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ORIGINAL PAPER

Largest small polygons: a sequential convex optimization approach

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Abstract
A small polygon is a polygon of unit diameter. The maximal area of a small polygon with $n = 2m$ vertices is not known when $m \geq 7$. Finding the largest small n-gon for a given number $n \geq 3$ can be formulated as a nonconvex quadratically constrained quadratic optimization problem. We propose to solve this problem with a sequential convex optimization approach, which is an ascent algorithm guaranteeing convergence to a locally optimal solution. Numerical experiments on polygons with up to $n = 128$ sides suggest that optimal solutions obtained are near-global. Indeed, for even $6 \leq n \leq 12$, the algorithm proposed in this work converges to known global optimal solutions found in the literature.

Keywords Planargeometry · Polygons · Isodiametric problem · Maximal area · Quadratically constrained quadratic optimization · Sequential convex optimization · Concave-convex procedure

1 Introduction

The *diameter* of a polygon is the largest Euclidean distance between pairs of its vertices. A polygon is said to be *small* if its diameter equals one. For a given integer $n \geq 3$, the maximal area problem consists in finding a small n-gon with the largest area. The problem was first investigated by Reinhardt [16] in 1922. He proved that

- when n is odd, the regular small n-gon is the unique optimal solution;
- when $n = 4$, there are infinitely many optimal solutions, including the small square;
- when $n \geq 6$ is even, the regular small n-gon is not optimal.

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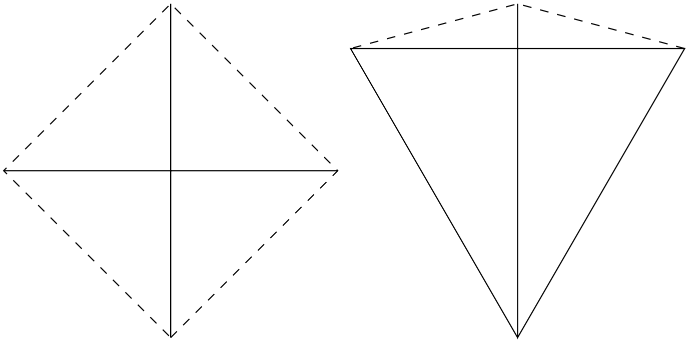
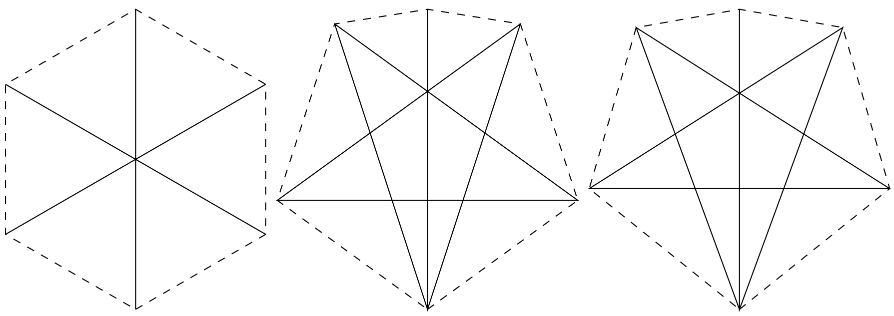


(**a**) $(\mathtt{R}_4, 0.5)$ (**b**) $(\mathtt{R}_3^+, 0.5)$

Fig. 1 Two small 4-gons $(\mathtt{P}_4, A(\mathtt{P}_4))$

When $n \geq 6$ is even, the maximal area problem is solved for $n \leq 12$. In 1961, Bieri [5] found the largest small 6-gon, assuming the existence of an axis of symmetry. In 1975, Graham [9] independently constructed the same 6-gon, represented in Fig. 2c. In 2002, Audet et al. [3] combined Graham's strategy with global optimization methods to find the largest small 8-gon, illustrated in Fig. 3c. In 2013, Henrion and Messine [10] found the largest small 10- and 12-gons by also solving globally a non-convex quadratically constrained quadratic optimization problem. They also found the largest small axially symmetrical 14- and 16-gons. In 2017, Audet [1] showed that the regular small polygon has the maximal area among all equilateral small polygons. In 2021, Audet et al. [4] determined analytically the largest small axially symmetrical 8-gon.

