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ORIGINAL PAPER

Spatial price equilibrium networks with flow-dependent arc multipliers

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Abstract

The spatial price equilibrium modeling framework, which emphasizes the importance of transportation costs between markets, has been utilized in agricultural, energy, mineral as well as financial applications. In this paper, we construct static and dynamic spatial price equilibrium networks with flow-dependent arc multipliers, which expand the reach of applications. The static model is formulated and analyzed as a variational inequality problem, whereas the dynamic one is formulated as a projected dynamical system, whose set of stationary points coincides with the set of solutions of the variational inequality. Qualitative results are presented, along with an algorithm, the Euler method, which yields a time-discretization of the continuous-time adjustment processes associated with the product shipments from supply markets to demand markets. The algorithm is implemented and applied to compute solutions to numerical examples with flow-dependent arc multipliers addressing losses and/or gains, inspired by perishable agricultural products, and by financial investments. The results in this paper add to the literature on generalized networks as well as that on commodity trade.

Keywords Spatial price equilibrium · Commodity trade · Perishable products · Generalized networks · Variational inequalities · Projected dynamical systems

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1 Introduction

Generalized networks provide a rich formalism for the modeling of a plethora of problems in diverse applications, ranging from transportation, agriculture, and energy to economics and finance. Originating in the classical work of [9, 17], such networks utilize arc multipliers to capture gains and/or losses as flows propagate over links. Examples of gains in applications include growth in financial investments through interest rates and/or currency exchanges, the increase in volume due to breeding or related biological and chemical processes, as well as the increase in value of products through manufacturing and other processes. Losses in applications can be due to product perishability, theft, seepage, evaporation, leakage, etc.

Interestingly, in the majority of generalized network models the arc multipliers are fixed [1, 10, 32, 36, 52]; that is, they are not flow-dependent. Truemper in [50] recognized that, to that date, most of the generalized networks that had been studied had fixed arc multipliers, which he noted correspond to linear, as opposed to, nonlinear, functions of flow on the arcs. He argued that it was important to consider nonlinear functions, since there are real-world problems, including some associated with chemical processes, that could not be modeled using, in effect, fixed arc multipliers for gains (see [53]). Subsequently, Shigeno in [43] also studied such generalized networks, wherein, specifically, the flow leaving an arc is an increasing concave function of the flow entering it. More recently, Vegh in [51], building on the earlier work in [43, 50], established that the resulting general convex programming model is relevant for several market equilibrium problems, including the linear Fisher market model and its various extensions. Vegh in [51] constructed a polynomial time combinatorial algorithm for solving corresponding flow maximization problems, which also yields a new algorithm for linear generalized flows. Generalized networks with convex outflow functions, as opposed to concave ones, were studied in [14, 15], where it was noted that having such more general outflow functions allows one to model processes in which the effectiveness increases with the load. The authors emphasized various financial trading applications in which better rates are obtained if larger amounts are traded.

In this paper, we construct spatial price network models, in static and dynamic versions, that incorporate flow-dependent arc multipliers. Generalized networks, with fixed arc multipliers, were first applied to spatial price equilibrium problems in [49] and, subsequently, in [29], where variational inequality was used for for modeling, qualitative analysis, and algorithmic development. Variational inequalities, in pure network flow settings for spatial price equilibrium problems, were also utilized in [5–7, 12, 33–35, 38]. This paper adds to the literature on spatial price equilibrium problems, which originated in the classical work of [42, 45, 46], as well as to that of generalized networks.

Recent contributions to the modeling, analysis, and solution of spatial price equilibrium problems have included, among others, the work of [31], who introduced quality into spatial price equilibrium models, and that of [30], who demonstrated how tariff rate quotas, along with tariffs and quotas, could be incorporated into spatial price equilibrium models, using the theory of variational inequalities. However, these contributions did not handle perishability of products (see also [22, 36, 54]), a feature that has garnered increased attention in the pandemic (see [26]). Nagurney in [28] incorporated tariffs and quotas in a multicommodity spatial price equilibrium model with perishability but it did not include flow-dependent arc multipliers. Furthermore, the model therein assumed, in contrast to the model in this paper, that the arc multipliers are in the range (0, 1] and, hence, only possible losses, and not gains, were considered. The spatial price equilibrium framework, as noted by [31], is relevant to many agricultural product industries (see [2, 16, 18, 40, 44, 47]) as well as to mineral and energy ones (see [21]). Furthermore, it has been utilized also in financial applications (see [23, 27, 48]). Recently, flow-dependent arc multipliers to capture gains have also been applied to bitcoin exchanges (see [39]), where, clearly, the use of more general arc multipliers can enhance modeling capabilities; for background on blockchain technologies, see [41]. In this paper, we consider flow-dependent arc multipliers that can correspond to losses or to gains. For additional background on networks in economics and finance, see [19, 27].

The novelty of our work lies in the following:

1. As far as we know, this is the first work in the literature that takes up the challenge to provide a mathematical framework for flow-dependent arc multipliers in spatial price network equilibrium models. Moreover, to-date, there has only been very limited work on flow-dependent arc multipliers in any network context.
2. This paper studies the complex phenomena of modeling flow-dependent arc multipliers in static and dynamic spatial price equilibrium network models.
3. The work represents the first use of both variational inequality theory and projected dynamical systems theory in the formulation of networks with flow-dependent arc multipliers.
4. The theoretical constructs, the algorithm, and the numerical studies demonstrate the wide applicability of such a complex operations research problem through mathematical elegance.

