Optimization Letters (2021) 15:2181–2199
https://doi.org/10.1007/s11590-020-01678-w

ORIGINAL PAPER

Explicit extragradient-like method with adaptive stepsizes for pseudomonotone variational inequalities

Duong Viet Thong[1] · Jun Yang[2] · Yeol Je Cho[3,4] · Themistocles M. Rassias[5]

Received: 20 July 2019 / Accepted: 2 December 2020 / Published online: 2 January 2021

Abstract

The purpose of this paper is to introduce a new modified subgradient extragradient method for finding an element in the set of solutions of the variational inequality problem for a pseudomonotone and Lipschitz continuous mapping in real Hilbert spaces. It is well known that for the existing subgradient extragradient methods, the step size requires the line-search process or the knowledge of the Lipschitz constant of the mapping, which restrict the applications of the method. To overcome this barrier, in this work we present a modified subgradient extragradient method with adaptive stepsizes and do not require extra projection or value of the mapping. The advantages of the proposed method only use one projection to compute and the strong convergence proved without the prior knowledge of the Lipschitz constant of the inequality variational mapping. Numerical experiments illustrate the performances of our new algorithm and provide a comparison with related algorithms.

✉ Duong Viet Thong
duongvietthong@duytan.edu.vn

Jun Yang
xysyyangjun@163.com

Yeol Je Cho
yjcho@gnu.ac.kr

Themistocles M. Rassias
trassias@math.ntua.gr

1 Institute of Research and Development, Duy Tan University, Da Nang 550000, Vietnam

2 School of Mathematics and Statistics, Xidian University, Xi'an, People's Republic of China

3 Department of Mathematics Education and the RINS, Gyeongsang National University, Jinju 52828, Korea

4 School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, Sichuan, People's Republic of China

5 Department of Mathematics, National Technical University of Athens, Zografou Campus, 15780 Athens, Greece

Keywords Subgradient extragradient method · Mann type method · Variational inequality problem · Pseudomonotone mapping

1 Introduction

In this paper, we consider the classical *variational inequality problem* (VI) of Fichera [15,16] and Stampacchia [37] (see also Kinderlehrer and Stampacchia [21]) in real Hilbert spaces. The **(VI)** is formulated as follows:

$$\text{Find a point } x^* \in C \text{ such that } \langle Ax^*, x - x^* \rangle \geq 0 \text{ for all } x \in C,$$

where C is a nonempty closed convex subset in a real Hilbert space H, $A : H \to H$ is a single-valued mapping, $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ are the inner product and the norm in H, respectively. Let us denote $VI(C, A)$ by the solution set of the problem **(VI)**.

In the last years, the techniques for the variational inequality problem have been applied to a variety of diverse areas, such as, operations research, nonlinear equations, and network equilibrium problems, see, for instance, [2,3,5,14,21,24–28] and the extensive list of references therein.

Many authors have proposed and analyzed several methods for solving the problem **(VI)**. One of the most popular methods is the *extragradient method* introduced by Korpelevich [23] which was called the extragradient method:

$$x_0 \in C, \quad y_n = P_C(x_n - \lambda Ax_n), \quad x_{n+1} = P_C(x_n - \lambda Ay_n),$$

where the mapping $A : C \to H$ is monotone and L-Lipschitz continuous, $\lambda \in \big(0, \frac{1}{L}\big)$. The algorithm converges to an element of $VI(C, A)$ provided that $VI(C, A)$ is nonempty.

In recent years, the extragradient method has been received great attention by many authors in various ways (see, for example, [6,9–12,19,29,30,36,39,43] and the references therein).

Censor et al. [10,11] proposed the *subgradient extragradient method*:

$$x_0 \in H, y_n = P_C(x_n - \lambda Ax_n), \quad x_{n+1} = P_{T_n}(x_n - \lambda Ay_n), \tag{1}$$

where $T_n = \{x \in H | \langle x_n - \lambda Ax_n - y_n, x - y_n \rangle \leq 0\}$, the mapping $A : H \to H$ is monotone and L-Lipschitz continuous, $\lambda \in \big(0, \frac{1}{L}\big)$. This method replaces two projections onto C by one projection onto C and one onto a half-space. Since the projection onto the half-space T_n can be explicitly calculated, the subgradient extragradient requires only one projection per iteration. For this, recently, the subgradient extragradient method [10,11] has been received great attention by many authors, they improved and extended it in various ways to obtain the weak and strong convergence of this method (see [8,12,22,34,35,38,39,41,42] and the references therein).

