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ORIGINAL PAPER

Robust optimality, duality and saddle points for multiobjective fractional semi-infinite optimization with uncertain data

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Abstract
This paper is devoted to the investigation of a class of uncertain multiobjective fractional semi-infinite optimization problems (UMFP, for brevity). We first obtain, by combining robust optimization and scalarization methodologies, necessary and sufficient optimality conditions for robust approximate weakly efficient solutions of (UMFP). Then, we introduce a Mixed type approximate dual problem for (UMFP) and investigate their robust approximate duality relationships. Moreover, we obtain some robust approximate weak saddle point theorems for an uncertain multiobjective Lagrangian function related to (UMFP).

Keywords Robust optimization · Approximate efficient solutions · Multiobjective semi-infinite optimization

1 Introduction

Fractional optimization is an important model of nonlinear optimization problems, where the objective function is a ratio of two functions. It is an active research topic

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over the last several decades due to a wide range of applications in different fields of engineering and economics. There are many papers devoted to the investigation of fractional optimization problems from several different perspectives; see, e.g., [1–14] and the references therein.

As we know, many optimization problems that arise in applications involve uncertainties due to measurement and/or manufacturing errors, imprecise information, fluctuations or disturbances. Consequently, a great deal of attention has been focused on optimization problems with uncertain data. Robust optimization [15,16] is one of the most effective methods to solve optimization problems with uncertain data. The aim of robust optimization is to find the worst-case solution, which is immunized against the data uncertainty, to the optimization problem. Recently, many researchers have been attracted to work on the theory and applications of uncertain optimization problem in terms of robust optimization methodology, see [17–29] and the references therein. It appears that there are few papers devoting to the investigation of uncertain fractional optimization problems via robust optimization. More precisely, results on optimality conditions and duality theorems of robust approximate optimal solutions for an uncertain fractional optimization problem are considered in [19]. Robust strong duality for an uncertain fractional optimization problem and its uncertain Wolfe dual optimization problem is obtained in [20] in the framework of robust optimization. Some characterizations of robust optimal solution sets for a class of fractional optimization problems with uncertainty appearing in both the objective and constraints are obtained in [23]. Optimality conditions and duality theorems for an uncertain minimax convex-concave fractional programming problem are established in [25] under a robust type subdifferential constraint qualification condition. Some new results of robust approximate optimal solutions for fractional semi-infinite optimization problems with uncertain data are obtained in [26] under a robust type constraint qualification condition.

These papers mentioned above are mainly devoted to the investigation of single-objective fractional optimization problems with uncertain data. However, many practical optimization models have various goals due to many decision makers, and they have different optimization criteria. Therefore, it is necessary and interesting to deal with multiobjective fractional optimization problems. In the last few decades, multiobjective fractional optimization problems have been studied by many scholars without taking into account data uncertainty. For example, Long et al. [6,7] obtained optimality conditions, duality and saddle-point results for nondifferentiable multiobjective fractional optimization problems under some generalized convexity assumptions. Verma [8] established some parametric sufficient efficiency conditions for multiobjective fractional optimization problems in terms of generalized invexity assumptions. Using first- and second-order approximations as generalized derivatives, Khanh and Tung [10] derived optimality conditions for nonsmooth multiobjective fractional optimization problems. Chuong [12] established optimality conditions and duality theorems for nondifferentiable fractional semi-infinite multiobjective optimization problems. More recently, some results on optimality and duality for a nonsmooth and nonconvex multiobjective fractional programming problem are obtained in [14] in terms of contingent derivatives.

However, in contrast to the deterministic case, it appears that there exist few papers in the literature devoting to the study of multiobjective fractional optimization problems with uncertain data, see [23]. This is the main motivation to investigate a class of uncertain multiobjective fractional semi-infinite optimization problems (UMFP, for brevity). Our main concern is to provide some new characterizations of robust approximate weakly efficient solutions for (UMFP). Our contributions can be more specifically stated as follows. By using a robust type constraint qualification condition, we first establish necessary and sufficient optimality conditions for robust approximate weakly efficient solutions for (UMFP). The obtained results can be regarded as the generalizations of the optimality conditions for approximate weakly efficient solutions of deterministic multiobjective fractional optimization problems. Based on the obtained optimality conditions, we give a Mixed type robust approximate dual problem of (UMFP). Then, we investigate the weak and strong duality relations between (RUMFP) and the optimistic counterpart of the uncertain dual problem of (UMFP). Moreover, we introduce an uncertain approximate multiobjective Lagrangian function related to (UMFP) and obtain a nonsmooth robust saddle point theorem.

The rest of the paper is organized as follows. Section 2 contains some basic definitions and auxiliary results. In Sect. 3, we obtain necessary and sufficient optimality conditions for robust approximate weakly efficient solutions of (UMFP). In Sect. 4, we first introduce a Mixed type robust dual problems for (UMFP), and then discuss the robust approximate duality properties. We also deal with robust approximate saddle point of the uncertain approximate multiobjective Lagrangian function for (UMFP).

2 Preliminaries

In this section, we give some basic notations and preliminary results from [30], which will be used in the sequel. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space equipped with the usual Euclidean norm $\|\cdot\|$. The nonnegative orthant of $\mathbb{R}^n$ is defined by $\mathbb{R}^n_+ := \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_i \geq 0\}$. The inner product in $\mathbb{R}^n$ is denoted by $\langle x, y\rangle$ for any $x, y \in \mathbb{R}^n$. For a set D in $\mathbb{R}^n$, the closure and the convex hull of D are denoted by cl$D$ and co$D$, respectively. The indicator function $\delta_D : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of D is defined by

$$\delta_D = \begin{cases} 0, & \text{if } x \in D, \\ +\infty, & \text{if } x \notin D. \end{cases}$$

Let T be a nonempty infinite index set, $\mathbb{R}^{(T)}$ be the following linear space, which has been used for semi-infinite programming [31],

$$\mathbb{R}^{(T)} := \{\eta = (\eta_t)_{t\in T} \mid \eta_t = 0 \text{ for all } t \in T \text{ except for finitely many } \eta_t \neq 0\}.$$

The nonnegative cone of $\mathbb{R}^{(T)}$ is defined by

$$\mathbb{R}^{(T)}_+ := \{\eta \in \mathbb{R}^{(T)} \mid \eta_t \geq 0, \forall t \in T\}.$$

Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended real value function. The effective domain and the epigraph of φ are defined respectively by

$$\text{dom}\varphi := \{x \in \mathbb{R}^n \mid \varphi(x) < +\infty\} \text{ and } \text{epi}\varphi := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid \varphi(x) \leq r\}.$$

The Frenchel conjugate function φ^* of φ is defined by

$$\varphi^*(x^*) := \sup_{x \in \mathbb{R}^n} \{\langle x^*, x\rangle - \varphi(x)\}.$$

The function φ is said to be proper if its effective domain is nonempty, and φ is said to be convex if epiφ is a convex set, or equivalently, $\varphi(\alpha x+(1-\alpha)y) \leq \alpha\varphi(x)+(1-\alpha)\varphi(y)$, for any $x, y \in \mathbb{R}^n$ and $\alpha \in [0, 1]$. The function φ is said to be concave if $-\varphi$ is convex. Moreover, φ is said to be lower semicontinuous if epiφ is closed. The subdifferential of φ at $\bar{x} \in \text{dom}\varphi$ is defined by

$$\partial\varphi(\bar{x}) := \{x^* \in \mathbb{R}^n \mid \varphi(x) - \varphi(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle, \forall x \in \mathbb{R}^n\}.$$

For any $\varepsilon \in \mathbb{R}_+$, the ε-subdifferential of φ at $\bar{x} \in \text{dom}\varphi$ is the convex set given by

$$\partial_\varepsilon\varphi(\bar{x}) := \{x^* \in \mathbb{R}^n \mid \varphi(x) - \varphi(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - \varepsilon, \forall x \in \mathbb{R}^n\}.$$

Obviously, if φ is a proper lower semicontinuous convex function, and if $\bar{x} \in \text{dom}\varphi$, then

$$\text{epi}\varphi^* = \bigcup_{\varepsilon \in \mathbb{R}_+} \{(\xi, \langle \xi, \bar{x}\rangle + \varepsilon - \varphi(\bar{x})) \mid \xi \in \partial_\varepsilon\varphi(\bar{x})\}. \tag{1}$$

The following important properties will be used in the sequel.

Lemma 2.1 [32,33] *Let $\varphi_1, \varphi_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper convex function such that* $\text{dom}\varphi_1 \cap \text{dom}\varphi_2 \neq \emptyset$.