This paper is organized as follows. In Sect. 2, the static spatial price equilibrium model with flow-dependent arc multipliers is first presented, along with the governing equilibrium conditions, followed by the derivation of the variational inequality formulation. The variational inequality formulation, with variables that are product shipments, enables an effective and easy to implement computational scheme. Furthermore, existence of a solution is guaranteed. We then propose a dynamic adjustment process for the evolution of the product shipments and provide the associated projected dynamical system (cf. [37]), which are accompanied by additional qualitative results under monotonicity conditions. We also describe the value of the flow-dependent arc multiplier modeling extensions for spatial price networks, in terms of several applications, specifically, to perishable products and to finance. In Sect. 3, the Euler method is detailed, along with its realization for the solution of the spatial equilibrium model. The algorithm, at each iteration, yields explicit formulae for the product shipments. We provide convergence results accompanied by a series of numerical examples of increasing complexity. Section 4 summarizes the results, presents our conclusions, and discusses several promising directions for future research.

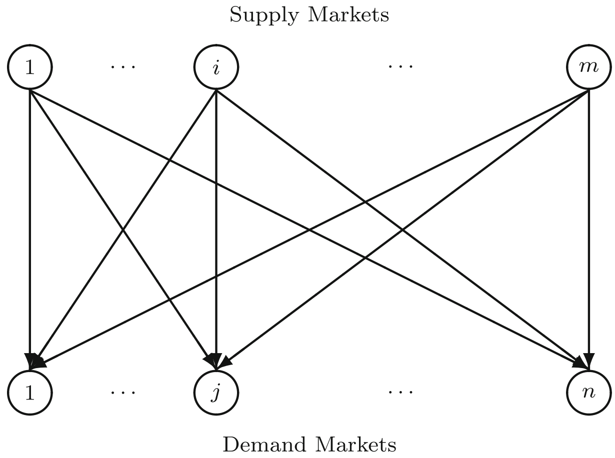

Fig. 1 The bipartite structure of the spatial price network equilibrium problem

2 Spatial price equilibrium networks with flow-dependent arc multipliers

In this section, we develop the spatial price equilibrium network models with flow-dependent arc multipliers. In Sect. 2.1, we introduce the static model, state the governing equilibrium conditions, and derive the variational inequality formulation. In Sect. 2.2, we describe the underlying dynamics associated with the product shipments and present the projected dynamical systems model whose set of stationary points corresponds to the set of solutions of the variational inequality problem governing the static spatial price equilibrium model. In Sect. 2.3, we discuss how the models expand the scope of applications. The algorithm that we will apply to compute solutions is a time-discretization of the continuous time adjustment processes.

See Fig. 1 for the underlying bipartite network structure of the spatial price equilibrium problem (SPEP).

There are m supply markets and n demand markets that are spatially separated and are engaged, respectively, in the production and consumption of a homogeneous product. Recall that the spatial price equilibrium framework is that of perfect competition.

The major notation for the model is given in Table 1. All vectors are assumed to be column vectors.

2.1 The static spatial price equilibrium model with flow-dependent arc multipliers

We now construct the static model. The conservation of flow equations are:

$$s_i = \sum_{j=1}^{n} Q_{ij}, \quad i = 1, \ldots, m; \tag{1}$$

$$d_j = \sum_{i=1}^{m} \alpha_{ij}(Q_{ij}) Q_{ij}, \quad j = 1, \ldots, n; \tag{2}$$

Table 1 Notation for the spatial price equilibrium network models with flow-dependent arc multipliers

Notation	Parameter definition
u_{ij}	The upper bound on the product shipment from supply market i to demand market j; $i = 1, \ldots, m$; $j = 1, \ldots, n$
Notation	**Variable definition**
s_i	The supply of the product at supply market i; $i = 1, \ldots, m$. We group all the supplies into the vector $s \in R^m_+$
d_j	The demand for the product at demand market j; $j = 1, \ldots, n$. We group all the demands into the vector $d \in R^n_+$
Q_{ij}	The shipment of the product from supply market i to demand market j; $i = 1, \ldots, m$; $j = 1, \ldots, n$. We group all the product shipments into the vector $Q \in R^{mn}_+$
Notation	**Function definition**
$\pi_i(s)$	The supply price function at supply market i; $i = 1, \ldots, m$. We group all these functions into the vector $\pi(s) \in R^m$
$\rho_j(d)$	The demand price function at demand market j; $j = 1, \ldots, n$. We group all the demand price functions into the vector $\rho(d) \in R^n$
$c_{ij}(Q)$	The unit transportation cost associated with shipping the product from supply market i to demand market j; $i = 1, \ldots, m$; $j = 1, \ldots, n$. We group the unit transportation costs for all supply/demand market pairs into the vector $c(Q) \in R^{mn}$
$\alpha_{ij}(Q_{ij})$	The multiplier function associated with arc/link (i, j); $i = 1, \ldots, m$; $j = 1, \ldots, n$

$$0 \leq Q_{ij} \leq u_{ij}, \quad i = 1, \ldots, m; \; j = 1, \ldots, n. \tag{3}$$

According to Eq. (1), the supply of the product at each supply market is equal to the sum of the amounts of the product shipped to all the demand markets. Equation (2) expresses that the quantity of the product consumed at a demand market is equal to the sum of the amounts of the product that actually arrive at the demand market. Finally, (3) guarantees that the product shipments are nonnegative and do not exceed the capacity of the links.

Note that a special case of the above flow-dependent arc multipliers is that of a fixed value for each (i, j), which we denote by α_{ij}. For example, if $\alpha_{ij} = .97$, this means that with a product shipment of Q_{ij} starting at supply market node i, one is left with $.97Q_{ij}$ units of the product when it reaches demand market node j. If the arc multiplier is greater than 1, there is a gain; if all the multipliers are equal to 1, then the problem is a pure network problem; if an arc multiplier is less than 1, then there is a loss.