The diameter graph of a small polygon is the graph with the vertices of the polygon, and an edge between two vertices exists only if the distance between these vertices equals one. Graham [9] conjectured that, for even $n \geq 6$, the diameter graph of a small n-gon with maximal area has a cycle of length $n-1$ and one additional edge from the remaining vertex. The case $n = 6$ was proven by Graham himself [9] and the case $n = 8$ by Audet et al. [3]. In 2007, Foster and Szabo [8] proved Graham's conjecture for all



(**a**) $(\mathtt{R}_6, 0.649519)$ (**b**) $(\mathtt{R}_5^+, 0.672288)$ (**c**) $(\mathtt{U}_6, 0.674981)$

Fig. 2 Three small 6-gons $(\mathtt{P}_6, A(\mathtt{P}_6))$

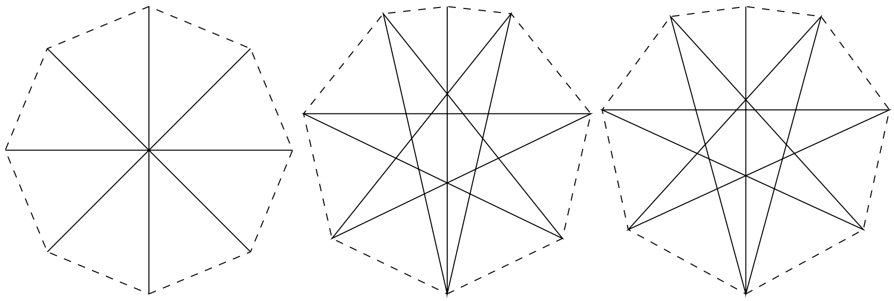


Fig. 3 Three small 8-gons $(\mathtt{P}_8, A(\mathtt{P}_8))$

even $n \geq 6$. Figures 1, 2, and 3 show diameter graphs of some small polygons. The solid lines illustrate pairs of vertices which are unit distance apart.

In addition to exact results and bounds, uncertified largest small polygons have been obtained both by metaheurisitics and nonlinear optimization. Assuming Graham's conjecture and the existence of an axis of symmetry, Mossinghoff [13] in 2006 constructed large small n-gons for even $6 \leq n \leq 20$. In 2020, using a formulation based on polar coordinates, Pintér [14] presented numerical solutions estimates of the maximal area for even $6 \leq n \leq 80$. The polar coordinates-based formulation was recently used by Pintér et al. [15] to obtain estimates of the maximal area for even $6 \leq n \leq 1000$.

The maximal area problem can be formulated as a nonconvex quadratically constrained quadratic optimization problem. In this work, we propose to solve it with a sequential convex optimization approach, also known as the concave-convex procedure [11, 12]. This approach is an ascent algorithm guaranteeing convergence to a locally optimal solution. Numerical experiments on polygons with up to $n = 128$ sides suggest that optimal solutions obtained are near-global. Indeed, without assuming Graham's conjecture nor the existence of an axis of symmetry in our quadratic formulation, optimal n-gons obtained with the algorithm proposed in this work verify both conditions within the limit of numerical computations. Moreover, for even $6 \leq n \leq 12$, this algorithm converges to known global optimal solutions. The algorithm is implemented as a MATLAB-based package, OPTIGON, which is available on GitHub [6]. Using the algorithm in MATLAB requires that CVX [7] be installed.

The remainder of this paper is organized as follows. In Sect. 2, we recall principal results on largest small polygons. Section 3 presents the quadratic formulation of the maximal area problem and the sequential convex optimization approach to solve it. We report in Sect. 4 computational results. Section 5 concludes the paper.

2 Largest small polygons

Let $A(\mathtt{P})$ denote the area of a polygon $\mathtt{P}$. Let $\mathtt{R}_n$ denote the regular small n-gon. We have

$$A(\mathtt{R}_n) = \begin{cases} \frac{n}{2}\left(\sin\frac{\pi}{n} - \tan\frac{\pi}{2n}\right) & \text{if } n \text{ is odd,} \\ \frac{n}{8}\sin\frac{2\pi}{n} & \text{if } n \text{ is even.} \end{cases}$$

We remark that $A(\mathtt{R}_n) < A(\mathtt{R}_{n-1})$ for all even $n \geq 6$ [2]. This suggests that $\mathtt{R}_n$ does not have maximum area for any even $n \geq 6$. Indeed, when n is even, we can construct a small n-gon with a larger area than $\mathtt{R}_n$ by adding a vertex at distance 1 along the mediatrix of an angle in $\mathtt{R}_{n-1}$. We denote this n-gon by $\mathtt{R}_{n-1}^+$ and we have

$$A(\mathtt{R}_{n-1}^+) = \frac{n-1}{2}\left(\sin\frac{\pi}{n-1} - \tan\frac{\pi}{2n-2}\right) + \sin\frac{\pi}{2n-2} - \frac{1}{2}\sin\frac{\pi}{n-1}.$$

Theorem 1 (Reinhardt [16]) *For all $n \geq 3$, let A_n^* denote the maximal area among all small n-gons and let* $\overline{A}_n := \frac{n}{2}\left(\sin\frac{\pi}{n} - \tan\frac{\pi}{2n}\right)$.