(i) If φ_1 and φ_2 are lower semicontinuous, then

$$\text{epi}(\varphi_1 + \varphi_2)^* = \text{cl}(\text{epi}\varphi_1^* + \text{epi}\varphi_2^*).$$

(ii) If one of φ_1 and φ_2 is continuous at some $\bar{x} \in \text{dom}\varphi_1 \cap \text{dom}\varphi_2$, then

$$\text{epi}(\varphi_1 + \varphi_2)^* = \text{epi}\varphi_1^* + \text{epi}\varphi_2^*.$$

At the end of this section, we recall some notations of multiobjective fractional semi-infinite optimization problems used in this paper. Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set, and let T be a nonempty infinite index set. In this paper, we focus on the following multiobjective fractional semi-infinite optimization problem

$$\text{(MFP)} \qquad \min_{x \in C} \left\{\left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)}\right) \mid h_t(x) \leq 0, \forall t \in T\right\},$$

where $f_i, g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, ..., p$, and $h_t : \mathbb{R}^n \to \mathbb{R}$, $t \in T$, are real-valued functions. The problem (MFP) with uncertain data in the constraint functions can be captured by the following uncertain multiobjective fractional semi-infinite optimization problem

$$\text{(UMFP)} \qquad \min_{x \in C} \left\{ \left(\frac{f_1(x)}{g_1(x)}, ..., \frac{f_p(x)}{g_p(x)} \right) \mid h_t(x, v_t) \leq 0, \forall t \in T \right\},$$

where v_t, $t \in T$, are uncertain parameters from the uncertainty set $V_t \subseteq \mathbb{R}^q$, $t \in T$. $h_t : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $t \in T$, are given functions.

We consider the robust counterpart of (UMFP), namely

$$\text{(RUMFP)} \qquad \min_{x \in C} \left\{ \left(\frac{f_1(x)}{g_1(x)}, ..., \frac{f_p(x)}{g_p(x)} \right) \mid h_t(x, v_t) \leq 0, \forall v_t \in V_t, t \in T \right\},$$

where the feasible set of (RUMFP) is defined by $F := \{x \in C \mid h_t(x, v_t) \leq 0, \forall v_t \in V_t, t \in T\}$. In this paper, we assume that f_i, $i = 1, ..., p$, are continuous convex functions with $f_i(x) \geq 0$, for all $x \in F$, and g_i, $i = 1, ..., p$, are continuous concave functions with $g_i(x) > 0$, for all $x \in F$.

3 Robust approximate optimality conditions

This section is devoted to derive necessary and sufficient optimality conditions of robust ϵ-weakly efficient solutions for (UMFP). It is worth noting that, in this paper, we only deal with robust ϵ-weakly efficient solutions for (UMFP), since other kinds of robust approximate efficient solutions can be dealt with similarly. For convenience, we denote $\epsilon := (\epsilon_1, ..., \epsilon_p) \in \mathbb{R}^p_+$.

Now, we recall some important concepts which will be used later in this paper.

Definition 3.1 Let $\epsilon \in \mathbb{R}^p_+$. A point $\bar{x} \in F$ is said to be a robust ϵ-weakly efficient solution of (UMFP) iff there does not exist $x \in F$ such that

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(\bar{x})}{g_i(\bar{x})} - \epsilon_i, \text{ for all } i = 1, ..., p.$$

Definition 3.2 [34] We say that robust type constraint qualification condition (RCQC) holds iff

$$\bigcup_{v \in V, \eta \in \mathbb{R}^{(T)}_+} \text{epi} \left(\sum_{t \in T} \eta_t h_t(\cdot, v_t) \right)^* + \text{epi}\delta^*_C \text{ is a closed convex set,}$$

where $\eta := (\eta_t)_{t \in T} \in \mathbb{R}^{(T)}_+$ and $v \in V$ means that v is a selection of V, i.e., $v : T \to \mathbb{R}^p$ and $v_t \in V_t$ for all $t \in T$.

The following result provides a robust version of Farkas-type results for infinite convex systems with uncertain data.

Lemma 3.1 [34] *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a continuous convex function, and let $h_t : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $t \in T$, be continuous functions such that, for any $v_t \in \mathbb{R}^q$, $h_t(\cdot, v_t)$ is a convex function. Let $V_t \subseteq \mathbb{R}^q$, $t \in T$, be compact and let $F \neq \emptyset$. Then the following statements are equivalent:*

(i) $\{x \in C \mid h_t(x, v_t) \leq 0, \forall v_t \in V_t, t \in T\} \subseteq \{x \in C \mid \phi(x) \geq 0\}$.

(ii) $(0, 0) \in \text{epi}\phi^* + \text{clco}\left(\bigcup_{v \in V, \eta \in \mathbb{R}_+^{(T)}} \text{epi}\left(\sum_{t \in T} \eta_t h_t(\cdot, v_t)\right)^* + \text{epi}\delta_C^*\right)$.

In order to investigate the robust ϵ-weakly efficient solution of (UMFP), we associate (RUMFP) with the following optimization problem:

$$(\text{RUMFP})_\mu \qquad \min_{x \in C} \left\{\Big(f_1(x) - \mu_1 g_1(x), ..., f_p(x) - \mu_p g_p(x)\Big) \mid h_t(x, v_t) \leq 0, \forall v_t \in V_t, t \in T\right\},$$

where the parametric $\mu := (\mu_1, ..., \mu_p) \in \mathbb{R}_+^p$.

The following relation between the ϵ-weakly efficient solution of (RUMFP) and $(\text{RUMFP})_\mu$ can be easily obtained by using similar methods of [14, Proposition 1] and [19, Lemma 2.1]. The proof is included here for the sake of completeness.

Lemma 3.2 *Let $\bar{x} \in F$ and $\epsilon \in \mathbb{R}_+^p$. Let $\bar{\mu}_i = \frac{f_i(\bar{x})}{g_i(\bar{x})} - \epsilon_i \geq 0$, $i = 1, ..., p$. Then $\bar{x} \in F$ is a robust ϵ-weakly efficient solution of* (UMFP) *if and only if $\bar{x} \in F$ is an $\bar{\epsilon}$-weakly efficient solution of* $(\text{RUMFP})_{\bar{\mu}}$*, where $\bar{\mu} := (\bar{\mu}_1, ..., \bar{\mu}_p)$ and $\bar{\epsilon} := (\epsilon_1 g_1(\bar{x}), ..., \epsilon_p g_p(\bar{x}))$.*

Proof ($\Rightarrow$) On the contrary, we assume that $\bar{x} \in F$ is not a robust $\bar{\epsilon}$-weakly efficient solution of $(\text{RUMFP})_{\bar{\mu}}$. Then, there exists $\hat{x} \in F$ such that

$$f_i(\hat{x}) - \bar{\mu}_i g_i(\hat{x}) < f_i(\bar{x}) - \bar{\mu}_i g_i(\bar{x}) - \bar{\epsilon}_i, \text{ for all } i = 1, .., p.$$

Then, together with $\bar{\mu}_i = \frac{f_i(\bar{x})}{g_i(\bar{x})} - \epsilon_i$ and $\bar{\epsilon}_i = \epsilon_i g_i(\bar{x})$, $i = 1, ..p$, we obtain

$$\frac{f_i(\hat{x})}{g_i(\hat{x})} < \frac{f_i(\bar{x})}{g_i(\bar{x})} - \epsilon_i, \text{ for all } i = 1, .., p,$$

Therefore, $\bar{x}$ is not a robust ϵ-weakly efficient solution of (UMFP). We arrive at a contradiction. Thus, $\bar{x} \in F$ is an $\bar{\epsilon}$-weakly efficient solution of $(\text{RUMFP})_{\bar{\mu}}$.

($\Leftarrow$) Assume that $\bar{x} \in F$ is an $\bar{\epsilon}$-weakly efficient solution of $(\text{RUMFP})_{\bar{\mu}}$. Similarly, it is easy to show that $\bar{x} \in F$ is a robust ϵ-weakly efficient solution of (UMFP). The proof is complete. □

Now, we establish necessary and sufficient optimality conditions of robust ϵ-weakly efficient solutions for (UMFP).