We assume that the supply price, demand price, and unit transportation cost functions are continuous and that the supply price functions and the unit transportation cost functions are monotone increasing, whereas the demand price functions are monotone decreasing (see also [24]).

We note that [49] was the first to use arc multipliers in a spatial equilibrium model but it had separable price functions, fixed unit transportation costs, and the arc multipliers were not flow-dependent. Nagurney and Aronson in [29] utilized variational inequality theory for the formulation of a multiperiod spatial price equilibrium model with gains and losses using arc multipliers, but, again, the arc multipliers were fixed. Arc multipliers, with a focus on losses, and also fixed, for a variety of supply chain network models and applications, optimization-based, as well as game theoretic, are constructed in [36]. Note that here we do not limit ourselves to arc multipliers that are less than 1.

We define the feasible set $\mathcal{K}^1 \equiv \{(s, Q, d) | (1), (2), \text{ and } (3) \text{ hold}\}$.

Definition 1 *Spatial price equilibrium with flow-dependent arc multipliers.* A supply, product shipment, and demand pattern $(s^*, Q^*, d^*) \in \mathcal{K}^1$ is a spatial price equilibrium with flow-dependent arc multipliers if it satisfies the following conditions: for each pair of supply and demand markets (i, j); $i = 1, \ldots, m$; $j = 1, \ldots, n$:

$$\pi_i(s^*) + c_{ij}(Q^*) \begin{cases} \leq \alpha_{ij}(Q^*_{ij})\rho_j(d^*), & \text{if } Q^*_{ij} = u_{ij}, \\ = \alpha_{ij}(Q^*_{ij})\rho_j(d^*), & \text{if } 0 < Q^*_{ij} < u_{ij}, \\ \geq \alpha_{ij}(Q^*_{ij})\rho_j(d^*), & \text{if } Q^*_{ij} = 0. \end{cases} \tag{4}$$

The above equilibrium conditions express the following: if there is a positive quantity of the product shipped from a supply market to a demand market, and this volume is not at its upper bound, then, in equilibrium, the value of the surviving volume at the demand market must cover the supply market price plus the unit transportation cost (cf. [49]). If the volume is at the upper bound, then the value at the demand market of the product can exceed the supply price plus the unit transportation cost. If the supply price plus the unit transportation cost is greater than the value of the product at the demand market, then there will be no shipment of the product between the pair of supply and demand markets.

We now provide a variational inequality of the above governing spatial price equilibrium conditions in which the variables are exclusively shipments. Such a formulation enables a more direct determination of the evolution of the product shipments over time via a projected dynamical system (PDS). The constructed PDS will then be used to propose an algorithm, which corresponds to a discretization of the continuous-time adjustment processes provided by the PDS.

We define supply price functions and demand price functions, denoted, respectively, by $\hat{\pi}_i(Q)$ for $i = 1, \ldots, m$, and by $\hat{\rho}_j(Q)$ for $j = 1, \ldots, n$, that are functions of the product shipments. This can be done because of constraints (1) and (2). We, hence, have that:

$$\hat{\pi}_i = \hat{\pi}_i(Q) \equiv \pi_i(s), \quad i = 1, \ldots, m, \tag{5}$$

and

$$\hat{\rho}_j = \hat{\rho}_j(Q) \equiv \rho_j(d), \quad j = 1, \dots, n. \tag{6}$$

We define the feasible set $\mathcal{K}^2 \equiv \{Q | Q \in R^{mn}_+, \text{ such that } 0 \leq Q_{ij} \leq u_{ij}, \forall i, j\}$. We assume that each arc multiplier function $\alpha_{ij}(Q_{ij})$, $\forall i, j$, is continuous and is positive over the range of Q in the feasible set $\mathcal{K}^2$.

Theorem 1 Variational inequality formulation of spatial price equilibrium with flow-dependent arc multipliers. *A shipment pattern $Q^* \in \mathcal{K}^2$ is a spatial price equilibrium with flow-dependent arc multipliers according to Definition 1 if and only if it satisfies the variational inequality problem:*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(\hat{\pi}_i(Q^*) + c_{ij}(Q^*) - \alpha_{ij}(Q^*_{ij})\hat{\rho}_j(Q^*)) \times (Q_{ij} - Q^*_{ij}) \geq 0, \quad \forall Q \in \mathcal{K}^2. \tag{7}$$

Proof We first establish necessity, that is, if $Q^* \in \mathcal{K}^2$ satisfies the spatial price equilibrium conditions (4) according to Definition 1, then it satisfies variational inequality (7).

Observe that, for a fixed pair of supply and demand markets (i, j), (4) implies that

$$(\pi_i(s^*) + c_{ij}(Q^*) - \alpha_{ij}(Q^*_{ij})\rho_j(d^*)) \times (Q_{ij} - Q^*_{ij}) \geq 0,$$
$$\forall Q_{ij}, \text{ such that } 0 \leq Q_{ij} \leq u_{ij}, \forall i, j. \tag{8}$$

Indeed, since, if $u_{ij} > Q^*_{ij} > 0$, we know, from the equilibrium conditions, that the expression to the left of the multiplication sign in (8) is equal to zero, so (8) holds true. Also, if $Q^*_{ij} = 0$, then the expression preceding and following the multiplication sign in (8) will be nonnegative and, hence, the product is also nonnegative and (8) holds true for this case, as well. Finally, if $Q^*_{ij} = u_{ij}$, then (8) also holds true.