Inspired by the results in [10,11], Kraikaew and Saejung [22] introduced the Halpern subgradient extragradient method for solving monotone variational inequalities as follows:

$$x_0 \in H, \quad y_n = P_C(x_n - \lambda A x_n), \quad x_{n+1} = \gamma_n x_0 + (1-\gamma_n) P_{T_n}(x_n - \lambda A y_n), \quad (2)$$

where $T_n = \{x \in H : \langle x_n - \lambda A x_n - y_n, x - y_n \rangle \leq 0\}$, the mapping $A : H \to H$ is monotone and L-Lipschitz continuous and $\lambda \in (0, \frac{1}{L})$, $\{\gamma_n\} \subset (0, 1)$, $\lim_{n\to\infty} \gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = +\infty$, and they proved that the sequence $\{x_n\}$ is generated by (2) converges strongly to a point x^*, where $x^* = P_{\mathrm{VI}(C,A)} x_0$.

It is worth pointing out that the main shortcoming of algorithms (1) and (2) is that it requires to know the Lipschitz constant or at least to know some estimation of it. Very recently, in [42], motivated and inspired by the algorithms in [10,11], they introduced a modified subgradient extragradient method for solving monotone variational inequalities with a new step size. It is worth pointing out that the convergence analysis of the algorithm in [42] doesn't require either the prior knowledge of the Lipschitz constant of the variational inequality mapping or any additional evaluation of P_C.

The pseudomonotone mappings in the sense of Karamardian were introduced in [20] as a generalization of the monotone mappings. The notion of the pseudomonotone mapping has found many applications in variational inequalities and economics.

It is a known fact that, in [12], Censor et al. showed that the subgradient extragradient method can be successfully applied for solving the pseudomonotone variational inequality in a finite dimensional Euclidean space. Since, in infinite dimensional spaces, the norm convergence is often more desirable, a natural question raises as follows:

How to design and extend the result of Censor et al. in [12] such that strong convergence is obtained in infinite dimensional Hilbert spaces?

To answer this question, in this paper, we develop a new version of the subgradient extragradient method with the technique of choosing stepsizes in [42] for finding an element of the set of solutions of a pseudomonotone and Lipschitz-continuous variational inequality problem in Hilbert spaces and prove that the sequence generated by the proposed algorithm converges strongly to a solution of the pseudomonotone variational inequality.

This paper is organized as follows: In Sect. 2, we recall some definitions and preliminary results for further use. In Sect. 3, we deal with analyzing the convergence of the proposed algorithm. Finally, in Sect. 4, we give several numerical experiments to illustrate the convergence of the proposed algorithm and compare it with previously known algorithms.

2 Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}$ to x is written as $x_n \to x$ as $n \to \infty$. For each $x, y, z \in H$, we have

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle. \quad (3)$$

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2 \quad (4)$$

for all $\alpha, \beta, \gamma \in [0; 1]$ with $\alpha + \beta + \gamma = 1$.

Definition 2.1 Let $T : H \to H$ be a mapping. Then we have the following:

(1) T is called *L-Lipschitz continuous* with $L > 0$ if

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in H.$$

(2) T is called *monotone* if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall x, y \in H.$$

(3) T is called *pseudomonotone* if

$$\langle Tx, y - x\rangle \geq 0 \implies \langle Ty, y - x\rangle \geq 0, \quad \forall x, y \in H.$$

(4) T is called *sequentially weakly continuous* if, for each sequence $\{x_n\}$ in H with $x_n \rightharpoonup x$ as $n \to \infty$, $Tx_n \rightharpoonup Tx$.

It is easy to see that every monotone operator T is pseudomonotone, but the converse is not true.

Now, we present an academic example of the variational inequality problem in an infinite dimensional space, where the cost function A is pseudomonotone, L-Lipschitz continuous and sequentially weakly continuous on C, but A fails to be a monotone mapping on H.