- When n is odd, $A_n^* = \overline{A}_n$ is only achieved by $\mathtt{R}_n$.
- $A_4^* = 1/2 < \overline{A}_4$ is achieved by infinitely many 4-gons, including $\mathtt{R}_4$ and $\mathtt{R}_3^+$ illustrated in Figure 1.
- When $n \geq 6$ is even, $A(\mathtt{R}_n) < A_n^* < \overline{A}_n$.

When $n \geq 6$ is even, the maximal area A_n^* is known for even $n \leq 12$. Using geometric arguments, Graham [9] determined analytically the largest small 6-gon, represented in Fig. 2c. Its area $A_6^* \approx 0.674981$ is about 3.92% larger than $A(\mathtt{R}_6) = 3\sqrt{3}/8$. The approach of Graham, combined with methods of global optimization, has been followed by [3] to determine the largest small 8-gon, represented in Fig. 3c. Its area $A_8^* \approx 0.726868$ is about 2.79% larger than $A(\mathtt{R}_8) = \sqrt{2}/2$. Henrion and Messine [10] found that $A_{10}^* \approx 0.749137$ and $A_{12}^* \approx 0.760730$.

For all even $n \geq 6$, let $\mathtt{U}_n$ denote an optimal small n-gon.

Theorem 2 (Graham [9], Foster and Szabo [8]) *For even $n \geq 6$, the diameter graph of $\mathtt{U}_n$ has a cycle of length $n-1$ and one additional edge from the remaining vertex.*

Conjecture 1 *For even $n \geq 6$, $\mathtt{U}_n$ has an axis of symmetry corresponding to the pending edge in its diameter graph.*

From Theorem 2, we note that $\mathtt{R}_{n-1}^+$ has the same diameter graph as the largest small n-gon $\mathtt{U}_n$. Conjecture 1 is only proven for $n = 6$ and this is due to Yuan [17]. However, the largest small polygons obtained by [3] and [10] are a further evidence that the conjecture may be true.

3 Nonconvex quadratically constrained quadratic optimization

We use cartesian coordinates to describe an n-gon $\mathtt{P}_n$, assuming that a vertex $\mathtt{v}_i$, $i = 0, 1, \dots, n-1$, is positioned at abscissa x_i and ordinate y_i. Placing the vertex $\mathtt{v}_0$ at the origin, we set $x_0 = y_0 = 0$. We also assume that the n-gon $\mathtt{P}_n$ is in the half-plane $y \ge 0$ and the vertices $\mathtt{v}_i$, $i = 1, 2, \dots, n-1$, are arranged in a counterclockwise order as illustrated in Fig. 4, i.e., $y_{i+1}x_i \ge x_{i+1}y_i$ for all $i = 1, 2, \dots, n-2$. The maximal area problem can be formulated as follows

$$\max_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{u}} \quad \sum_{i=1}^{n-2} u_i \tag{1a}$$

$$\text{s.t.} \quad (x_j - x_i)^2 + (y_j - y_i)^2 \le 1 \quad \forall 1 \le i < j \le n-1, \tag{1b}$$

$$x_i^2 + y_i^2 \le 1 \quad \forall 1 \le i \le n-1, \tag{1c}$$

$$y_i \ge 0 \quad \forall 1 \le i \le n-1, \tag{1d}$$

$$2u_i \le y_{i+1}x_i - x_{i+1}y_i \quad \forall 1 \le i \le n-2, \tag{1e}$$

$$u_i \ge 0 \quad \forall 1 \le i \le n-2. \tag{1f}$$

At optimality, for all $i = 1, 2, \dots, n-2$, $u_i = (y_{i+1}x_i - x_{i+1}y_i)/2$, which corresponds to the area of the triangle $\mathtt{v}_0\mathtt{v}_i\mathtt{v}_{i+1}$. It is important to note that, unlike what was done in [3, 10], this formulation does not make the assumption of Graham's conjecture, nor of the existence of an axis of symmetry.