Theorem 3.1 *Let $\bar{x} \in F$, $\epsilon \in \mathbb{R}_+^p$, and $\bar{\mu}_i = \frac{f_i(\bar{x})}{g_i(\bar{x})} - \epsilon_i \geq 0$, $i = 1, ..., p$. Let $h_t(\cdot, v_t) : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $t \in T$, be continuous functions such that, for any $v_t \in V_t$, $h_t(\cdot, v_t)$ is a convex function. If* (RCQC) *holds, then $\bar{x}$ is a robust ϵ-weakly efficient* *solution of* (UMFP) *if and only if there exist* $\bar{\lambda}_i \geq 0$, $\sum_{i=1}^{p} \bar{\lambda}_i = 1$, $\epsilon_i^{'} \geq 0$, $\epsilon_i^{''} \geq 0$, $i = 1, ..., p$, $\bar{\eta}_t \geq 0$, $\bar{v}_t \in V_t$, $\epsilon_t^0 \geq 0$, $t \in T$, *and* $\epsilon^c \geq 0$, *such that*

$$0 \in \sum_{i=1}^{p} \left(\partial_{\epsilon_i^{'}} (\bar{\lambda}_i f_i)(\bar{x}) + \partial_{\epsilon_i^{''}} (-\bar{\lambda}_i \bar{\mu}_i g_i)(\bar{x}) \right) + \sum_{t \in T} \partial_{\epsilon_t^0} \left(\bar{\eta}_t h_t(\cdot, \bar{v}_t)\right)(\bar{x}) + \partial_{\epsilon^c} \delta_C(\bar{x}), \tag{2}$$

and

$$\sum_{i=1}^{p} \left(\epsilon_i^{'} + \epsilon_i^{''} \right) + \sum_{t \in T} \epsilon_t^0 + \epsilon^c - \sum_{i=1}^{p} \bar{\lambda}_i \epsilon_i g_i(\bar{x}) = \sum_{t \in T} \bar{\eta}_t h_t(\bar{x}, \bar{v}_t). \tag{3}$$

Proof ($\Rightarrow$) Let $\bar{x} \in F$ be a robust ϵ-weakly efficient solution of (UMFP). It follows from Lemma 3.2 that $\bar{x}$ is an $\bar{\epsilon}$-weakly efficient solution of $(\text{RUMFP})_{\bar{\mu}}$, where $\bar{\mu} = (\bar{\mu}_1, ..., \bar{\mu}_p)$ and $\bar{\epsilon} = (\epsilon_1 g_1(\bar{x}), ..., \epsilon_p g_p(\bar{x}))$. By [22, Proposition 5.3], there exist $\bar{\lambda}_i \geq 0$, $i = 1, ..., p$, and $\sum_{i=1}^{p} \bar{\lambda}_i = 1$, such that

$$\sum_{i=1}^{p} \bar{\lambda}_i (f_i(x) - \bar{\mu}_i g_i(x)) \geq \sum_{i=1}^{p} \bar{\lambda}_i (f_i(\bar{x}) - \bar{\mu}_i g_i(\bar{x})) - \sum_{i=1}^{p} \bar{\lambda}_i \epsilon_i g_i(\bar{x}) = 0, \forall x \in F.$$

For any $x \in C$, set $\phi(x) := \sum_{i=1}^{p} \bar{\lambda}_i \left(f_i(x) - \bar{\mu}_i g_i(x) \right)$. Then,

$$h_t(x, v_t) \leq 0, \forall v_t \in V_t, t \in T, x \in C \Longrightarrow \phi(x) \geq 0.$$

By Lemma 3.1, we have

$$(0, 0) \in \text{epi}\phi^* + \text{clco} \left(\bigcup_{v \in V, \eta \in \mathbb{R}_+^{(T)}} \text{epi} \left(\sum_{t \in T} \eta_t h_t(\cdot, v_t) \right)^* + \text{epi}\delta_C^* \right).$$

Since (RCQC) holds, it follows that

$$(0, 0) \in \text{epi}\phi^* + \bigcup_{v \in V, \eta \in \mathbb{R}_+^{(T)}} \text{epi} \left(\sum_{t \in T} \eta_t h_t(\cdot, v_t) \right)^* + \text{epi}\delta_C^*. \tag{4}$$

By Lemma 2.1, we obtain

$$\text{epi}\phi^* = \sum_{i=1}^{p} \left(\text{epi}(\bar{\lambda}_i f_i)^* + \text{epi}(-\bar{\lambda}_i \bar{\mu}_i g_i)^* \right), \tag{5}$$

and

$$\operatorname{epi}\left(\sum_{t\in T}\eta_t h_t(\cdot,v_t)\right)^* = \sum_{t\in T}\operatorname{epi}\left(\eta_t h_t(\cdot,v_t)\right)^*. \tag{6}$$

Then, together with (4), (5), and (6), we obtain

$$(0,0)\in\sum_{i=1}^{p}\left(\operatorname{epi}(\bar\lambda_i f_i)^* + \operatorname{epi}(-\bar\lambda_i\bar\mu_i g_i)^*\right) + \bigcup_{v\in V,\eta\in\mathbb{R}_+^{(T)}}\sum_{t\in T}\operatorname{epi}\left(\eta_t h_t(\cdot,v_t)\right)^* + \operatorname{epi}\delta_C^*.$$

So, there exist $\bar\eta_t \geq 0$ and $\bar v_t \in V_t$, $t\in T$, such that

$$(0,0)\in\sum_{i=1}^{p}\left(\operatorname{epi}(\bar\lambda_i f_i)^* + \operatorname{epi}(-\bar\lambda_i\bar\mu_i g_i)^*\right) + \sum_{t\in T}\operatorname{epi}\left(\bar\eta_t h_t(\cdot,\bar v_t)\right)^* + \operatorname{epi}\delta_C^*.$$

This means that there exist $\left(\xi_i^{'}, r_i^{'}\right)\in\operatorname{epi}(\bar\lambda_i f_i)^*$, $\left(\xi_i^{''}, r_i^{''}\right)\in\operatorname{epi}(-\bar\lambda_i\bar\mu_i g_i)^*$, $i = 1,\ldots p$, $(\xi_t, r_t)\in\operatorname{epi}\left(\bar\eta_t h_t(\cdot,\bar v_t)\right)^*$, $t\in T$, and $(\xi_c, r_c)\in\operatorname{epi}\delta_C^*$, such that

$$(0,0)\in\left(\sum_{i=1}^{p}\left(\xi_i^{'}+\xi_i^{''}\right)+\sum_{t\in T}\xi_t+\xi_c,\ \sum_{i=1}^{p}\left(r_i^{'}+r_i^{''}\right)+\sum_{t\in T}r_t+r_c\right). \tag{7}$$

Moreover, by (1), there exist $\epsilon_i^{'}\geq 0$, $\epsilon_i^{''}\geq 0$, $i=1,\ldots,p$, and $\epsilon_t^0\geq 0$, $t\in T$, such that

$$\xi_i^{'}\in\partial_{\epsilon_i^{'}}(\bar\lambda_i f_i)(\bar x),\ r_i^{'}=\left\langle\xi_i^{'},\bar x\right\rangle+\epsilon_i^{'}-\bar\lambda_i f_i(\bar x),\quad i=1,\ldots,p,$$
$$\xi_i^{''}\in\partial_{\epsilon_i^{''}}(-\bar\lambda_i\bar\mu_i g_i)(\bar x),\ r_i^{''}=\left\langle\xi_i^{''},\bar x\right\rangle+\epsilon_i^{''}+\bar\lambda_i\bar\mu_i g_i(\bar x),\quad i=1,\ldots,p,$$
$$\xi_t\in\partial_{\epsilon_t^0}\left(\bar\eta_t h_t(\cdot,\bar v_t)\right)(\bar x),\ r_t=\langle\xi_t,\bar x\rangle+\epsilon_t^0-\bar\eta_t h_t(\bar x,\bar v_t),\quad t\in T,$$

and

$$\xi_c\in\partial_{\epsilon^c}\delta_C(\bar x),\ r_c=\langle\xi_c,\bar x\rangle+\epsilon^c.$$

It follows from (7) that

$$0\in\sum_{i=1}^{p}\left(\partial_{\epsilon_i^{'}}(\bar\lambda_i f_i)(\bar x)+\partial_{\epsilon_i^{''}}(-\bar\lambda_i\bar\mu_i g_i)(\bar x)\right)+\sum_{t\in T}\partial_{\epsilon_t^0}\left(\bar\eta_t h_t(\cdot,\bar v_t)\right)(\bar x)+\partial_{\epsilon^c}\delta_C(\bar x),$$

and

$$
\begin{aligned}
0 &= \sum_{i=1}^{p} \left(r_i^{'} + r_i^{''} \right) + \sum_{t \in T} r_t + r_c \\
&= \left\langle \sum_{i=1}^{p} \left(\xi_i^{'} + \xi_i^{''} \right) + \sum_{t \in T} \xi_t + \xi_c, \bar{x} \right\rangle + \sum_{i=1}^{p} \left(\epsilon_i^{'} + \epsilon_i^{''} \right) + \sum_{t \in T} \epsilon_t^0 + \epsilon^c \\
&\quad - \sum_{i=1}^{p} \left(\bar{\lambda}_i f_i(\bar{x}) - \bar{\lambda}_i \bar{\mu}_i g_i(\bar{x}) \right) - \sum_{t \in T} \bar{\eta}_t h_t(\bar{x}, \bar{v}_t) \\
&= \sum_{i=1}^{p} \left(\epsilon_i^{'} + \epsilon_i^{''} \right) + \sum_{t \in T} \epsilon_t^0 + \epsilon^c - \sum_{i=1}^{p} \bar{\lambda}_i \epsilon_i g_i(\bar{x}) - \sum_{t \in T} \bar{\eta}_t h_t(\bar{x}, \bar{v}_t).
\end{aligned}
$$

Thus, (2) and (3) hold.