Example 1 Consider a Hilbert space defined as follows:

$$H = l_2 := \left\{u = (u_1, u_2, \ldots, u_i, \ldots) : \sum_{i=1}^{\infty} |u_i|^2 < +\infty\right\}$$

equipped with the inner product and the induced norm on H:

$$\langle u, v\rangle = \sum_{i=1}^{\infty} u_i v_i, \quad \|u\| = \sqrt{\langle u, u\rangle}$$

for any $u = (u_1, u_2, \ldots, u_i, \ldots)$, $v = (v_1, v_2, \ldots, v_i, \ldots) \in H$, respectively. Let $\alpha, \beta \in \mathbb{R}$ such that $\beta > \alpha > \frac{\beta}{2} > 2$ and consider the set and the mapping:

$$C = \left\{u = (u_1, u_2, \ldots, u_i, \ldots) \in H : |u_i| \leq \frac{1}{i}, \forall i \geq 1\right\}, \quad Au = (\beta - \|u\|)u.$$

Then it is easy to see that $VI(C, A) \neq \emptyset$ since $0 \in VI(C, A)$. Moreover, let

$$C_\alpha := \{u \in H : \|u\| \leq \alpha\}.$$

It is known that A is pseudomonotone, $(\beta + 2\alpha)$-Lipschitz continuous on C_α and A fails to be a monotone mapping on H (see [18, Example 4.1]).

Now, we show that $C \subset C_\alpha$. Indeed, let $u = (u_1, u_2, \ldots, u_i, \ldots) \in C$. Then we have

$$\|u\|^2 = \sum_{i=1}^{\infty} |u_i|^2 \leq \sum_{i=1}^{\infty} \frac{1}{i^2} = 1 + \sum_{i=2}^{\infty} \frac{1}{i^2} \leq 1 + \sum_{i=2}^{\infty} \frac{1}{i^2 - 1} = 1 + \frac{3}{4} = \frac{7}{4},$$

which implies that $\|u\| \leq \alpha$, that is, $u \in C_\alpha$ and so $C \subset C_\alpha$.

Further, since $C \subset C_\alpha$, it follows that A is pseudomonotone and $\beta + 2\alpha$-Lipschitz continuous on C. On the other hand, since C is compact and A is continuous on H, A is sequentially weakly continuous on C.

Remark 2.1 (1) It should be noted here that the mapping A is not sequentially weakly continuous on C_α since C_α is not compact on H.

(2) An example on noncompact sets can be found in [4, Example 2.1], where the mapping A is pseudomonotone, Lipschitz continuous and sequentially weakly continuous.

For all point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

Then P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive.

Lemma 2.1 ([17]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Then, for any $x \in H$ and $z \in C$,*

$$z = P_C x \iff \langle x - z, z - y \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.2 ([7]) *For any $x \in H$ and $v \in H$ with $v \neq 0$, let $T = \{z \in H : \langle v, z - x \rangle \leq 0\}$. Then, for all $u \in H$, the projection $P_T(u)$ is defined by*

$$P_T(u) = u - \max\left\{0, \frac{\langle v, u - x \rangle}{||v||^2}\right\} v.$$

In particular, if $u \notin T$, then we have

$$P_T(u) = u - \frac{\langle v, u - x \rangle}{||v||^2} v.$$

Lemma 2.2 gives us an explicit formula of the projection of any point onto a half-space. For more some properties of the metric projection, the interested reader can refer to Sect. 3 in [17] and Chapter 4 in [7].

The following lemmas are useful for the convergence of our proposed method:

Lemma 2.3 ([13, Lemma 2.1]) *Consider the solution set $VI(C, A)$ of the problem* (VI)*, where C is a nonempty closed convex subset of a real Hilbert space H and $A : C \to H$ is pseudomonotone and continuous. Then $x^* \in VI(C, A)$ if and only if*

$$\langle Ax, x - x^* \rangle \geq 0, \quad \forall x \in C.$$

Lemma 2.4 ([33]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\gamma_n\}$ be a sequence of real numbers in $(0, 1)$ with $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n b_n, \quad \forall n \geq 1.$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying

$$\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0,$$

then $\lim_{n\to\infty} a_n = 0$.