Problem (1) is a nonconvex quadratically constrained quadratic optimization problem and can be reformulated as a difference-of-convex optimization (DCO) problem of the form

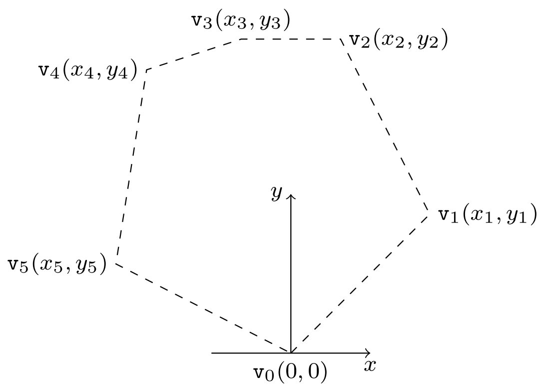


Fig. 4 Definition of variables: case of $n = 6$ vertices

$$\max_{z} \quad g_0(z) - h_0(z) \tag{2a}$$

$$\text{s.t.} \quad g_i(z) - h_i(z) \geq 0 \quad \forall 1 \leq i \leq m, \tag{2b}$$

where $g_0, \ldots, g_m$ and $h_0, \ldots, h_m$ are convex quadratic functions. We note that the feasible set

$$\Omega := \{z : g_i(z) - h_i(z) \geq 0, i = 1, 2, \ldots, m\}$$

is compact with a nonempty interior, which implies that $g_0(z) - h_0(z) < \infty$ for all $z \in \Omega$.

For a fixed $\boldsymbol{c}$, we have

$$\underline{g}_i(z;\boldsymbol{c}) := g_i(\boldsymbol{c}) + \nabla g_i(\boldsymbol{c})^T (z - \boldsymbol{c}) \leq g_i(z)$$

for all $i = 0, 1, \ldots, m$. Then the following problem

$$\max_{z} \quad \underline{g}_0(z;\boldsymbol{c}) - h_0(z) \tag{3a}$$

$$\text{s.t.} \quad \underline{g}_i(z;\boldsymbol{c}) - h_i(z) \geq 0 \quad \forall 1 \leq i \leq m \tag{3b}$$

is a convex restriction of the DCO problem (2) as stated by Proposition 1. Constraint (1e) is equivalent to

$$(y_{i+1} - x_i)^2 + (x_{i+1} + y_i)^2 + 8u_i \leq (y_{i+1} + x_i)^2 + (x_{i+1} - y_i)^2$$

for all $i = 1, 2, \ldots, n-2$. For a fixed $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$, if we replace (1e) in (1) by

$$\begin{aligned}(y_{i+1} - x_i)^2 + (x_{i+1} + y_i)^2 + 8u_i &\leq 2(b_{i+1} + a_i)(y_{i+1} + x_i) \\ &- (b_{i+1} + a_i)^2 + 2(a_{i+1} - b_i)(x_{i+1} - y_i) - (a_{i+1} - b_i)^2\end{aligned}$$

for all $i = 1, 2, \ldots, n-2$, we obtain a convex restriction of the maximal area problem.

Proposition 1 *If z is a feasible solution of (3) then z is a feasible solution of (2).*

Proof Let z be a feasible solution of (3), i.e., $\underline{g}_i(z;\boldsymbol{c}) - h_i(z) \geq 0$ for all $i = 1, 2, \ldots, m$. Then $g_i(z) - h_i(z) \geq \underline{g}_i(z;\boldsymbol{c}) - h_i(z) \geq 0$ for all $i = 1, 2, \ldots, m$. Thus, z is a feasible solution of (2). □

Proposition 2 *If $\boldsymbol{c}$ is a feasible solution of (2) then (3) is a feasible problem. Moreover, if z^* is an optimal solution of (3) then $g_0(\boldsymbol{c}) - h_0(\boldsymbol{c}) \leq g_0(z^*) - h_0(z^*)$.*

Proof Let $\boldsymbol{c}$ be a feasible solution of (2), i.e., $g_i(\boldsymbol{c}) - h_i(\boldsymbol{c}) \geq 0$ for all $i = 1, 2, \ldots, m$. Then there exists $z = \boldsymbol{c}$ such that $\underline{g}_i(\boldsymbol{c};\boldsymbol{c}) - h_i(\boldsymbol{c}) = g_i(\boldsymbol{c}) - h_i(\boldsymbol{c}) \geq 0$ for all

$i = 1, 2, \dots, m$. Thus, (3) is a feasible problem. Moreover, if z^* is an optimal solution of (3), we have $g_0(\boldsymbol{c}) - h_0(\boldsymbol{c}) = \underline{g}_0(\boldsymbol{c};\boldsymbol{c}) - h_0(\boldsymbol{c}) \le \underline{g}_0(z^*;\boldsymbol{c}) - h_0(z^*) \le g_0(z^*) - h_0(z^*)$. □

From Proposition 2, the optimal small n-gon $(\boldsymbol{x}, \boldsymbol{y})$ obtained by solving a convex restriction of Problem (1) constructed around a small n-gon $(\boldsymbol{a}, \boldsymbol{b})$ has a larger area than this one. Proposition 3 states that if $(\boldsymbol{a}, \boldsymbol{b})$ is the optimal n-gon of the convex restriction constructed around itself, then it is a local optimal n-gon for the maximal area problem.