But, making use of (5) and (6), (8) may be rewritten as:

$$(\hat{\pi}_i(Q^*) + c_{ij}(Q^*) - \alpha_{ij}(Q^*_{ij})\hat{\rho}_j(Q^*)) \times (Q_{ij} - Q^*_{ij}) \geq 0,$$
$$\forall Q_{ij}, \text{ such that } 0 \leq Q_{ij} \leq u_{ij}, \forall i, j. \tag{9}$$

Summing now (9) over all supply markets i, and over all demand markets j, yields:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(\hat{\pi}_i(Q^*) + c_{ij}(Q^*) - \alpha_{ij}(Q^*_{ij})\hat{\rho}_j(Q^*)) \times (Q_{ij} - Q^*_{ij}) \geq 0,$$
$$\forall Q_{ij}, \text{ such that } 0 \leq Q_{ij} \leq u_{ij}, \forall i, j, \tag{10}$$

which corresponds to variational inequality (7).

We now establish sufficiency, that is, if $Q^* \in \mathcal{K}^2$ satisfies variational inequality (7) then it also satisfies the spatial price equilibrium conditions (4).

Let $Q_{ij} = Q^*_{ij}, \forall (i,j) \neq (h,l)$; where (h,l) is set arbitrarily and substitute into (7). The resultant is:

$$(\hat{\pi}_h(Q^*) + c_{hl}(Q^*) - \alpha_{hl}(Q^*_{hl})\hat{\rho}_l(Q^*)) \times (Q_{hl} - Q^*_{hl}) \geq 0,$$
$$\forall Q_{hl}, \text{ such that } u_{hl} \geq Q_{hl} \geq 0. \tag{11}$$

But (11) implies that, if $Q^*_{hl} = 0$ then $(\hat{\pi}_h(Q^*)+c_{hl}(Q^*)-\alpha_{hl}(Q^*_{hl})\hat{\rho}_l(Q^*)) \geq 0$, and, if $u_{hl} > Q^*_{hl} > 0$, then, for (11) to hold, $(\hat{\pi}_h(Q^*)+c_{hl}(Q^*)-\alpha_{hl}(Q^*_{hl})\hat{\rho}_l(Q^*)) = 0$. Finally, (11) implies that for $Q^*_{hl} = u_{hl}$, then $(\hat{\pi}_h(Q^*)+c_{hl}(Q^*)-\alpha_{hl}(Q^*_{hl})\hat{\rho}_l(Q^*)) \leq 0$. Since these results hold for any pair (h,l), and making use of (5) and (6), we conclude that the equilibrium conditions (4) are satisfied by the shipment pattern satisfying (7).

The proof is complete. □

We now put variational inequality (7) into standard form [24]: determine $X^* \in \mathcal{K}$, such that

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{12}$$

where $\mathcal{K}$ is the feasible set, which must be closed and convex. The vector X is an N-dimensional vector, as is $F(X)$, with $F(X)$ being continuous and given, and maps X from $\mathcal{K}$ into R^N. $\langle \cdot, \cdot \rangle$ denotes the inner product in N-dimensional Euclidean space. We define the vector $X \equiv Q$ and the vector $F(X)$ with components $F_{ij}(X) = \hat{\pi}_i(Q) + c_{ij}(Q) - \alpha_{ij}(Q_{ij})\hat{\rho}_j(Q)$; $i = 1, \ldots, m$; $j = 1, \ldots, n$. Here $N = mn$. Also, we define the feasible set $\mathcal{K} \equiv \mathcal{K}^2$. Then, variational inequality (7) can be placed into standard form (12).

Remark

Since the feasible set $\mathcal{K}^2$ is compact and the functions that enter the respective variational inequalities are assumed to be continuous, existence of a solution to variational inequality (7) is guaranteed from the classical theory of variational inequalities (see [20]). It also follows from the classical theory that if $F(X)$ in (12) is strictly monotone, that is,

$$\langle F(X^1) - F(X^2), X^1 - X^2 \rangle > 0, \quad \forall X^1, X^2 \in \mathcal{K}, \quad X^1 \neq X^2, \tag{13}$$

then the solution X^*, which recall is equal to Q^*, is unique.

For additional background on the variational inequality problem, we refer the reader to the book by [24].

2.2 The projected dynamical system spatial price model

We now describe a dynamic adjustment process for the evolution of the product shipments. For a current shipment at time t, $X(t) = Q(t)$, $-F_{ij}(X(t)) = \alpha_i(Q_{ij}(t)\hat{\rho}_j(Q(t)) - c_{ij}(Q(t)) - \hat{\pi}_i(Q(t))$ is the excess value of the product between demand market j and supply market i. In our framework, the rate of change of the product shipment between a supply and demand market pair (i,j), which is denoted by $\dot{Q}_{ij}$, is in proportion to $-F_{ij}(X)$, as long as the product shipment Q_{ij} is positive, and not at its upper bound; that is, when $u_{ij} > Q_{ij} > 0$:

$$\dot{Q}_{ij} = \alpha_{ij}(Q_{ij})\hat{\rho}_j(Q) - c_{ij}(Q) - \hat{\pi}_i(Q). \tag{14}$$

When Q_{ij} falls on the boundary, that is, is at level zero or is at its upper bound, then we have that

$$\dot{Q}_{ij} = \min\{u_{ij}, \max\{0, \alpha_{ij}(Q_{ij})\hat{\rho}_j(Q) - c_{ij}(Q) - \hat{\pi}_i(Q)\}\}. \tag{15}$$

We can write (14) and (15) succinctly as:

$$\dot{Q}_{ij} = \begin{cases} \alpha_{ij}(Q_{ij})\hat{\rho}_j(Q) - c_{ij}(Q) - \hat{\pi}_i(Q), & \text{if } 0 < Q_{ij} < u_{ij} \\ \min\{u_{ij}, \max\{0, \alpha_{ij}(Q_{ij})\hat{\rho}_j(Q) - c_{ij}(Q) - \hat{\pi}_i(Q)\}\}, & \text{otherwise.} \end{cases} \tag{16}$$