($\Leftarrow$) Suppose that there exist $\bar{\lambda}_i \geq 0$, $\sum_{i=1}^{p} \bar{\lambda}_i = 1$, $\epsilon_i^{'} \geq 0$, $\epsilon_i^{''} \geq 0$, $i = 1, ..., p$, $\bar{\eta}_t \geq 0$, $\bar{v}_t \in V_t$, $\epsilon_t^0 \geq 0$, $t \in T$, and $\epsilon^c \geq 0$, such that (2) and (3) hold. By (2), there exist $\xi_i^{'} \in \partial_{\epsilon_i^{'}} (\bar{\lambda}_i f_i)(\bar{x})$, $\xi_i^{''} \in \partial_{\epsilon_i^{''}} (-\bar{\lambda}_i \bar{\mu}_i g_i)(\bar{x})$, $i = 1, ..., p$, $\xi_t \in \partial_{\epsilon_t^0} \left(\bar{\eta}_t h_t(\cdot, \bar{v}_t) \right)(\bar{x})$, $t \in T$, and $\xi_c \in \partial_{\epsilon^c} \delta_C(\bar{x})$, such that

$$
\sum_{i=1}^{p} \left(\xi_i^{'} + \xi_i^{''} \right) + \sum_{t \in T} \xi_t + \xi_c = 0. \tag{8}
$$

From $\xi_i^{'} \in \partial_{\epsilon_i^{'}} (\bar{\lambda}_i f_i)(\bar{x})$, $\xi_i^{''} \in \partial_{\epsilon_i^{''}} (-\bar{\lambda}_i \bar{\mu}_i g_i)(\bar{x})$, $i = 1, ..., p$, $\xi_t \in \partial_{\epsilon_t^0} \left(\bar{\eta}_t h_t(\cdot, \bar{v}_t) \right)(\bar{x})$, $t \in T$, and $\xi_c \in \partial_{\epsilon^c} \delta_C(\bar{x})$, it follows that, for any $x \in F$,

$$
\begin{aligned}
&\bar{\lambda}_i f_i(x) - \bar{\lambda}_i f_i(\bar{x}) \geq \left\langle \xi_i^{'}, x - \bar{x} \right\rangle - \epsilon_i^{'}, \quad - \bar{\lambda}_i \bar{\mu}_i g_i(x) + \bar{\lambda}_i \bar{\mu}_i g_i(\bar{x}) \\
&\quad \geq \left\langle \xi_i^{''}, x - \bar{x} \right\rangle - \epsilon_i^{''}, i = 1, ..., p, \\
&\bar{\eta}_t h_t(x, \bar{v}_t) - \bar{\eta}_t h_t(\bar{x}, \bar{v}_t) \geq \langle \xi_t, x - \bar{x} \rangle - \epsilon_t^0, t \in T, \text{ and } \delta_C(x) - \delta_C(\bar{x}) \\
&\quad \geq \langle \xi_c, x - \bar{x} \rangle - \epsilon^c.
\end{aligned}
$$

Then, adding these inequalities yields

$$
\begin{aligned}
&\sum_{i=1}^{p} (\bar{\lambda}_i f_i(x) - \bar{\lambda}_i f_i(\bar{x}) - \bar{\lambda}_i \bar{\mu}_i g_i(x) + \bar{\lambda}_i \bar{\mu}_i g_i(\bar{x})) + \sum_{t \in T} \bar{\eta}_t h_t(x, \bar{v}_t) - \sum_{t \in T} \bar{\eta}_t h_t(\bar{x}, \bar{v}_t) \\
&\geq \left\langle \sum_{i=1}^{p} \left(\xi_i^{'} + \xi_i^{''} \right) + \sum_{t \in T} \xi_t + \xi_c, x - \bar{x} \right\rangle - \sum_{i=1}^{p} \left(\epsilon_i^{'} + \epsilon_i^{''} \right) - \sum_{t \in T} \epsilon_t^0 - \epsilon^c, \forall x \in F.
\end{aligned}
$$

Together with $\sum_{t\in T} \bar{\eta}_t h_t(x, \bar{v}_t) \leq 0$ and (8), we obtain

$$\sum_{i=1}^{p} \left(\bar{\lambda}_i f_i(x) - \bar{\lambda}_i f_i(\bar{x}) - \bar{\lambda}_i \bar{\mu}_i g_i(x) + \bar{\lambda}_i \bar{\mu}_i g_i(\bar{x})\right) - \sum_{t\in T} \bar{\eta}_t h_t(\bar{x}, \bar{v}_t)$$
$$\geq -\sum_{i=1}^{p} \left(\epsilon_i^{'} + \epsilon_i^{''}\right) - \sum_{t\in T} \epsilon_t^0 - \epsilon^c, \qquad \forall x \in F. \tag{9}$$

From (3) and (9), it gives

$$\sum_{i=1}^{p} \bar{\lambda}_i (f_i(x) - \bar{\mu}_i g_i(x)) \geq \sum_{i=1}^{p} \bar{\lambda}_i (f_i(\bar{x}) - \bar{\mu}_i g_i(\bar{x})) - \sum_{i=1}^{p} \bar{\lambda}_i \epsilon_i g_i(\bar{x}), \forall x \in F.$$

Therefore, it follows from [22, Proposition 5.3] that $\bar{x}$ is an $\bar{\epsilon}$-weakly efficient solution of (RUMFP)$_{\bar{\mu}}$, where $\bar{\epsilon} = (\epsilon_1 g_1(\bar{x}), ..., \epsilon_p g_p(\bar{x}))$. Thus, by Lemma 3.2, $\bar{x}$ is a robust ϵ-weakly efficient solution of (UMFP). The proof is complete. □

Remark 3.1 Theorem 3.1 encompasses [26, Theorem 3.1] for the case of scalar optimization problem, where the corresponding results were obtained by applying similar approaches.

Now, we give an example to illustrate the results obtained in Theorem 3.1.

Example 3.1 Let $x \in \mathbb{R}$, $C{:=}\mathbb{R}_+$ and $v_t \in V_t{:=}[-t+2, t+2]$ for any $t \in T{:=}[0, 1]$. Consider the following multiobjective fractional optimization problem

$$\text{(UFMP)} \quad \min_{x\in\mathbb{R}_+} \left\{\left(\tfrac{x^2+1}{x+2}, \tfrac{2x^2+1}{2x+2}\right) \mid tx^2 - 2v_t x \leq 0, t \in T\right\}.$$

Obviously, $f_1(x) = x^2 + 1$, $f_2(x) = 2x^2 + 1$, $g_1(x) = x + 2$, $g_2(x) = 2x + 2$, and $h_t(x, v_t) = tx^2 - 2v_t x, \forall t \in T$. It is easy to show that $F = [0, 2]$ and the assumptions of Theorem 3.1 are satisfied.

Let $\bar{x}{:=}0 \in F$ and $\epsilon{:=}(\frac{1}{4}, \frac{1}{4})$. Obviously, $\bar{\epsilon} = (\frac{1}{2}, \frac{1}{2})$, $\bar{\mu} = (\frac{1}{4}, \frac{1}{4})$, and $\bar{x}$ is a robust ϵ-weakly efficient solution of (UFMP). Moreover, there exist $\epsilon_i' = \epsilon_i'' = \epsilon_c = \frac{1}{32}$, $\bar{\lambda}_i = \frac{1}{2}, i = 1, 2$, $\bar{v}_t = t+2, t \in T$, $\bar{\eta}_t{:=}\begin{cases} 0, & \text{if } t \in (0, 1], \\ \frac{1}{32}, & \text{if } t = 0, \end{cases}$ and $\epsilon_t^0{:=}\begin{cases} 0, & \text{if } t \in (0, 1], \\ \frac{11}{32}, & \text{if } t = 0, \end{cases}$ such that

$$\partial_{\epsilon_1'}(\bar{\lambda}_1 f_1)(\bar{x}) = \left[-\frac{1}{4}, \frac{1}{4}\right], \partial_{\epsilon_2'}(\bar{\lambda}_2 f_2)(\bar{x}) = \left[-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right], \partial_{\epsilon_1''}(-\bar{\lambda}_1 \bar{\mu}_1 g_1)(\bar{x}) = \left\{-\frac{1}{8}\right\},$$
$$\partial_{\epsilon_2''}(-\bar{\lambda}_2 \bar{\mu}_2 g_2)(\bar{x}) = \left\{-\frac{1}{4}\right\}, \ \partial_{\epsilon_t^0}(\bar{\eta}_t h_t(\cdot, \bar{v}_t))(\bar{x}) = \begin{cases} 0, & \text{if } t \in (0, 1], \\ -\frac{1}{8}, & \text{if } t = 0, \end{cases} \ \text{and } \partial_{\epsilon^c}\delta_C(\bar{x}) = (-\infty, 0].$$

Clearly,

$$\sum_{i=1}^{2} \left(\partial_{\epsilon_i'}(\bar{\lambda}_i f_i)(\bar{x}) + \partial_{\epsilon_i''}(-\bar{\lambda}_i \bar{\mu}_i g_i)(\bar{x})\right) + \sum_{t\in T} \partial_{\epsilon_t^0}(\bar{\eta}_t h_t(\cdot, \bar{v}_t))(\bar{x}) + \partial_{\epsilon^c}\delta_C(\bar{x}) = \left(-\infty, \frac{1}{2\sqrt{2}} - \frac{1}{4}\right],$$

and

$$\sum_{i=1}^{2}\left(\epsilon_i^{'}+\epsilon_i^{''}\right)+\sum_{t\in T}\epsilon_t^0+\epsilon^c-\sum_{i=1}^{2}\bar{\lambda}_i\epsilon_i g_i(\bar{x})=\sum_{t\in T}\bar{\eta}_t h_t(\bar{x},\bar{v}_t).$$

Thus, Theorem 3.1 is applicable.