Proposition 3 *Let* $\boldsymbol{c}$ *be a feasible solution of* (2). *We suppose that* $\underline{\Omega}(\boldsymbol{c}) := \{z : \underline{g}_i(z;\boldsymbol{c}) - h_i(z) \ge 0, i = 1, 2, \dots, m\}$ *satisfies Slater condition. If* $\boldsymbol{c}$ *is an optimal solution of* (3) *then* $\boldsymbol{c}$ *is a critical point of* (2).

Proof If $\boldsymbol{c}$ is an optimal solution of (3) then there exist m scalars $\mu_1, \mu_2, \dots, \mu_m$ such that

$$\nabla \underline{g}_0(\boldsymbol{c};\boldsymbol{c}) + \sum_{i=1}^{m} \mu_i \nabla \underline{g}_i(\boldsymbol{c};\boldsymbol{c}) = \nabla h_0(\boldsymbol{c}) + \sum_{i=1}^{m} \mu_i \nabla h_i(\boldsymbol{c}),$$

$$\underline{g}_i(\boldsymbol{c};\boldsymbol{c}) \ge h_i(\boldsymbol{c}) \qquad \forall i = 1, 2, \dots, m,$$

$$\mu_i \ge 0 \qquad \forall i = 1, 2, \dots, m,$$

$$\mu_i \underline{g}_i(\boldsymbol{c};\boldsymbol{c}) = \mu_i h_i(\boldsymbol{c}) \qquad \forall i = 1, 2, \dots, m.$$

Since $\underline{g}_i(\boldsymbol{c};\boldsymbol{c}) = g_i(\boldsymbol{c})$ and $\nabla \underline{g}_i(\boldsymbol{c};\boldsymbol{c}) = \nabla g_i(\boldsymbol{c})$ for all $i = 0, 1, \dots, m$, we conclude that $\boldsymbol{c}$ is a critical point of (2). □

We propose to solve the DCO problem (2) with a sequential convex optimization approach given in Algorithm 1, also known as concave-convex procedure. A proof of showing that a sequence $\{z_k\}_{k=0}^{\infty}$ generated by Algorithm 1 converges to a critical point z^* of the original DCO problem (2) can be found in [11, 12].

Algorithm 1 Sequential convex optimization

```
Initialization: choose a feasible solution z_0 and a stopping criteria ε > 0.
z_1 ∈ arg max{g_0(z; z_0) − h_0(z): g_i(z; z_0) − h_i(z) ≥ 0, i = 1, 2, ..., m}
k := 1
while ||z_k − z_{k−1}|| / ||z_k|| > ε do
    z_{k+1} ∈ arg max{g_0(z; z_k) − h_0(z): g_i(z; z_k) − h_i(z) ≥ 0, i = 1, 2, ..., m}
    k := k + 1
end while
```

4 Computational results

Problem (1) was solved in MATLAB using CVX 2.2 with MOSEK 9.1.9 and `default precision` (tolerance $\epsilon = 1.49 \times 10^{-8}$). All the computations were carried out on an `Intel(R) Core(TM) i7-3540M CPU @ 3.00 GHz` computing platform. Algorithm 1 was implemented as a MATLAB package: OPTIGON, which is freely available on GitHub [6]. Using Algorithm 1 in MATLAB requires that CVX be installed. CVX is a MATLAB-based modeling system for convex optimization, which turns MATLAB into a modeling language, allowing constraints and objectives to be specified using standard MATLAB expression syntax [7].