Applying (16) to all supply and demand market pairs (i, j); $i = 1, \ldots, m$; $j = 1, \ldots, n$, yields the ordinary differential equation (ODE) for the adjustment processes of the shipments, in vector form:

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \tag{17}$$

where, since $\mathcal{K}$ is a convex polyhedron, according to [11], $\Pi_{\mathcal{K}}(X, -F(X))$ is the projection, with respect to $\mathcal{K}$, of the vector $-F(X)$ at X defined as

$$\Pi_{\mathcal{K}}(X, -F(X)) = \lim_{\delta \to 0} \frac{P_{\mathcal{K}}(X - \delta F(X)) - X}{\delta} \tag{18}$$

with $P_{\mathcal{K}}$ denoting the projection map:

$$P_{\mathcal{K}}(X) = \text{argmin}_{z \in \mathcal{K}} \|X - z\|, \tag{19}$$

and where $\|\cdot\| = \langle x, x \rangle$. Observe that (17) has a discontinuous right-hand side, which is in contrast to classical dynamical systems [13].

We now interpret the ODE (17) for the spatial model with flow-dependent arc multipliers. Note that the ODE (17) ensures that the product shipments are always nonnegative and that they never go above the respective upper bounds. ODE (17) keeps the interpretation that if X at time t lies in the interior of $\mathcal{K}$, then the rate at which X changes is greatest when the vector field $-F(X)$ is greatest. And, when the vector field $-F(X)$ pushes X to the boundary of the feasible set $\mathcal{K}$, then the projection $\Pi_{\mathcal{K}}$ ensures that X stays within $\mathcal{K}$.

The authors in [11] constructed the fundamental theory with regards to existence and uniqueness of projected dynamical systems as defined by (17). We recall the subsequent theorem from [11].

Theorem 2 *X^* solves the variational inequality problem (12), equivalently, (7), if and only if it is a stationary point of the ODE (17), that is,*

$$\dot{X} = 0 = \Pi_{\mathcal{K}}(X^*, -F(X^*)). \tag{20}$$

This theorem states that the necessary and sufficient condition for a product shipment pattern $X^* = Q^*$ to be a spatial price equilibrium, according to Definition 1, is that $X^* = Q^*$ is a stationary point of the adjustment process defined by ODE (17), that is, X^* is the point at which $\dot{X} = 0$. We refer to (17) as PDS $(F, \mathcal{K})$.

Lipschitz continuity of $F(X)$ [11, 37] ensures the existence of a unique solution to (17). Hence, $X^0(t)$ solves the initial value problem (IVP)

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \quad X(0) = X^0, \tag{21}$$

with $X^0(0) = X^0$.

Following [37], the following theorem is immediate.

Theorem 3 Stability analysis (i) *If $F(X)$ is monotone, then every spatial price equilibrium with flow-dependent arc multipliers, X^*, is a global monotone attractor for the the PDS$(F, \mathcal{K})$. If $F(X)$ is locally monotone at X^*, then it is a monotone attractor for the PDS$(F, \mathcal{K})$.*

(ii) *If $F(X)$ is strictly monotone, then the unique spatial price equilibrium with flow-dependent arc multipliers is a strictly global monotone attractor for the PDS$(F, \mathcal{K})$. If $F(X)$ is locally strictly monotone at X^*, then it is a strictly monotone attractor for the PDS$(F, \mathcal{K})$.*

(iii) *If $F(X)$ is strongly monotone, then the unique spatial price equilibrium with flow-dependent arc multipliers is globally exponentially stable for the PDS$(F, \mathcal{K})$. If $F(X)$ is locally strongly monotone at X^*, then X^* is exponentially stable.*

2.3 Flow-dependent arc multipliers and applications

In this section, we highlight how the modeling of certain applications is enriched through the use of flow-dependent arc multipliers within a spatial price equilibrium network context. We first consider applications in agriculture, specifically, that of fresh produce. Note that fresh produce, as in the case of fruits and vegetables, even in the best of circumstances, deteriorates and, hence, it is a perishable product. Both [3, 54] have emphasized this point, with the former constructing a perishable product supply chain network model for fresh produce, which is an example of imperfect competition, and the latter presenting formulae, based on temperature, chemistry, and time, to map the quality deterioration of fresh produce on supply chain pathways from origin nodes to destination nodes. Here, we consider flow-dependent arc multipliers to capture fresh produce perishability of the following form:

$$\alpha_{ij}(Q_{ij}) = \gamma_{ij} - \beta_{ij} Q_{ij}, \quad \forall i, j, \tag{22}$$

where both the γ_{ij}s and the β_{ij}s are positive parameters. Note that since the flow Q_{ij} would be multiplied by such an $\alpha_{ij}(Q_{ij})$ this would correspond to a nonlinear function, given by: $\gamma_{ij}Q_{ij} - \beta_{ij}Q_{ij}^2$, which is a concave function. Such a function is very reasonable since the greater the volume of fresh produce, the greater the likelihood of the product perishing in the transportation process.

On the other hand, as mentioned earlier, in the case of financial applications, a greater volume of a trade may result in a higher volume of financial flows and, therefore, in such an application, we can posit arc multiplier functions $\alpha_{ij}(Q_{ij})$ of the following form:

$$\alpha_{ij}(Q_{ij}) = \gamma_{ij} + \beta_{ij}Q_{ij}, \quad \forall i, j. \tag{23}$$

In this case, we, again, have a nonlinear function to represent the financial flow that originates at supply market i and is destined for demand market j, of the form: $\gamma_{ij}Q_{ij} + \beta_{ij}Q_{ij}^2$, but this function, in contrast to the one for perishable food, is convex. In financial applications, the $c_{ij}(Q)$s correspond to unit transaction costs.