4 Robust approximate duality and saddle point theorems

In this section, we introduce a *Mixed type* robust multiobjective dual problem for (UMFP), and then investigate the robust approximate weak and strong duality properties between them. Moreover, we also introduce an uncertain approximate multiobjective Lagrangian function related to (UMFP) and obtain a nonsmooth robust saddle point theorem. Here, we only focus on their robust ϵ-weakly efficient solutions since other kinds of robust approximate efficient solutions can be dealt with in a similar manner. For convenience, we use the notations $f:=(f_1,...,f_p)$, $g:=(g_1,...,g_p)$, $\eta:=(\eta_t)_t\in\mathbb{R}_+^{(T)}$, $\beta:=(\beta_t)_t\in\mathbb{R}_+^{(T)}$, $\lambda:=(\lambda_1,..,\lambda_p)\in\mathbb{R}_+^p\backslash\{0\}$ and $\mu:=(\mu_1,..,\mu_p)\in\mathbb{R}_+^p$.

Let $y\in C$, $\epsilon\in\mathbb{R}_+^p$, $\eta\in\mathbb{R}_+^{(T)}$, $\beta\in\mathbb{R}_+^{(T)}$, $\lambda\in\mathbb{R}_+^p\backslash\{0\}$ and $\mu\in\mathbb{R}_+^p$. For fixed $v_t\in V_t$, $t\in T$, the conventional *Mixed type* multiobjective dual problem (MD) of (UMFP) is given by

$$\begin{cases}\max\ (\mu_1,...,\mu_p)\\ s.t.\ \ 0\in\sum\limits_{i=1}^{p}\left(\partial_{\epsilon_i^{'}}(\lambda_i f_i)(y)+\partial_{\epsilon_i^{''}}(-\lambda_i\mu_i g_i)(y)\right)+\sum\limits_{t\in T}\partial_{\epsilon_t^0}\left((\eta_t+\beta_t)h_t(\cdot,v_t)\right)(y)+\partial_{\epsilon^c}\delta_C(y),\\ \quad\sum\limits_{i=1}^{p}(\lambda_i f_i(y)-\lambda_i\mu_i g_i(y))+\sum\limits_{t\in T}\eta_t h_t(y,v_t)\geq\sum\limits_{i=1}^{p}\lambda_i\epsilon_i g_i(y),\\ \quad\sum\limits_{i=1}^{p}\left(\epsilon_i^{'}+\epsilon_i^{''}\right)+\sum\limits_{t\in T}\epsilon_t^0+\epsilon^c-\sum\limits_{i=1}^{p}\lambda_i\epsilon_i g_i(y)\leq\sum\limits_{t\in T}\beta_t h_t(y,v_t),\\ \quad y\in C,\lambda_i>0,\mu_i\geq0,\epsilon_i^{'}\geq0,\epsilon_i^{''}\geq0,i=1,...,p,\epsilon^c\geq0,\eta_t\geq0,\beta_t\geq0,\epsilon_t^0\geq0,t\in T.\end{cases}$$

The optimistic counterpart (OMD) of (MD), called optimistic dual optimization problem, is a deterministic maximization problem given by

$$\begin{cases}\max\ (\mu_1,...,\mu_p)\\ s.t.\ \ 0\in\sum\limits_{i=1}^{p}\left(\partial_{\epsilon_i^{'}}(\lambda_i f_i)(y)+\partial_{\epsilon_i^{''}}(-\lambda_i\mu_i g_i)(y)\right)+\sum\limits_{t\in T}\partial_{\epsilon_t^0}\left((\eta_t+\beta_t)h_t(\cdot,v_t)\right)(y)+\partial_{\epsilon^c}\delta_C(y),\\ \quad\sum\limits_{i=1}^{p}(\lambda_i f_i(y)-\lambda_i\mu_i g_i(y))+\sum\limits_{t\in T}\eta_t h_t(y,v_t)\geq\sum\limits_{i=1}^{p}\lambda_i\epsilon_i g_i(y),\\ \quad\sum\limits_{i=1}^{p}\left(\epsilon_i^{'}+\epsilon_i^{''}\right)+\sum\limits_{t\in T}\epsilon_t^0+\epsilon^c-\sum\limits_{i=1}^{p}\lambda_i\epsilon_i g_i(y)\leq\sum\limits_{t\in T}\beta_t h_t(y,v_t),\\ \quad y\in C,\lambda_i>0,\mu_i\geq0,\epsilon_i^{'}\geq0,\epsilon_i^{''}\geq0,i=1,...,p,\epsilon^c\geq0,\eta_t\geq0,\beta_t\geq0,\epsilon_t^0\geq0,v_t\in V_t,t\in T.\end{cases}$$

Here, the feasible set of (OMD) is defined by $F_{(\mathrm{OMD})}$.

Remark 4.1 Obviously, if $\eta_t = 0$, $t \in T$, (OMD) collapses to *Mond-Weir type* optimistic dual optimization problem of (UMFP) as follows:

$$\begin{cases} \max\ (\mu_1, ..., \mu_p) \\ s.t.\ \ 0 \in \sum\limits_{i=1}^{p} \left(\partial_{\epsilon_i^{'}}(\lambda_i f_i)(y) + \partial_{\epsilon_i^{''}}(-\lambda_i \mu_i g_i)(y)\right) + \sum\limits_{t \in T} \partial_{\epsilon_t^0}(\beta_t h_t(\cdot, v_t))(y) + \partial_{\epsilon^c}\delta_C(y), \\ \quad \sum\limits_{i=1}^{p}(\lambda_i f_i(y) - \lambda_i \mu_i g_i(y)) \geq \sum\limits_{i=1}^{p} \lambda_i \epsilon_i g_i(y), \\ \quad \sum\limits_{i=1}^{p}\left(\epsilon_i^{'} + \epsilon_i^{''}\right) + \sum\limits_{t \in T} \epsilon_t^0 + \epsilon^c - \sum\limits_{i=1}^{p} \lambda_i \epsilon_i g_i(y) \leq \sum\limits_{t \in T} \beta_t h_t(y, v_t), \\ \quad y \in C, \lambda_i > 0, \mu_i \geq 0, \epsilon_i^{'} \geq 0, \epsilon_i^{''} \geq 0, i = 1, ..., p, \epsilon^c \geq 0, \beta_t \geq 0, \epsilon_t^0 \geq 0, v_t \in V_t, t \in T. \end{cases}$$

On the other hand, if $\beta_t = 0$, $t \in T$, (OMD) collapses to *Wolfe type* optimistic dual optimization problem of (UMFP) as follows:

$$\begin{cases} \max\ (\mu_1, ..., \mu_p) \\ s.t.\ \ 0 \in \sum\limits_{i=1}^{p} \left(\partial_{\epsilon_i^{'}}(\lambda_i f_i)(y) + \partial_{\epsilon_i^{''}}(-\lambda_i \mu_i g_i)(y)\right) + \sum\limits_{t \in T} \partial_{\epsilon_t^0}(\eta_t h_t(\cdot, v_t))(y) + \partial_{\epsilon^c}\delta_C(y), \\ \quad \sum\limits_{i=1}^{p}(\lambda_i f_i(y) - \lambda_i \mu_i g_i(y)) + \sum\limits_{t \in T} \eta_t h_t(y, v_t) \geq \sum\limits_{i=1}^{p} \lambda_i \epsilon_i g_i(y), \\ \quad \sum\limits_{i=1}^{p}\left(\epsilon_i^{'} + \epsilon_i^{''}\right) + \sum\limits_{t \in T} \epsilon_t^0 + \epsilon^c - \sum\limits_{i=1}^{p} \lambda_i \epsilon_i g_i(y) \leq 0, \\ \quad y \in C, \lambda_i > 0, \mu_i \geq 0, \epsilon_i^{'} \geq 0, \epsilon_i^{''} \geq 0, i = 1, ..., p, \epsilon^c \geq 0, \eta_t \geq 0, \epsilon_t^0 \geq 0, v_t \in V_t, t \in T. \end{cases}$$

Now, similar to the concept of robust ϵ-weakly efficient solutions of (UMFP) in Definition 3.1, we give hereafter such ϵ-weakly efficient solutions for (OMD). In what follows, for $w_1, w_2 \in \mathbb{R}^p$, we use the following notations for convenience.

$$w_1 \prec w_2 \Leftrightarrow w_2 - w_1 \in \text{int}\, \mathbb{R}^p_+, \quad w_1 \not\prec w_2 \text{ is the negation of } w_1 \prec w_2.$$

Here, int $\mathbb{R}^p_+ := \{(x_1, ..., x_p) \in \mathbb{R}^p \mid x_i > 0\}$.