We chose the following values as initial solution:

$$\begin{aligned} a_0 &= 0, & b_0 &= 0, \\ a_i &= \frac{\sin \frac{2i\pi}{n-1}}{2\cos \frac{\pi}{2n-2}} = -a_{n-i}, & b_i &= \frac{1-\cos \frac{2i\pi}{n-1}}{2\cos \frac{\pi}{2n-2}} = b_{n-i} \quad \forall i = 1,\dots,n/2-1, \\ a_{n/2} &= 0, & b_{n/2} &= 1, \end{aligned}$$

which define the n-gon $\mathtt{R}^+_{n-1}$, and the stopping criteria $\varepsilon = 10^{-5}$. At each iteration, we forced MOSEK to solve the dual problem by setting the parameter `MSK_IPAR_INTPNT_SOLVE_FORM` to `MSK_SOLVE_DUAL`. Table 1 shows the areas of the optimal n-gons $\hat{\mathtt{P}}_n$ for even numbers $n = 6, 8, \dots, 128$, along with the areas of the initial n-gons $\mathtt{R}^+_{n-1}$, the best lower bounds $\underline{A}_n$ found in the literature, and the upper bounds $\overline{A}_n$. We also report the number k of iterations of Algotithm 1 for each n. The results in Table 1 support the following keypoints:

1. For $6 \le n \le 12$, $\underline{A}_n - A(\hat{\mathtt{P}}_n) \le 10^{-8}$, i.e., Algorithm 1 converges to the best known optimal solutions found in the literature.
2. For $32 \le n \le 80$, $\underline{A}_n < A(\mathtt{R}^+_{n-1}) < A(\hat{\mathtt{P}}_n)$, i.e., it appears that the solutions obtained by Pintér [14] are suboptimal.
3. For all n, the solutions $\hat{\mathtt{P}}_n$ obtained with Algorithm 1 verify, within the limit of the numerical computations, Theorem 2 and Conjecture 1, i.e.,

$$\begin{aligned} x_{n/2} &= 0, & y_{n/2} &= 1, \\ \|\mathtt{v}_{n/2-1}\| &= 1, & \|\mathtt{v}_{n/2+1}\| &= 1, \\ \|\mathtt{v}_{i+n/2} - \mathtt{v}_i\| &= 1, & \|\mathtt{v}_{i+n/2+1} - \mathtt{v}_i\| &= 1 \quad \forall i = 1,2,\dots,n/2-2, \\ \|\mathtt{v}_{n-1} - \mathtt{v}_{n/2-1}\| &= 1, \\ x_{n-i} &= -x_i, & y_{n-i} &= y_i \quad \forall i = 1,2,\dots,n/2-1. \end{aligned}$$

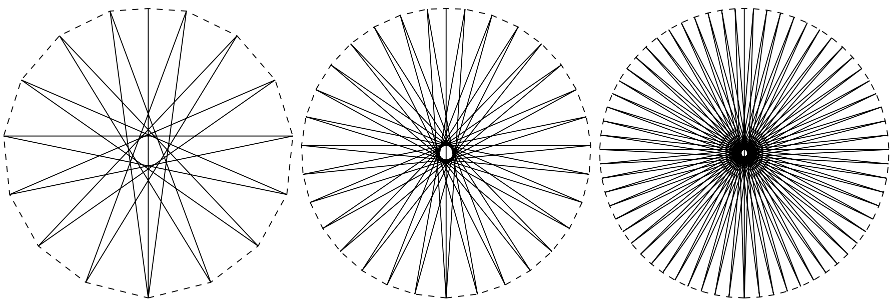
We illustrate the largest small 16-, 32- and 64-gons in Fig. 5. Furthermore, we remark that Theorem 2 and Conjecture 1 are verified by each polygon of the sequence generated by Algorithm 1. All 6-gons generated by the algorithm are represented in Fig. 6 and the coordinates of their vertices are given in Table 2.

