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ORIGINAL PAPER

Trigonometric approximation of the Max-Cut polytope is star-like

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Abstract
The Max-Cut polytope appears in the formulation of many difficult combinatorial optimization problems. These problems can also be formulated as optimization problems over the so-called *trigonometric approximation* which possesses an algorithmically accessible description but is not convex. Hirschfeld conjectured that this trigonometric approximation is star-like. In this article, we provide a proof of this conjecture.

Keywords Max-Cut polytope · Trigonometric approximation

Mathematics Subject Classification 90C20 · 90C27

1 Introduction

A common problem in combinatorial optimization is the maximization of a quadratic form over $\{-1, 1\}^n$

$$\max_{x\in\{-1,1\}^n} x^T Ax = \max_{\substack{X=xx^T \\ x\in\{-1,1\}^n}} \langle A, X\rangle \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product on real symmetric matrices of size n.

The decision problem associated to this optimization problem is NP-complete. Indeed the Max-Cut problem, one of Karp's 21 NP-complete problems, can be reduced in polynomial time to the maximization of a quadratic form over $\{-1, 1\}^n$ [3]. The reformulation in the form of (1) of several common hard combinatorial optimization problems such as vertex cover, knapsack, traveling salesman, etc, can be found in [4].

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Consider the set

$$\mathcal{SR} = \{X \succeq 0 \mid \operatorname{diag} X = 1\}$$

in the space of real symmetric $n \times n$ matrices, where $X \succeq 0$ means that X is a positive semidefinite matrix. It serves as a simple and convex outer approximation of the *Max-Cut polytope*

$$\mathcal{MC} = \operatorname{conv}\{X \in \mathcal{SR} \mid \operatorname{rk} X = 1\},$$

where conv denotes the convex envelope and rk X denotes the rank of X.

Note that $\{X \in \mathcal{SR} \mid \operatorname{rk} X = 1\} = \{X \mid \exists x \in \{-1, 1\}^n,\ X = xx^T\}$. Indeed a positive semidefinite matrix X has rank 1 if and only if there exists a nonzero vector x such that $X = xx^T$. Then the condition diag $X = 1$ implies that $x_i^2 = 1$ for every $i \in \{1, ..., n\}$, i.e., $x_i = \pm 1$, and conversely.

The maximal value of a linear functional $\langle A, .\rangle$ over a set E does not change if the set E is replaced by its convex envelope conv E. Therefore

$$\max_{\substack{X = xx^T \\ x \in \{-1,1\}^n}} \langle A, X\rangle = \max_{X \in \mathcal{MC}} \langle A, X\rangle.$$

However, the Max-Cut polytope is a difficult polytope. Indeed, "due to the NP-completeness of the max-cut problem, it follows from a result of Karp and Papadimitriou [1982] that there exists no polynomially concise linear description of $\mathcal{MC}$ unless NP = co-NP" [1, Section 4.4]. A good review of results on the Max-Cut polytope can be found in [1].

Maximizing $\langle A, X\rangle$ over $\mathcal{SR}$ instead of $\mathcal{MC}$ for $A \succeq 0$ approximates the exact solution of the problem with relative accuracy $\mu = \frac{\pi}{2} - 1$ [5]:

$$\frac{2}{\pi} \max_{X \in \mathcal{SR}} \langle A, X\rangle \le \max_{X \in \mathcal{MC}} \langle A, X\rangle \le \max_{X \in \mathcal{SR}} \langle A, X\rangle.$$

Define a function $f : [-1, 1] \to [-1, 1]$ by $f(x) = \frac{2}{\pi} \arcsin x$. Let $\mathbf{f}$ be the operator which applies f element-wise to a matrix. A non-convex inner approximation of $\mathcal{MC}$ is given by the *trigonometric approximation* [3, Section 4]

$$\mathcal{TA} = \{\mathbf{f}(X) \mid X \in \mathcal{SR}\}.$$

Nesterov proved in [5, Theorem 2.5] that

$$\max_{X \in \mathcal{TA}} \langle A, X\rangle = \max_{X \in \mathcal{MC}} \langle A, X\rangle.$$

Although not convex, $\mathcal{TA}$ is simpler than $\mathcal{MC}$ in the sense that checking whether a matrix X is in $\mathcal{TA}$ can be done in polynomial time by computing $\mathbf{f}^{-1}(X)$ and checking

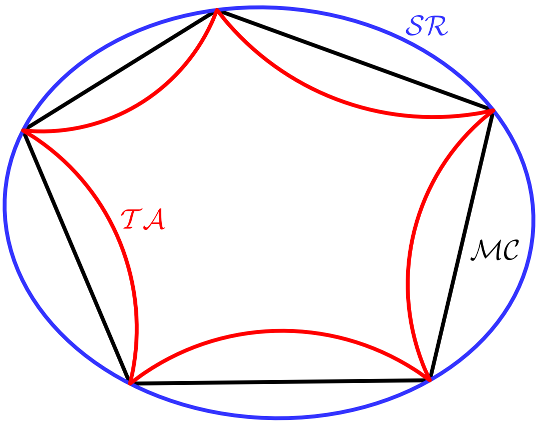

Fig. 1 $\mathcal{TA}$, $\mathcal{MC}$ and $\mathcal{SR}$

whether $\mathbf{f}^{-1}(X)$ is in $\mathcal{SR}$. This allows to reformulate the initial difficult problem (1) as an optimization problem over the algorithmically accessible set $\mathcal{TA}$. The complexity of the problem in this form arises solely from the non-convexity of this set.