Definition 4.1 Let $\epsilon \in \mathbb{R}^p_+$. We say that $(\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{\eta}, \bar{\beta}, \bar{v}) \in F_{(\text{OMD})}$ is an ϵ-weakly efficient solution of (OMD) if there does not exist $(y, \lambda, \mu, \eta, \beta, v) \in F_{(\text{OMD})}$, such that $\bar{\mu} \prec \mu - \epsilon$.

Remark 4.2 Clearly, if $\epsilon = 0$ and V_t, $t \in T$, are singletons, the ϵ-weakly efficient solution of (OMD) reduces to the usual weakly efficient solution of the corresponding deterministic optimization problem. For more details, see [12,14].

The following two theorems describes ϵ-duality relations between (RUMFP) and (OMD).

Theorem 4.1 ($\epsilon-$ *weak duality*) *Let* $\epsilon \in \mathbb{R}^p_+$. *For any feasible* x *of* (RUMFP) *and any feasible* $(y, \lambda, \mu, \eta, \beta, v)$ *of* (OMD), *it holds that* $\left(\frac{f_1(x)}{g_1(x)}, ..., \frac{f_p(x)}{g_p(x)}\right) \not\prec \mu - \epsilon$.

Proof On the contrary, we assume that

$$\left(\frac{f_1(x)}{g_1(x)}, ..., \frac{f_p(x)}{g_p(x)}\right) \prec \mu - \epsilon. \tag{10}$$

Since $(y, \lambda, \mu, \eta, \beta, v)$ is a feasible solution of (OMD), we have $y \in C$, $\lambda_i > 0$, $\mu_i \geq 0$, $\epsilon_i^{'} \geq 0$, $\epsilon_i^{''} \geq 0$, $i = 1, ..., p$, $\epsilon^c \geq 0$, $\eta_t \geq 0$, $\beta_t \geq 0$, $\epsilon_t^0 \geq 0$, $v_t \in V_t$, $t \in T$, and

$$0 \in \sum_{i=1}^{p} \left(\partial_{\epsilon_i^{'}} (\lambda_i f_i)(y) + \partial_{\epsilon_i^{''}} (-\lambda_i \mu_i g_i)(y) \right) + \sum_{t \in T} \partial_{\epsilon_t^0} \left((\eta_t + \beta_t) h_t(\cdot, v_t) \right)(y) + \partial_{\epsilon^c} \delta_C(y), \tag{11}$$

$$\sum_{i=1}^{p} (\lambda_i f_i(y) - \lambda_i \mu_i g_i(y)) + \sum_{t \in T} \eta_t h_t(y, v_t) \geq \sum_{i=1}^{p} \lambda_i \epsilon_i g_i(y), \tag{12}$$

and

$$\sum_{i=1}^{p} \left(\epsilon_i^{'} + \epsilon_i^{''} \right) + \sum_{t \in T} \epsilon_t^0 + \epsilon^c - \sum_{i=1}^{p} \lambda_i \epsilon_i g_i(y) \leq \sum_{t \in T} \beta_t h_t(y, v_t). \tag{13}$$

It follows from (11) that there exist $\xi_i^{'} \in \partial_{\epsilon_i^{'}} (\lambda_i f_i)(y)$, $\xi_i^{''} \in \partial_{\epsilon_i^{''}} (-\lambda_i \mu_i g_i)(y)$, $i = 1, ..., p$, $\xi_t \in \partial_{\epsilon_t^0} \left((\eta_t + \beta_t) h_t(\cdot, v_t) \right)(y)$, $t \in T$, and $\xi_c \in \partial_{\epsilon^c} \delta_C(y)$, such that

$$\sum_{i=1}^{p} \left(\xi_i^{'} + \xi_i^{''} \right) + \sum_{t \in T} \xi_t + \xi_c = 0. \tag{14}$$

Note that for any $x \in F$, we have $(\eta_t + \beta_t) h_t(x, v_t) \leq 0$, $\forall t \in T$, and $g(x) > 0$. Together with (12), (13), and (14), it gives

$$\begin{aligned}
&\sum_{i=1}^{p} \lambda_i (f_i(x) - \mu_i g_i(x) + \epsilon_i g_i(x)) \\
&\geq \sum_{i=1}^{p} \left(\lambda_i f_i(y) + \left\langle \xi_i^{'}, x - y \right\rangle - \epsilon_i^{'} - \lambda_i \mu_i g_i(y) + \left\langle \xi_i^{''}, x - y \right\rangle - \epsilon_i^{''} + \lambda_i \epsilon_i g_i(x) \right) \\
&= \sum_{i=1}^{p} (\lambda_i f_i(y) - \lambda_i \mu_i g_i(y)) - \left\langle \sum_{t \in T} \xi_t, x - y \right\rangle - \langle \xi_c, x - y \rangle + \sum_{i=1}^{p} \left(-\epsilon_i^{'} - \epsilon_i^{''} + \lambda_i \epsilon_i g_i(x) \right) \\
&\geq \sum_{i=1}^{p} (\lambda_i f_i(y) - \lambda_i \mu_i g_i(y)) - \sum_{t \in T} (\eta_t + \beta_t) h_t(x, v_t) \\
&\quad + \sum_{t \in T} (\eta_t + \beta_t) h_t(y, v_t) - \sum_{t \in T} \epsilon_t^0 - \epsilon^c + \sum_{i=1}^{p} \left(-\epsilon_i^{'} - \epsilon_i^{''} + \lambda_i \epsilon_i g_i(x) \right) \\
&\geq \sum_{i=1}^{p} (\lambda_i f_i(y) - \lambda_i \mu_i g_i(y)) + \sum_{t \in T} \eta_t h_t(y, v_t) \\
&\quad + \sum_{t \in T} \beta_t h_t(y, v_t) - \sum_{t \in T} \epsilon_t^0 - \epsilon^c + \sum_{i=1}^{p} \left(-\epsilon_i^{'} - \epsilon_i^{''} \right)
\end{aligned}$$

$$\geq \sum_{i=1}^{p} (\lambda_i f_i(y) - \lambda_i \mu_i g_i(y)) + \sum_{t\in T} \eta_t h_t(y, v_t) - \sum_{i=1}^{p} \lambda_i \epsilon_i g_i(y)$$
$$\geq 0.$$

This implies that there exists $i_0 \in \{1, ..., p\}$ such that $f_{i_0}(x) - \mu_{i_0} g_{i_0}(x) + \epsilon_{i_0} g_{i_0}(x) \geq 0$. The inequality is equivalent to

$$\frac{f_{i_0}(x)}{g_{i_0}(x)} \geq \mu_{i_0} - \epsilon_{i_0},$$

which is a contradiction to (10). The proof is complete. □

Theorem 4.2 (ϵ *− strong duality*) *Let* $\bar{x} \in F$ *and* $\epsilon \in \mathbb{R}^p_+$. *Let* $h_t : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $t \in T$, *be continuous functions such that, for any* $v_t \in V_t$, $h_t(\cdot, v_t)$ *is a convex function. Assume that* (RCQC) *holds. If* $\bar{x}$ *is a robust* ϵ*-weakly efficient solution of* (UMFP) *and* $\frac{f_i(\bar{x})}{g_i(\bar{x})} - \epsilon_i \geq 0$, *then there exist* $\bar{\lambda} \in \mathbb{R}^p_+ \backslash \{0\}$, $\bar{\mu} \in \mathbb{R}^p_+$, $\bar{\eta} \in \mathbb{R}^{(T)}_+$, $\bar{v} \in V$, *such that* $(\bar{x}, \bar{\lambda}, \bar{\mu}, 0, \bar{\eta}, \bar{v})$ *is a* 2ϵ*-weakly efficient solutions of* (OMD).