Table 1 Maximal area problem

n	$A(\mathrm{R}^{+}_{n-1})$	$\underline{A}_n$	$\overline{A}_n$	$A(\hat{\mathrm{P}}_n)$	# iter. k
6	0.6722882584	0.6749814429 [5, 9, 13]	0.6961524227	0.6749814387	5
8	0.7253199909	0.7268684828 [3, 13]	0.7350842599	0.7268684802	10
10	0.7482573378	0.7491373459 [10, 13]	0.7531627703	0.7491373454	16
12	0.7601970055	0.7607298734 [10, 13]	0.7629992851	0.7607298710	24
14	0.7671877750	0.7675310111 [13]	0.7689359584	0.7675310093	33
16	0.7716285345	0.7718613220 [13]	0.7727913493	0.7718613187	43
18	0.7746235089	0.7747881651 [13]	0.7754356273	0.7747881619	55
20	0.7767382147	0.7768587560 [13]	0.7773275822	0.7768587517	68
22	0.7782865351	0.7783773308 [14]	0.7787276939	0.7783773228	81
24	0.7794540033	0.7795240461 [14]	0.7797927529	0.7795240330	95
26	0.7803559816	0.7804111201 [14]	0.7806217145	0.7804111058	109
28	0.7810672517	0.7811114192 [14]	0.7812795297	0.7811114002	122
30	0.7816380102	0.7816739255 [14]	0.7818102598	0.7816739044	136
32	0.7821029651	0.7818946320 [14]	0.7822446490	0.7821325276	148
34	0.7824867354	0.7823103007 [14]	0.7826046775	0.7825113660	159
36	0.7828071755	0.7826513767 [14]	0.7829063971	0.7828279054	169
38	0.7830774889	0.7829526627 [14]	0.7831617511	0.7830950955	177
40	0.7833076096	0.7832011589 [14]	0.7833797744	0.7833226804	183
42	0.7835051276	0.7834135187 [14]	0.7835674041	0.7835181187	185
44	0.7836759223	0.7835966860 [14]	0.7837300377	0.7836871900	184
46	0.7838246055	0.7837554636 [14]	0.7838719255	0.7838344336	179
48	0.7839548353	0.7838942710 [14]	0.7839964516	0.7839634510	172
50	0.7840695435	0.7840161496 [14]	0.7841063371	0.7840771278	162
52	0.7841711020	0.7841233641 [14]	0.7842037903	0.7841778072	150
54	0.7842614465	0.7842192995 [14]	0.7842906181	0.7842674010	138
56	0.7843421691	0.7843044654 [14]	0.7843683109	0.7843474779	128
58	0.7844145892	0.7843807534 [14]	0.7844381066	0.7844193386	118
60	0.7844798073	0.7844492943 [14]	0.7845010402	0.7844840717	109
62	0.7845387477	0.7845111362 [14]	0.7845579827	0.7845425886	101
64	0.7845921910	0.7834620877 [14]	0.7846096710	0.7845956631	94
66	0.7846408000	0.7845910589 [14]	0.7846567322	0.7846439473	88
68	0.7846851407	0.7846139029 [14]	0.7846997026	0.7846880001	82
70	0.7847256986	0.7846403575 [14]	0.7847390429	0.7847283036	77
72	0.7847628920	0.7847454020 [14]	0.7847751508	0.7847652718	72
74	0.7847970830	0.7845564840 [14]	0.7848083708	0.7847992622	68
76	0.7848285863	0.7847585719 [14]	0.7848390031	0.7848305850	64
78	0.7848576763	0.7845160579 [14]	0.7848673094	0.7848595143	61
80	0.7848845934	0.7848252941 [14]	0.7848935195	0.7848862871	58
82	0.7849095487	–	0.7849178354	0.7849111119	55
84	0.7849327284	–	0.7849404352	0.7849341725	52
86	0.7849542969	–	0.7849614768	0.7849556352	50
88	0.7849744002	–	0.7849811001	0.7849756425	48

Table 1 (continued)

n	$A(\mathtt{R}_{n-1}^{+})$	$\underline{A}_n$	$\overline{A}_n$	$A(\hat{\mathtt{P}}_n)$	# iter. k
90	0.7849931681	–	0.7849994298	0.7849943223	46
92	0.7850107163	–	0.7850165772	0.7850117894	44
94	0.7850271482	–	0.7850326419	0.7850281477	42
96	0.7850425565	–	0.7850477130	0.7850434878	40
98	0.7850570245	–	0.7850618708	0.7850578951	39
100	0.7850706272	–	0.7850751877	0.7850714422	38
102	0.7850834323	–	0.7850877290	0.7850841941	36
104	0.7850955008	–	0.7850995538	0.7850962152	35
106	0.7851068883	–	0.7851107156	0.7851075587	34
108	0.7851176450	–	0.7851212630	0.7851182747	33
110	0.7851278167	–	0.7851312404	0.7851284086	32
112	0.7851374450	–	0.7851406881	0.7851380017	31
114	0.7851465680	–	0.7851496430	0.7851470916	30
116	0.7851552203	–	0.7851581386	0.7851557129	29
118	0.7851634339	–	0.7851662060	0.7851639010	29
120	0.7851712379	–	0.7851738734	0.7851716781	28
122	0.7851786591	–	0.7851811668	0.7851790741	27
124	0.7851857221	–	0.7851881101	0.7851861129	26
126	0.7851924497	–	0.7851947255	0.7851928211	26
128	0.7851988626	–	0.7852010332	0.7851992126	25



(a) $(\hat{\mathtt{P}}_{16}, 0.771861)$ **(b)** $(\hat{\mathtt{P}}_{32}, 0.782133)$ **(c)** $(\mathtt{U}_{64}, 0.784596)$