Remark

We emphasize that the spatial equilibrium conditions (4) can also handle certain trade policies. For example, the upper bounds can correspond to quotas, which are increasingly being instituted in the Covid-19 pandemic, for example, to reduce exports of certain foods (cf. [26, 28]). Also, it is important to recognize that the unit transportation costs can serve the role of unit transaction costs, which can include unit tariffs (see, e.g., [30]).

3 The algorithm and numerical examples

The projected dynamical system (17) may be interpreted as a continuous-time adjustment process in product shipments. Nevertheless, for computational purposes, a discrete-time algorithm, which serves as an approximation to the continuous-time trajectories is essential. We now present the algorithm, the Euler method, along with convergence and results, followed by a series of numerical examples, which are solved using the Euler method.

3.1 The Euler method

We recall the Euler method, which is induced by the general iterative scheme of [11]. Specifically, iteration τ of the Euler method (see also [37]) is given by:

$$X^{\tau+1} = P_{\mathcal{K}}(X^\tau - a_\tau F(X^\tau)), \tag{24}$$

where recall that $P_{\mathcal{K}}$ is the projection on the feasible set $\mathcal{K}$ and F is the function that enters the variational inequality problem (7).

As established in [11, 37], for convergence of the general iterative scheme, which induces the Euler method, the sequence $\{a_\tau\}$ must satisfy: $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$,

$a_\tau \to 0$, as $\tau \to \infty$. Conditions for convergence of this algorithm in the context of other network models as well as additional computational work can be found in [4, 25, 37] and in [8, 31].

Explicit formulae for the Euler method applied to the spatial price equilibrium model with flow-dependent arc multipliers
The algorithm yields explicit formulae for the product shipments at each iteration. Specifically, we have the following closed form expression for the product shipments $i = 1, \ldots, m$; $j = 1, \ldots, n$:

$$Q_{ij}^{\tau+1} = \min\{u_{ij}, \max\{0, Q_{ij}^{\tau} + a_\tau(\alpha_{ij}(Q_{ij}^{\tau})\hat{\rho}_j(Q^\tau) - c_{ij}(Q^\tau) - \hat{\pi}_i(Q^\tau))\}\}. \quad (25)$$

Expression (25) has an interpretation as a discrete-time adjustment process.

We now provide the convergence result. The proof is direct from Theorem 6.10 in [37].

Theorem 4 *Convergence If in the spatial price equilibrium problem with flow-dependent arc multipliers the $F(X)$ is strictly monotone at any equilibrium pattern and F is Lipschitz continuous, that is,*

$$\|F(X^1) - F(X^2)\| \leq L\|X^1 - X^2\|, \quad \forall X^1, X^2 \in \mathcal{K}, \quad (26)$$

where L is a positive number known as the Lipschitz constant, then there exists a unique equilibrium shipment pattern $Q^* \in \mathcal{K}$ and any sequence generated by the Euler method as given by (24), where $\{a_\tau\}$ satisfies $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, as $\tau \to \infty$ converges to Q^*.

Note that $F(X)$ is strictly monotone if its Jacobian is positive-definite over the feasible set $\mathcal{K}$. This will hold, for example, if $\left[\frac{\partial \hat{\pi}}{\partial Q}\right]$ is positive-semidefinite over $\mathcal{K}$, $\left[\frac{\partial c}{\partial Q}\right]$ is positive-definite over $\mathcal{K}$, and $\left[\frac{\partial \tilde{\rho}}{\partial Q}\right]$ is negative-semidefinite over $\mathcal{K}$, where we define $\tilde{\rho}_{ij}(Q) \equiv \alpha_{ij}(Q_{ij})\hat{\rho}_j(Q)$, $\forall i, j$, and $\tilde{\rho}(Q)$ is the vector with mn components, with the (i, j) component equal to $\alpha_{ij}(Q_{ij})\hat{\rho}_j(Q)$.

3.2 Numerical examples

Examples 1, 2, 3, 4, 5, and 6 consist of two supply markets and three demand markets, as illustrated in Fig. 2. Example 1 is the baseline and, hence, it has arc multipliers that are fixed. The function and arc multiplier data for Example 1 are as in Example 3.4 in [28].

The supply price functions are:

$$\pi_1(s) = 5s_1 + s_2 + 2, \quad \pi_2(s) = 2s_2 + 1.5s_1 + 1.5.$$

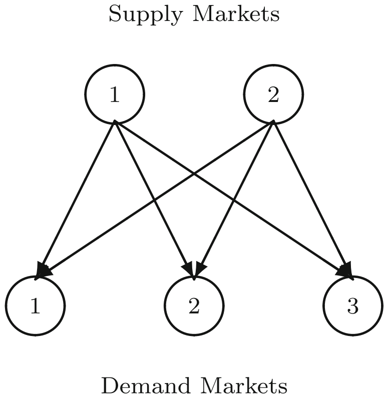

Fig. 2 The network structure of the spatial price equilibrium examples 1, 2, 3, 4, 5, and 6

The unit transportation cost functions are:

$$c_{11}(Q) = .01Q_{11}^2 + Q_{11} + 10, \quad c_{12}(Q) = .02Q_{12}^2 + 2Q_{12} + 13.5,$$
$$c_{13}(Q) = .02Q_{13}^2 + 2Q_{13} + 14.5,$$
$$c_{21}(Q) = .03Q_{21}^2 + 3Q_{21} + 14.25, \quad c_{22}(Q) = .02Q_{22}^2 + 2Q_{22} + 11.5,$$
$$c_{23}(Q) = .03Q_{23}^2 + 3Q_{23} + 15.$$

The demand price functions are:

$$\rho_1(d) = -2d_1 - 1.5d_2 + 380, \quad \rho_2(d) = -4d_2 - d_1 + 410,$$
$$\rho_3(d) = -3d_3 - d_2 + 350.$$

The arc multipliers for Example 1 are:

$$\alpha_{11} = .98, \quad \alpha_{12} = .95, \quad \alpha_{13} = .97, \quad \alpha_{21} = .95, \quad \alpha_{22} = .99, \quad \alpha_{23} = .97.$$

All the upper bounds are set to 50.