Proof Let $\bar{\mu}_i := \frac{f_i(\bar{x})}{g_i(\bar{x})} - \epsilon_i \geq 0$, $i = 1, ..., p$. Then,

$$\sum_{i=1}^{p} (f_i(\bar{x}) - \bar{\mu}_i g_i(\bar{x})) = \sum_{i=1}^{p} \epsilon_i g_i(\bar{x}). \tag{15}$$

By Theorem 3.1, there exist $\bar{\lambda}_i \geq 0$, $\sum_{i=1}^{p} \bar{\lambda}_i = 1$, $\epsilon_i^{'} \geq 0$, $\epsilon_i^{''} \geq 0$, $i = 1, ..., p$, $\bar{\eta}_t \geq 0$, $\bar{v}_t \in V_t$, $\epsilon_t^0 \geq 0$, $t \in T$, and $\epsilon^c \geq 0$, such that

$$0 \in \sum_{i=1}^{p} \left(\partial_{\epsilon_i^{'}} (\bar{\lambda}_i f_i)(\bar{x}) + \partial_{\epsilon_i^{''}} (-\bar{\lambda}_i \bar{\mu}_i g_i)(\bar{x}) \right) + \sum_{t\in T} \partial_{\epsilon_t^0} (\bar{\eta}_t h_t(\cdot, \bar{v}_t))(\bar{x}) + \partial_{\epsilon^c} \delta_C(\bar{x}), \tag{16}$$

and

$$\sum_{i=1}^{p} \left(\epsilon_i^{'} + \epsilon_i^{''} \right) + \sum_{t\in T} \epsilon_t^0 + \epsilon^c - \sum_{i=1}^{p} \bar{\lambda}_i \epsilon_i g_i(\bar{x}) = \sum_{t\in T} \bar{\eta}_t h_t(\bar{x}, \bar{v}_t). \tag{17}$$

From (15), (16), and (17), we can deduce that $(\bar{x}, \bar{\lambda}, \bar{\mu}, 0, \bar{\eta}, \bar{v})$ is a robust feasible solution of (OMD). By Theorem 4.1, for any feasible solution $(y, \lambda, \mu, \eta, \beta, v)$ of (OMD),

$$\bar{\mu} - \mu = \left(\frac{f_1(\bar{x})}{g_1(\bar{x})}, ..., \frac{f_p(\bar{x})}{g_p(\bar{x})} \right) - \epsilon - \mu \not< \mu - \epsilon - \epsilon - \mu = -2\epsilon.$$

Thus, $(\bar{x}, \bar{\lambda}, \bar{\mu}, 0, \bar{\eta}, \bar{v})$ is a 2ϵ-weakly efficient solutions of (OMD). The proof is complete. □

At the end of this section, we give an ϵ-weak saddle point theorem for (UMFP). We first define an uncertain multiobjective Lagrangian-type function related to (UMFP) as follows:

$$\begin{aligned} L_\mu(x,\eta,v) &:= f(x) - \mu g(x) + \sum_{t\in T}\eta_t h_t(x,v_t)e \\ &= \left(f_1(x) - \mu_1 g_1(x) + \sum_{t\in T}\eta_t h_t(x,v_t), ..., f_p(x) - \mu_p g_p(x) + \sum_{t\in T}\eta_t h_t(x,v_t)\right), \end{aligned}$$

where $x \in C$, $\mu := (\mu_1, .., \mu_p) \in \mathbb{R}^p_+$, $\eta := (\eta_t)_{t\in T} \in \mathbb{R}^{(T)}_+$, $v := (v_t)_{t\in T} \in V$ and $e := (1, ..., 1) \in \mathbb{R}^p$.

Now, we introduce a new concept of robust ϵ-weak saddle points of L_μ related to (UMFP).

Definition 4.2 Let $\epsilon \in \mathbb{R}^p_+$ and $C \subseteq \mathbb{R}^n$. We say that $(\bar{x}, \bar{\eta}, \bar{v}) \in C \times \mathbb{R}^{(T)}_+ \times V$ is a robust ϵ-weak saddle points of L_μ related to (UMFP), iff for any $(x, \eta, v) \in C \times \mathbb{R}^{(T)}_+ \times V$,

$$L_{\bar{\mu}}(x,\bar{\eta},\bar{v}) + \bar{\epsilon} \not< L_{\bar{\mu}}(\bar{x},\bar{\eta},\bar{v}) \not< L_{\bar{\mu}}(\bar{x},\eta,v) - \bar{\epsilon},$$

where $\bar{\epsilon} = (\epsilon_1 g_1(\bar{x}), ..., \epsilon_p g_p(\bar{x}))$ and $\bar{\mu} = (\bar{\mu}_1, ..., \bar{\mu}_p) = \left(\frac{f_1(\bar{x})}{g_1(\bar{x})} - \epsilon_1, ..., \frac{f_p(\bar{x})}{g_p(\bar{x})} - \epsilon_p\right)$.

Now, we present an example to explain the existence of robust ϵ-weak saddle point.

Example 4.1 Let $C := [-1, 1]$, $x \in \mathbb{R}$, and $v_t \in V_t := [2 - t, 2 + t]$, $t \in T := [0, 1)$. Consider the functions

$$f(x) = (f_1(x), f_2(x)) = \left(x^2+1, x^2+1\right),\ g(x) = (g_1(x), g_2(x)) = \left(-x^2+2, -x^2+2\right),$$

and

$$h_t(x, v_t) = tx^2 - 2v_t x - 3t.$$

Then, for any $(x, \eta, v) \in C \times \mathbb{R}^{(T)}_+ \times V$ and $\mu \in \mathbb{R}^p_+$, the uncertain multiobjective Lagrangian-type function L_μ related to (UMFP) is

$$\begin{aligned} L_\mu(x,\eta,v) = &\Big(x^2 + 1 - \mu_1(-x^2+2) + \sum_{t\in T}\eta_t(tx^2 - 2v_t x - 3t), \\ &x^2 + 1 - \mu_2(-x^2+2) + \sum_{t\in T}\eta_t(tx^2 - 2v_t x - 3t)\Big). \end{aligned}$$

Now, let $\bar{x} := 0$ and $\epsilon = (\epsilon_1, \epsilon_2) := (\frac{1}{4}, \frac{1}{4})$. Then, $\bar{\epsilon} = (\bar{\epsilon}_1, \bar{\epsilon}_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\bar{\mu} = \left(\frac{f_1(\bar{x})}{g_1(\bar{x})} - \epsilon_1, \frac{f_2(\bar{x})}{g_2(\bar{x})} - \epsilon_2\right) = (\frac{1}{4}, \frac{1}{4})$. By selecting $\bar{\eta}_t := \begin{cases} 0, & \text{if } t \in (0,1), \\ \frac{1}{4}, & \text{if } t = 0, \end{cases}$ and $\bar{v}_t :=$

$2-t, t \in [0,1)$. It is easy to show that

$$L_{\bar{\mu}}(\bar{x}, \bar{\eta}, \bar{v}) = \left(\frac{1}{2}, \frac{1}{2}\right),\ L_{\bar{\mu}}(\bar{x}, \eta, v) = \left(\frac{1}{2} - \sum_{t\in T} 3t\eta_t, \frac{1}{2} - \sum_{t\in T} 3t\eta_t\right),$$

and

$$L_{\bar{\mu}}(x, \bar{\eta}, \bar{v}) = \left(\frac{5}{4}x^2 - x + \frac{1}{2}, \frac{5}{4}x^2 - x + \frac{1}{2}\right).$$

Obviously, for any $(x, \eta, v) \in C \times \mathbb{R}_+^{(T)} \times V$,

$$L_{\bar{\mu}}(x, \bar{\eta}, \bar{v}) + \bar{\epsilon} \not< L_{\bar{\mu}}(\bar{x}, \bar{\eta}, \bar{v}) \not< L_{\bar{\mu}}(\bar{x}, \eta, v) - \bar{\epsilon}.$$

This means that $(\bar{x}, \bar{\eta}, \bar{v})$ is a robust ϵ-weak saddle point.

The following theorem is a robust ϵ-weak saddle point theorem of L_μ related to (UMFP). This is a new result on robust ϵ-saddle point theorem for multiobjective fractional optimization not yet being considered in the literature.

Theorem 4.3 *Let $\bar{x} \in F$, $\epsilon \in \mathbb{R}_+^p$, and $\bar{\mu}_i = \frac{f_i(\bar{x})}{g_i(\bar{x})} - \epsilon_i \geq 0$, $i = 1, ..., p$. Let $h_t(\cdot, v_t) : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $t \in T$, be continuous functions such that, for any $v_t \in V_t$, $h_t(\cdot, v_t)$ is a convex function. Assume that* (RCQC) *holds. If $\bar{x} \in F$ is a robust ϵ-weakly efficient solution of* (UMFP)*, then there exist $\bar{\eta} = (\bar{\eta}_t)_{t\in T} \in \mathbb{R}_+^{(T)}$, and $\bar{v} = (\bar{v}_t)_{t\in T} \in V$, such that $(\bar{x}, \bar{\eta}, \bar{v}) \in C \times \mathbb{R}_+^{(T)} \times V$ is a robust ϵ-weak saddle point of L_μ related to* (UMFP).