Fig. 5 Three largest small n-gons $(\hat{\mathtt{P}}_n, A(\hat{\mathtt{P}}_n))$

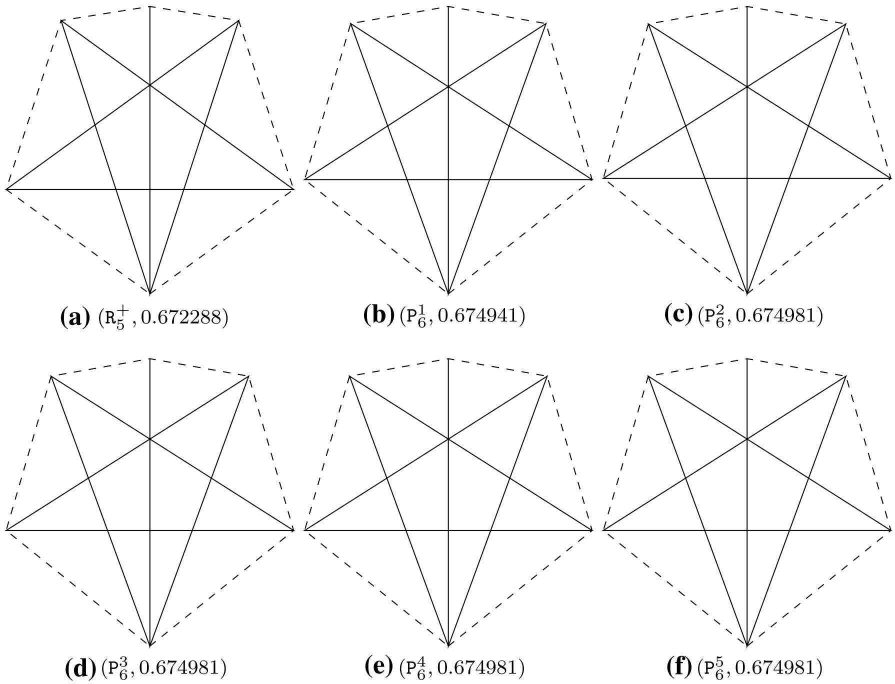


Fig. 6 All 6-gons ($\mathtt{P}_6^k, A(\mathtt{P}_6^k)$) generated by Algorithm 1

5 Conclusion

We proposed a sequential convex optimization approach to find the largest small n-gon for a given even number $n \geq 6$, which is formulated as a nonconvex quadratically constrained quadratic optimization problem. The algorithm, also known as the concave-convex procedure, guarantees convergence to a locally optimal solution.

Without assuming Graham's conjecture nor the existence of an axis of symmetry in our quadratic formulation, numerical experiments on polygons with up to $n = 128$ sides showed that each optimal n-gon obtained with the algorithm proposed verifies both conditions within the limitation of the numerical computations. Futhermore, for even $6 \leq n \leq 12$, the n-gons obtained correspond to the known largest small n-gons.

Table 2 Vertices of 6-gons generated by Algorithm 1

6-gon	Coordinates (x_i, y_i)					Area
	(x_1, y_1)	(x_2, y_2)	(x_3, y_3)	(x_4, y_4)	(x_5, y_5)	
R_5^+	(0.500000, 0.363271)	(0.309017, 0.951057)	(0.000000, 1.000000)	(− 0.309017, 0.951057)	(− 0.500000, 0.363271)	0.6722882584
P_6^1	(0.500000, 0.397460)	(0.339680, 0.940541)	(0.000000, 1.000000)	(− 0.339680, 0.940541)	(− 0.500000, 0.397460)	0.6749414624
P_6^2	(0.500000, 0.401764)	(0.343285, 0.939231)	(0.000000, 1.000000)	(− 0.343285, 0.939231)	(− 0.500000, 0.401764)	0.6749808685
P_6^3	(0.500000, 0.402283)	(0.343715, 0.939074)	(0.000000, 1.000000)	(− 0.343715, 0.939074)	(− 0.500000, 0.402283)	0.6749814310
P_6^4	(0.500000, 0.402345)	(0.343766, 0.939055)	(0.000000, 1.000000)	(− 0.343766, 0.939055)	(− 0.500000, 0.402345)	0.6749814386
P_6^5	(0.500000, 0.402352)	(0.343773, 0.939053)	(0.000000, 1.000000)	(− 0.343773, 0.939053)	(− 0.500000, 0.402352)	0.6749814387

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