The Euler method converges in 154 iterations and yields the equilibrium product shipments, along with the incurred supply prices, unit transportation costs, arc multipliers (which in this example are fixed), and the demand prices reported in Table 2. The equilibrium product supplies are: $s_1^* = 31.31$ and $s_2^* = 60.32$, and the equilibrium demands are: $d_1^* = 36.71$, $d_2^* = 30.25$, and $d_3^* = 22.30$.

Example 2 has the same data as Example 1 except that now we modified the fixed arc multipliers to be flow-dependent as follows. Each of the fixed arc multipliers is expanded with the term $-.01 \times Q_{ij}$ for all i, j. Therefore, the arc multipliers are now of the form (22). For example, we now have: $\alpha_{11}(Q_{11}) = .98 - .01Q_{11}$, $\alpha_{12}(Q_{12}) = .95 - .01Q_{12}$, and so on. Hence, Example 2 corresponds to the case in which the functions on the arcs are concave, which is associated with losses as in the deterioration/perishability of fresh produce agricultural products. The Euler method converges in 136 iterations to the equilibrium solution reported in Table 2.

Table 2 Equilibrium Solution for Examples 1, 2, 3, 4, 5, and 6

Product flows	*Ex*.1	Ex. 2	Ex. 3	Ex. 4	Ex. 5	Ex. 6
Q_{11}^*	22.17	15.63	33.66	10.15	10.00	7.47
Q_{12}^*	3.52	8.98	0.00	0.00	11.22	7.24
Q_{13}^*	5.62	7.03	0.00	25.10	8.44	6.86
Q_{21}^*	15.77	15.54	7.96	24.34	10.00	7.67
Q_{22}^*	27.18	22.12	29.81	32.17	23.58	8.36
Q_{23}^*	17.37	14.99	23.13	0.00	15.61	7.73
Supply prices	*Ex*.1	Ex. 2	Ex. 3	Ex. 4	Ex. 5	Ex. 6
$\pi_1(s^*)$	218.88	212.84	231.21	234.77	199.47	133.61
$\pi_2(s^*)$	169.11	154.25	173.78	167.39	144.36	81.37
Transportation costs	*Ex*.1	Ex. 2	Ex. 3	Ex. 4	Ex. 5	Ex. 6
$c_{11}(Q^*)$	37.09	28.07	55.00	21.18	21.00	18.03
$c_{12}(Q^*)$	20.78	33.07	13.50	13.50	38.45	29.03
$c_{13}(Q^*)$	26.38	29.55	14.50	77.30	32.81	29.15
$c_{21}(Q^*)$	79.03	78.13	50.03	115.04	57.25	49.01
$c_{22}(Q^*)$	80.64	65.52	88.88	96.53	69.79	29.63
$c_{23}(Q^*)$	76.15	66.70	100.43	15.00	69.12	39.98
Arc multipliers	*Ex*.1	Ex. 2	Ex. 3	Ex. 4	Ex. 5	Ex. 6
$\alpha_{11}(Q_{11}^*)$	.98	.82	1.32	1.08	.88	.42
$\alpha_{12}(Q_{12}^*)$	.95	.86	.95	.95	.84	.43
$\alpha_{13}(Q_{13}^*)$	.97	.90	.97	2.23	.89	.50
$\alpha_{21}(Q_{21}^*)$	.95	.79	1.03	1.19	.85	.36
$\alpha_{22}(Q_{22}^*)$	.99	.77	1.29	1.31	.75	.29
$\alpha_{23}(Q_{23}^*)$	.97	.82	1.20	.97	.81	.37
Demand prices	*Ex*.1	Ex. 2	Ex. 3	Ex. 4	Ex. 5	Ex. 6
$\rho_1(d^*)$	261.20	292.46	217.38	236.66	304.63	359.88
$\rho_2(d^*)$	252.28	285.86	203.92	201.21	283.97	382.02
$\rho_3(d^*)$	252.85	269.42	228.26	140.26	262.29	325.56

The equilibrium supplies are now: $s_1^* = 31.64$ and $s_2^* = 52.65$, and the equilibrium demands are: $d_1^* = 25.22$, $d_2^* = 24.73$, and $d_3^* = 18.62$.

It is interesting to see the value of the flow-dependent arc multipliers at the equilibrium—all have lower values than their fixed counterparts in Example 1. The supply market prices decrease at both supply markets, whereas the demand market prices increase at all the demand markets. The demands are now lower at each demand market, as compared to the respective value in Example 1. Example 2 illustrates that it is important to invest in preserving the fresh produce in the transportation processes, so that it does not perish, since this also results in higher prices to consumers at the demand markets.

Example 3, in turn, has the same data as Example 2 but now we change the flow-dependent arc multipliers so that, rather than a minus sign in front of each $.01Q_{ij}$, there is now a plus sign. This allows one to investigate, for example, the impact of enhanced investments. The arc multipliers in Example 3 are of the form (23). Thus, in Example 3, $\alpha_{11}(Q_{11}) = .98 + .01Q_{11}$, $\alpha_{12}(Q_{12}) = .95 + .01Q_{12}$, and so on.