3 Main results

In this section, we introduce a modified subgradient extragradient algorithm for solving the pseudomonotone variational inequality problem. Under mild assumptions, the sequence generated by the proposed method converges strongly to $x^* \in VI(C, A)$, where

$$\|x^*\| = \min\{\|z\| : z \in VI(C, A)\}.$$

First, the following conditions are assumed for the convergence of the method:

Condition 1 *The feasible set C is a nonempty closed convex subset of a real Hilbert space H.*

Condition 2 *The mapping $A : H \to H$ is L-Lipschitz continuous, pseudomonotone on H and the mapping $A : H \to H$ satisfies the following condition*

$$\text{whenever } \{x_n\} \subset C, x_n \rightharpoonup z, \text{ one has } \|Az\| \leq \liminf_{n\to\infty} \|Ax_n\|. \tag{5}$$

Condition 3 *The solution set of the problem* **(VI)** *is nonempty, that is, $VI(C, A) \neq \emptyset$.*

Condition 4 *Assume that* $\{\gamma_n\}$ *and* $\{\beta_n\}$ *are two real sequences in* $(0, 1)$ *such that* $\{\beta_n\} \subset (a, 1-\gamma_n)$ *for some* $a > 0$ *and*

$$\lim_{n\to\infty} \gamma_n = 0, \quad \sum_{n=1}^{\infty} \gamma_n = \infty.$$

Now, we present our algorithm.

Algorithm 3.1

Initialization: *Given* $\lambda_0 > 0$, $\mu \in (0, 1)$. *Let* $x_0 \in H$ *be arbitrary*

Iterative Steps: *Calculate* x_{n+1} *as follows:*

Step 1. *Compute*

$$y_n = P_C(x_n - \lambda_n A x_n),$$

If $x_n = y_n$ *or* $Ay_n = 0$ *then stop and* y_n *is a solution of the problem* **(VI)**. *Otherwise,*
Step 2. *Compute*

$$w_n = P_{T_n}(x_n - \lambda_n A y_n),$$

where $T_n := \{x \in H : \langle x_n - \lambda_n A x_n - y_n, x - y_n\rangle \le 0\}$.
Step 3. *Compute*

$$x_{n+1} = (1 - \gamma_n - \beta_n)x_n + \beta_n w_n,$$

and update

$$\lambda_{n+1} = \begin{cases} \min\{\mu \dfrac{\|x_n - y_n\|^2 + \|w_n - y_n\|^2}{2\langle Ax_n - Ay_n, w_n - y_n\rangle}, \lambda_n\} & \text{if } \langle Ax_n - Ay_n, w_n - y_n\rangle > 0, \\ \lambda_n & \text{otherwise}. \end{cases} \tag{6}$$

Set $n := n + 1$ *and go to* ***Step 1***.

Remark 3.2 It is easy to show that condition (5) is weaker than the sequential weak continuity of the mapping A (see [1,32]), which is frequently assumed in recent works on pseudomonotone variational inequality problems, (see, [40]). Indeed, if A is sequentially weakly continuous, then due to the weak lower semicontinuity of the norm, condition (5) is fulfilled. Conversely, let $Ax = \|x||x$ and suppose that $x_n \rightharpoonup x$. Due to the weak lower continuiuty of the norm, one has $\|x\| \le \liminf_{n\to\infty} \|x_n\|$, hence,

$$\|Ax\| = \|x\|^2 \le (\liminf_{n\to\infty} \|x_n\|)^2 \le \liminf_{n\to\infty} \|x_n\|^2 = \liminf_{n\to\infty} \|Ax_n\|,$$

Thus, condition (5) is satisfied. However, A is not sequentially weakly continuous. Indeed, let $x_n = e_n + e_1$, where $\{e_n\}$ is an orthonormal system in H. Then $x_n \rightharpoonup e_1$. For $n > 1$, $Ax_n = \sqrt{2}(e_n + e_1) \rightharpoonup \sqrt{2}e_1 \ne A(e_1) = e_1$.

Lemma 3.5 ([42]) *Assume that Conditions 1–3 hold. Then the sequence* $\{\lambda_n\}$ *generated by (6) is a non-increasing sequence and*

$$\lim_{n\to\infty}\lambda_n=\lambda\geq\min\left\{\lambda_0,\frac{\mu}{L}\right\}.$$

The following lemmas are quite helpful to analyze the convergence of algorithm:

Lemma 3.6 *Assume that Conditions 1–3 hold. Let* $\{w_n\}$ *be a sequence generated by Algorithm 3.1 Then we have*

$$\|w_n-x^*\|^2\leq\|x_n-x^*\|^2-\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|y_n-x_n\|^2-\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|w_n-y_n\|^2 \tag{7}$$

for all $x^*\in VI(C,A)$.

