (*EFF*), *linearity* (*L*), *inessential game* (*IN*), *and monotonicity in the average worth* (*M*).

Proof First, Proposition 1 ensures that the LS^E-prenucleolus for v satisfies the four mentioned properties.

Now, take φ a value also satisfying these four conditions. Them, using Lemma 2, φ can be expressed as in (17) for a certain $\beta \geq 0$. Besides, if we take an inessential game, we have that

$$\begin{aligned}\sum_{\substack{(S;P) \in \mathcal{EC}(N)\\ k \in S}} v(S;P) &= \sum_{\substack{S \subseteq N\\ k \in S}} \sum_{P\in\Pi(N\setminus S)} v(S;P) = \sum_{\substack{S \subseteq N\\ k \in S}} \sum_{P\in\Pi(N\setminus S)} \sum_{j\in S} v(j;\lfloor N \setminus j\rfloor) \\ &= v(k;\lfloor N \setminus k\rfloor)\left(\sum_{s=1}^{n-1}\binom{n-2}{s-1}B_{n-s}\right) \\ &\quad + v(N;\emptyset)\left(\sum_{s=2}^{n}\binom{n-2}{s-2}B_{n-s}\right),\end{aligned} \tag{19}$$

and, hence,

$$\sum_{j\in N}\left(\sum_{\substack{(S;P)\in\mathcal{EC}(N)\\ j\in S}} v(S;P)\right)=\left(\sum_{j\in N} v(j;\lfloor N\backslash j\rfloor)\right)\left(\sum_{s=1}^{n-1}\binom{n-2}{s-1}B_{n-s}\right) + nv(N;\emptyset)\left(\sum_{s=2}^{n}\binom{n-2}{s-2}B_{n-s}\right). \tag{20}$$

Considering the expressions in (19) and in (20) as well as the property of inessential game, we can rewrite the formulation in (17) by

$$\varphi_k(v)=\frac{v(N;\emptyset)}{n}+\beta\left(\sum_{s=1}^{n-1}\binom{n-2}{s-1}B_{n-s}\right)\big(nv(k;\lfloor N\setminus k\rfloor)-v(N;\emptyset)\big).$$

and, by the inessential game property, it also satisfies that $\varphi_k(v)=v(k;\lfloor N\setminus k\rfloor)$. Thus, it holds that

$$\frac{v(N;\emptyset)}{n}+\beta\left(\sum_{s=1}^{n-1}\binom{n-2}{s-1}B_{n-s}\right)\big(nv(k;\lfloor N\setminus k\rfloor)-v(N;\emptyset)\big) = v(k;\lfloor N\setminus k\rfloor)$$

and, consequently,

$$n\beta\left(\sum_{s=1}^{n-1}\binom{n-2}{s-1}B_{n-s}\right)\big(nv(k;\lfloor N\setminus k\rfloor)-v(N;\emptyset)\big) = nv(k;\lfloor N\setminus k\rfloor)-v(N;\emptyset).$$

Then, if $nv(k;\lfloor N\setminus k\rfloor)-v(N;\emptyset)\neq 0$, we obtain that $\beta=\frac{1}{n\sum_{s=1}^{n-1}\binom{n-2}{s-1}B_{n-s}}$. Now, if we use this value of β on the formulation of φ in (17), the unique value for games with externalities satisfying efficiency (EFF), linearity (L), inessential game (IN) and monotonicity in the average worth (M) is given, for all $k\in N$, by

$$\varphi_k(v)=\frac{v(N;\emptyset)}{n}+\frac{1}{n\sum\limits_{s=1}^{n-1}\binom{n-2}{s-1}B_{n-s}}\left(n\sum_{\substack{(S;P)\in\mathcal{EC}(N)\\ k\in S}} v(S;P)-\sum_{i\in N}\left(\sum_{\substack{(S;P)\in\mathcal{EC}(N)\\ i\in S}} v(S;P)\right)\right),$$

that coincides with the formulation of the LS^E-prenucleolus for games with externalities shown in Remark 1. □

Next, the independence of the four axioms that are considered in Theorem 2 is shown. If we define for all $v\in\mathcal{G}(N)$ the value

$$\tilde{\varphi}^1(v)=2LS^E(v),$$

it satisfies linearity (L), inessential game (IN) and monotonicity in the average worth (M), but not efficiency (EFF).

To prove the independence of (L), we define the value

$$\tilde{\varphi}^2(v) = \begin{cases} \frac{v(N;\emptyset)}{|N|}, & \text{if } v \text{ is a non-additive game,} \\ LS^E(v), & \text{otherwise,} \end{cases}$$

for all $v \in \mathcal{G}(N)$. Clearly, it satisfies (EFF), (IN) and (M), but no (L).

To prove the independence of (IN), we define, for all $v \in \mathcal{G}(N)$ and for all $i \in N$, the value

$$\tilde{\varphi}_i^3(v) = \frac{v(N;\emptyset)}{|N|}$$

for all $v \in \mathcal{G}(N)$ and for all $i \in N$. This solution does not satisfy (IN) but it satisfies the rest of the properties.

Finally, we define the value

$$\tilde{\varphi}_i^4(v) = v(\{i\}, \lfloor N \setminus \{i\} \rfloor) + \frac{1}{|N|}\left(v(N;\emptyset) - \sum_{j \in N} v(\{j\}, \lfloor N \setminus \{j\} \rfloor) \right)$$

for all $v \in \mathcal{G}(N)$ and for all $i \in N$. Then, $\tilde{\varphi}^4(v)$ only fails (M).

5 Concluding remarks

The prenucleolus and the nucleolus for TU games are solution concepts based on the excess vector that can be associated to any allocation. Using this idea, we explore a new solution concept for games with externalities that provides that efficient allocation whose associated excess is the closest to the average excess under the least square criterion. Thus, we call the least square prenucleolus for games with externalities to such efficient allocation, that minimizes the variance of the resulting excesses over the set of all embedded coalitions.

In this paper, we provide the exact expression of the least square prenucleolus and we analyse its properties. Besides, an axiomatic characterization of the least square prenucleolus is established. We have also studied its relationship with analogous solution concepts for TU games, which assume the absence of externalities, and that were originally considered in [10, 11] under an analogous perspective. Specifically, this new value for games with externalities can be interpreted as the weighted least square prenucleolus for the TU game in [2], that is built using an "average approach" (cf. [7]). A consistency property has been formalized to characterize the family of weigthed least square values for TU games studied in [10] and in [11]. A similar solution concept for games with externalities, although different in its nature, is the optimistic (pre)-nucleolus for games with externalities (cf. [3]), that is also characterized by means of a consistency property. In any case, our solution concept does not satisfy either of these perspectives.

Finally, it is noteworthy that the minimization problem for computing the least square nucleolus for games with externalities can be analogously raised. The theoretical analysis of the consistency property for the least square prenucleolus, the study of the nucleolus and the study of alternative solution concepts based on the "average approach" will be part of further research.

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Declarations

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