Proof Since $\bar{x}$ is a robust ϵ-weakly efficient solution of (UMFP), it follows from Theorem 3.1 that there exist $\bar{\lambda}_i \geq 0$, $\sum_{i=1}^{p} \bar{\lambda}_i = 1$, $\epsilon_i^{'} \geq 0$, $\epsilon_i^{''} \geq 0$, $i = 1, ..., p$, $\bar{\eta}_t \geq 0$, $\bar{v}_t \in V_t$, $\epsilon_t^0 \geq 0$, $t \in T$, and $\epsilon^c \geq 0$, such that (2) and (3) hold. By (2), there exist $\xi_i^{'} \in \partial_{\epsilon_i^{'}}(\bar{\lambda}_i f_i)(\bar{x})$, $\xi_i^{''} \in \partial_{\epsilon_i^{''}}(-\bar{\lambda}_i \bar{\mu}_i g_i)(\bar{x})$, $i = 1, ..., p$, $\xi_t \in \partial_{\epsilon_t^0}(\bar{\eta}_t h_t(\cdot, \bar{v}_t))(\bar{x})$, $t \in T$, and $\xi_c \in \partial_{\epsilon^c}\delta_C(\bar{x})$, such that

$$\sum_{i=1}^{p}(\xi_i^{'} + \xi_i^{''}) + \sum_{t\in T}\xi_t + \xi_c = 0. \tag{18}$$

Moreover, from $\xi_i^{'} \in \partial_{\epsilon_i^{'}}(\bar{\lambda}_i f_i)(\bar{x})$, $\xi_i^{''} \in \partial_{\epsilon_i^{''}}(-\bar{\lambda}_i \bar{\mu}_i g_i)(\bar{x})$, $i = 1, ..., p$, $\xi_t \in \partial_{\epsilon_t^0}(\bar{\eta}_t h_t(\cdot, \bar{v}_t))(\bar{x})$, $t \in T$, and $\xi_c \in \partial_{\epsilon^c}\delta_C(\bar{x})$, it follows that, for any $x \in C$,

$$\begin{aligned}&\bar{\lambda}_i f_i(x) \geq \bar{\lambda}_i f_i(\bar{x}) + \left\langle \xi_i^{'}, x - \bar{x}\right\rangle - \epsilon_i^{'},\ \ -\bar{\lambda}_i\bar{\mu}_i g_i(x) \geq -\bar{\lambda}_i\bar{\mu}_i g_i(\bar{x}) + \left\langle \xi_i^{''}, x - \bar{x}\right\rangle \\ &\quad -\epsilon_i^{''},\ i = 1, ..., p, \bar{\eta}_t h_t(x, \bar{v}_t) \geq \bar{\eta}_t h_t(\bar{x}, \bar{v}_t) + \langle \xi_t, x - \bar{x}\rangle - \epsilon_t^0,\ t \in T,\ \text{and } \delta_C(x) \\ &\quad \geq \delta_C(\bar{x}) + \langle \xi_c, x - \bar{x}\rangle - \epsilon^c.\end{aligned}$$

These, together with (3) and (18), imply that for any $x \in F$,

$$\begin{aligned}
&\sum_{i=1}^{p} \bar{\lambda}_i (f_i(x) - \bar{\mu}_i g_i(x)) + \sum_{t \in T} \bar{\eta}_t h_t(x, \bar{v}_t) \\
&\geq \sum_{i=1}^{p} \bar{\lambda}_i (f_i(\bar{x}) - \bar{\mu}_i g_i(\bar{x})) + \sum_{t \in T} \bar{\eta}_t h_t(\bar{x}, \bar{v}_t) - \sum_{i=1}^{p} \left(\epsilon_i^{'} + \epsilon_i^{''}\right) - \sum_{t \in T} \epsilon_t^0 - \epsilon^c \\
&= \sum_{i=1}^{p} \bar{\lambda}_i (f_i(\bar{x}) - \bar{\mu}_i g_i(\bar{x})) - \sum_{i=1}^{p} \bar{\lambda}_i \epsilon_i g_i(\bar{x}).
\end{aligned}$$

Moreover, by $\bar{x} \in F$, we get $h_t(\bar{x}, \bar{v}_t) \leq 0, \forall t \in T$. Then,

$$\begin{aligned}
&\sum_{i=1}^{p} \bar{\lambda}_i (f_i(x) - \bar{\mu}_i g_i(x)) + \sum_{t \in T} \bar{\eta}_t h_t(x, \bar{v}_t) \\
&\geq \sum_{i=1}^{p} \bar{\lambda}_i (f_i(\bar{x}) - \bar{\mu}_i g_i(\bar{x})) + \sum_{t \in T} \bar{\eta}_t h_t(\bar{x}, \bar{v}_t) - \sum_{i=1}^{p} \bar{\lambda}_i \epsilon_i g_i(\bar{x}).
\end{aligned} \tag{19}$$

Assume that there exists $\hat{x} \in C$ such that $L_{\bar{\mu}}(\hat{x}, \bar{\eta}, \bar{v}) + \bar{\epsilon} \prec L_{\bar{\mu}}(\bar{x}, \bar{\eta}, \bar{v})$. That is

$$f_i(\hat{x}) - \bar{\mu}_i g_i(\hat{x}) + \sum_{t \in T} \bar{\eta}_t h_t(\hat{x}, \bar{v}_t) + \bar{\epsilon}_i < f_i(\bar{x}) - \bar{\mu}_i g_i(\bar{x}) + \sum_{t \in T} \bar{\eta}_t h_t(\bar{x}, \bar{v}_t).$$

Thus,

$$\begin{aligned}
&\sum_{i=1}^{p} \bar{\lambda}_i \left(f_i(\hat{x}) - \bar{\mu}_i g_i(\hat{x}) + \sum_{t \in T} \bar{\eta}_t h_t(\hat{x}, \bar{v}_t) + \epsilon_i g_i(\bar{x}) \right) \\
&< \sum_{i=1}^{p} \bar{\lambda}_i \left(f_i(\bar{x}) - \mu_i g_i(\bar{x}) + \sum_{t \in T} \bar{\eta}_t h_t(\bar{x}, \bar{v}_t) \right).
\end{aligned}$$

This contradicts (19).

On the other hand, assume to the contrary that there exist $\hat{\eta} \in \mathbb{R}_+^{(T)}$, and $\hat{v} \in V$ such that $L_{\bar{\mu}}(\bar{x}, \bar{\eta}, \bar{v}) \prec L_{\bar{\mu}}(\bar{x}, \hat{\eta}, \hat{v}) - \bar{\epsilon}$. This implies that

$$\sum_{t \in T} \bar{\eta}_t h_t(\bar{x}, \bar{v}_t) < \sum_{t \in T} \hat{\eta}_t h_t(\bar{x}, \hat{v}_t) - \epsilon_i g_i(\bar{x}), i = 1, ..., p. \tag{20}$$

Since $\sum_{i=1}^{p} \bar{\lambda}_i = 1$, we deduce from (20) that

$$\sum_{t \in T} \bar{\eta}_t h_t(\bar{x}, \bar{v}_t) < \sum_{t \in T} \hat{\eta}_t h_t(\bar{x}, \hat{v}_t) - \sum_{i=1}^{p} \bar{\lambda}_i \epsilon_i g_i(\bar{x}). \tag{21}$$

Now, we will get a contradiction. Indeed, from (3), we have

$$\sum_{t\in T}\bar{\eta}_t h_t(\bar{x},\bar{v}_t)+\sum_{i=1}^{p}\bar{\lambda}_i\epsilon_i g_i(\bar{x})=\sum_{i=1}^{p}\left(\epsilon_i^{'}+\epsilon_i^{''}\right)+\sum_{t\in T}\epsilon_t^0+\epsilon^c\geq 0.$$

By $\bar{x}\in F$, we get $h_t(\bar{x},\hat{v}_t)\leq 0, \forall t\in T$. Therefore,

$$\sum_{t\in T}\bar{\eta}_t h_t(\bar{x},\bar{v}_t)+\sum_{i=1}^{p}\bar{\lambda}_i\epsilon_i g_i(\bar{x})-\sum_{t\in T}\hat{\eta}_t h_t(\bar{x},\hat{v}_t)\geq 0,$$

which contradicts (21). The proof is complete. □

The following result can be easily obtained by using similar methods reported in [24, Theorem 6.7].

Theorem 4.4 *Let* $\epsilon\in\mathbb{R}_+^p$. *If* $(\bar{x},\bar{\eta},\bar{v})\in C\times\mathbb{R}_+^{(T)}\times V$ *is a robust* ϵ*-weak saddle point of* L_μ *related to* (UMFP)*, then,* $\bar{x}\in F$ *is a robust* ϵ*-weakly efficient solution of* (UMFP).

5 Conclusions

In this paper, following the framework of robust optimization, we consider robust ϵ-weakly efficient solutions for a class of nonsmooth multiobjective fractional semi-infinite optimization problems with uncertain data in the constraint functions. We employ a scalarization method and a robust type constraint qualification condition to establish necessary and sufficient conditions for the robust ϵ-weakly efficient solutions of this uncertain fractional semi-infinite optimization problem. We also obtain some robust ϵ-duality properties and ϵ-weak saddle point theorems. It is worth noting that the approach used in this paper is new, and some existing results in the literature can be obtained by the use of our approach. Moreover, the mathematical framework developed in this paper can be extended to the investigation of robust ϵ-quasi (weakly) efficient solutions for uncertain multiobjective fractional semi-infinite optimization.

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