The Euler method now converges in 184 iterations to the equilibrium pattern in Table 2. It is worth comparing these results to those in Examples 1 and 2. Observe that, in contrast to Examples 1 and 2, now the supply market prices increase for both supply markets, whereas the demand market prices decrease for all three demand markets. We also now encounter flows of zero value, and these are associated with Q^*_{12} and Q^*_{13}. The equilibrium supplies are now: $s^*_1 = 33.66$ and $s^*_2 = 60.8$, and the equilibrium demands are: $d^*_1 = 52.52$, $d^*_2 = 38.39$, and $d^*_3 = 27.28$. Note that, in Example 3, both $\alpha_{12}(Q^*_{12})$ and $\alpha_{13}(Q^*_{13})$ have values less than 1 at the equilibrium, and these correspond to losses, whereas the other alpha values at the equilibrium are all greater than 1, representing gains. Hence, it is reasonable that both Q^*_{12} and Q^*_{13} are equal to 0.00. This example shows that one may in the same generalized network problem have instances in which some links have gains, whereas others have losses.

In Example 4, we raise the β_{ij} (see (22)) for arcs (1, 2) and (1, 3) from .01 to .05 to see the impact of a higher return associated with the investments since the equilibrium flows on these links are zero in Example 3. All other data in Example 4 remain as in Example 3. The equilibrium is achieved in 201 iterations of the Euler method with the solution reported in Table 2. Q^*_{12} remains at 0.00, whereas Q^*_{13} is now positive. Q^*_{23} now drops to 0.00. The equilibrium supplies in Example 4 are: $s^*_1 = 35.25$ and $s^*_2 = 56.51$, and the equilibrium demands are: $d^*_1 = 40.03$, $d^*_2 = 42.19$, and $d^*_3 = 55.85$.

In Example 5, we return to Example 2. Example 5 has the identical data as Example 2 but now we investigate the impacts of the tightening of upper bounds. Specifically, we consider a disruption associated with shipments from the two supply markets to the first demand market. The corresponding upper bounds are now: $u_{11} = u_{21} = 10.00$. The Euler method converges to the equilibrium solution in Table 2. Both Q^*_{11} and Q^*_{21} are now at their upper bounds of 10.00. The computed equilibrium supplies for Example 5 are: $s^*_1 = 29.66$ and $s^*_2 = 49.19$, and the equilibrium demands are: $d^*_1 = 17.30$, $d^*_2 = 27.18$, and $d^*_3 = 20.18$. The supply market prices drop at both supply markets and the supplies of the product decrease at both supply markets. The demand market price increases at the first demand market. It is interesting to see, from Table 2, that the arc multipliers for both (1, 1) and (2, 1) increase, as compared to their respective values in Example 2. This makes sense, since with less of the perishable agricultural product to handle/transport, the preservation of the product is enhanced.

Example 6 has the same data as Example 5, except that we now make each of the flow-dependent arc multiplier functions nonlinear in that for each $\alpha_{ij}(Q_{ij})$ function as in Example 5, we raise the Q_{ij} to the second power. Hence, we now have that: $\alpha_{11}(Q_{11}) = .98 - .01Q^2_{11}$, $\alpha_{12}(Q_{ij}) = .95 - .01Q^2_{12}$, and so on. The Euler method now converges in 726 iterations to the equilibrium solution in Table 2. The computed equilibrium supplies for Example 6 are: $s^*_1 = 21.57$ and $s^*_2 = 23.76$, and the equilibrium demands are: $d_1^* = 5.93$, $d_2^* = 5.51$, and $d_3^* = 6.31$. The values of the flow-dependent arc multipliers at the equilibrium are now significantly lower for each arc, as compared to the corresponding values in Example 5. The equilibrium product shipments decline, as well as the product supply prices but the demand market prices increase at all the demand markets. There are no product shipments at a value of 0.00, and none of the product shipments are at the upper bounds. This numerical example further reinforces the importance of preserving perishable products in the transportation process since an increase in perishability affects consumers negatively in terms of both product quantity and product price. This example is also interesting from a methodological standpoint since each arc multiplier function is itself a nonlinear function of flow. Note that this example violates our assumption that the multiplier functions should remain positive over the feasible set $\mathcal{K}^2$ but a reasonable answer is, nevertheless, attained.

The above examples are stylized but, nevertheless, illustrate the utility of having a rigorous theoretical and computational framework for spatial price networks with flow-dependent arc multipliers.

4 Summary and conclusions

Generalized networks are applicable to important problems in many disciplines. Nevertheless, the investigation of extensions of fixed arc multipliers, as in the case of flow-dependent ones, has been only minimally explored and studied. In this paper, we construct some of the very first market equilibrium problems, in the form of static and dynamic spatial price equilibrium networks, with flow-dependent arc multipliers. These advances expand the scope of possible applications, since spatial price equilibrium problems have had wide use in agriculture, economics, and even in finance. The arc multipliers that we consider address gains and/or losses, with the latter being of special relevance to perishable food products, and the former, to financial investments.

We utilize the methodology of variational inequality theory for the formulation of the governing equilibrium conditions and projected dynamical systems theory for the construction of the dynamic counterpart. For both realizations of the spatial network model, we provide qualitative results. The variational inequality that we derive is in product shipments only, which allows for an easy to implement algorithm, with nice features for computations. Furthermore, the algorithm provides a time-discretization of the continuous-time adjustment processes associated with the product shipments from the supply markets to the demand markets. The algorithm is then implemented and applied to compute solutions to numerical spatial price network examples with flow-dependent arc multipliers, inspired by perishable products as well as financial applications.

Possible research in the future may include the construction of a variety of network equilibrium models with flow-dependent arc multipliers.

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