Hirschfeld studied $\mathcal{TA}$ in [3, Section 4]. In this work, we prove that $\mathcal{TA}$ possesses an additional beneficial property. Namely, we prove the conjecture of Hirschfeld that it is starlike, i.e., for every $X \in \mathcal{TA}$ and every $\lambda \in [0, 1]$, the convex combination $\lambda X + (1-\lambda)I$ of X and the central point I, the identity matrix, is in $\mathcal{TA}$ (Fig. 1).

Although this result does not directly lead to a better algorithm, it has the potential to do so because we know something about $\mathcal{TA}$ that we did not know before (a review of the properties and applications of starshaped sets can be found in [6] and [2]).

2 Hirschfeld's conjecture

In this section, we describe the conjecture and related results which have been obtained by Hirschfeld in his thesis [3, Section 4.3].

In order to show that $\mathcal{TA}$ is star-like, one has to prove that

$$\forall X \in \mathcal{SR},\ \forall \lambda \in [0, 1],\ \mathbf{f}^{-1}(\lambda \mathbf{f} X + (1-\lambda)I) \in \mathcal{SR}.$$

Note that the operator acting on X is nearly an element-wise one, defined by the function

$$\begin{aligned} f_\lambda : [-1, 1] &\longrightarrow [-1, 1] \\ x &\longmapsto f^{-1}(\lambda f(x)) = \sin(\lambda \arcsin x) \end{aligned}$$

acting on the off-diagonal elements, while the diagonal elements remain equal to 1, contrary to $f_\lambda(1) = f^{-1}(\lambda) = \sin \frac{\pi\lambda}{2}$. Thus one has to show that

$$\forall X \in \mathcal{SR}, \forall \lambda \in [0, 1],\ \mathbf{f}_\lambda(X) + \left(1 - \sin \frac{\pi\lambda}{2}\right) I \succeq 0.$$

A sufficient condition is that $\mathbf{f}_\lambda(X) \succeq 0$ for all $X \in \mathcal{SR}$ and for all $\lambda \in [0, 1]$, i.e., the element-wise operator $\mathbf{f}_\lambda$ is positivity preserving. Hirschfeld conjectured that this sufficient condition is verified [3, Conjecture 4.9].

Lemma 1

$$\forall X \in \mathcal{SR}, \forall \lambda \in [0, 1],\ \mathbf{f}_\lambda(X) \succeq 0$$

A sufficient (and necessary) condition for an operator of this type to be positivity preserving is that all of the Taylor coefficients of f_λ are nonnegative [7].

Lemma 1 proves the following theorem.

Theorem 1 *$\mathcal{TA}$ is star-like.*

3 Proof of the conjecture

In this section, we prove Lemma 1.

Proof Let $\lambda \in [0, 1]$ and write f_λ as a power series

$$f_\lambda(x) = \sum_{n\in\mathbb{N}} a_n(\lambda) x^n.$$

The first two derivatives of f_λ are given by

$$f_\lambda'(x) = \frac{\lambda}{\sqrt{1-x^2}} \cos(\lambda \arcsin x)$$

and

$$f_\lambda''(x) = \frac{x}{1-x^2} \frac{\lambda \cos(\lambda \arcsin x)}{\sqrt{1-x^2}} - \frac{\lambda^2}{1-x^2} \sin(\lambda \arcsin x).$$

Hence f_λ is a solution on $(-1, 1)$ of the differential equation

$$(1-x^2) f_\lambda'' - x f_\lambda' + \lambda^2 f_\lambda = 0.$$

Therefore, the Taylor coefficients of f_λ verify the recurrence relation

$$(n+2)(n+1)a_{n+2}(\lambda) - n(n-1)a_n(\lambda) - n a_n(\lambda) + \lambda^2 a_n(\lambda) = 0$$

which can be re-expressed as

$$a_{n+2}(\lambda) = \frac{n^2 - \lambda^2}{(n+2)(n+1)} a_n(\lambda) \tag{2}$$

with initial conditions

$$\begin{cases} a_0(\lambda) = 0 \\ a_1(\lambda) = \lambda \end{cases}.$$

Given that $\lambda \in [0, 1]$, a trivial induction shows that

$$\forall n \in \mathbb{N}, \quad a_n(\lambda) \geq 0.$$

□

Recursion (2) also proves that the roots of the polynomials $a_n(\lambda)$ are located at $0, \pm 1, ..., \pm n$ and are given by the polynomials $\widetilde{P}_n(\lambda)$ [3, eq. 4.23], as also conjectured by Hirschfeld.

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