Proof First, it is easy to see that, by the definition of $\{\lambda_n\}$,

$$2\langle Ax_n-Ay_n,w_n-y_n\rangle\leq\frac{\mu}{\lambda_{n+1}}\|x_n-y_n\|^2+\frac{\mu}{\lambda_{n+1}}\|w_n-y_n\|^2,\quad\forall n\geq 1. \tag{8}$$

Indeed, if $\langle Ax_n-Ay_n,w_n-y_n\rangle<0$, then the inequality (8) holds. Otherwise, from (6), we have

$$\lambda_{n+1}=\min\left\{\mu\frac{\|x_n-y_n\|^2+\|w_n-y_n\|^2}{2\langle Ax_n-Ay_n,w_n-y_n\rangle},\lambda_n\right\}\leq\mu\frac{\|x_n-y_n\|^2+\|w_n-y_n\|^2}{2\langle Ax_n-Ay_n,w_n-y_n\rangle}.$$

This implies that

$$2\langle Ax_n-Ay_n,w_n-y_n\rangle\leq\frac{\mu}{\lambda_{n+1}}\|x_n-y_n\|^2+\frac{\mu}{\lambda_{n+1}}\|w_n-y_n\|^2.$$

Therefore, the inequality (8) holds. Now, using the inequality (8) and $x^*\in VI(C,A)\subset C\subset T_n$, we prove that the inequality (7) holds. Indeed, we have

$$\begin{aligned}\|w_n-x^*\|^2&=\|P_{T_n}(x_n-\lambda_nAy_n)-P_{T_n}x^*\|^2\leq\langle w_n-x^*,x_n-\lambda_nAy_n-x^*\rangle\\&=\frac{1}{2}\|w_n-x^*\|^2+\frac{1}{2}\|x_n-\lambda_nAy_n-x^*\|^2-\frac{1}{2}\|w_n-x_n+\lambda_nAy_n\|^2\\&=\frac{1}{2}\|w_n-x^*\|^2+\frac{1}{2}\|x_n-x^*\|^2+\frac{1}{2}\lambda_n^2\|Ay_n\|^2-\langle x_n-x^*,\lambda_nAy_n\rangle\\&\quad-\frac{1}{2}\|w_n-x_n\|^2-\frac{1}{2}\lambda_n^2\|Ay_n\|^2-\langle w_n-x_n,\lambda_nAy_n\rangle\\&=\frac{1}{2}\|w_n-x^*\|^2+\frac{1}{2}\|x_n-x^*\|^2-\frac{1}{2}\|w_n-x_n\|^2-\langle w_n-x^*,\lambda_nAy_n\rangle.\end{aligned}$$

This implies that

$$\|w_n-x^*\|^2\leq\|x_n-x^*\|^2-\|w_n-x_n\|^2-2\langle w_n-x^*,\lambda_nAy_n\rangle. \tag{9}$$

Since x^* is the solution of the problem **(VI)**, we have $\langle Ax^*, x - x^*\rangle \geq 0$ for all $x \in C$. By the pseudomontonicity of A on C, we have $\langle Ax, x - x^*\rangle \geq 0$ for all $x \in C$. Taking $x := y_n \in C$, we get

$$\langle Ay_n, x^* - y_n\rangle \leq 0.$$

Thus we have

$$\langle Ay_n, x^* - w_n\rangle = \langle Ay_n, x^* - y_n\rangle + \langle Ay_n, y_n - w_n\rangle \leq \langle Ay_n, y_n - w_n\rangle. \tag{10}$$

From (9) and (10), it follows that

$$\begin{aligned}\|w_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|w_n - x_n\|^2 + 2\lambda_n\langle Ay_n, y_n - w_n\rangle \\ &= \|x_n - x^*\|^2 - \|w_n - y_n\|^2 - \|y_n - x_n\|^2 \\ &\quad - 2\langle w_n - y_n, y_n - x_n\rangle + 2\lambda_n\langle Ay_n, y_n - w_n\rangle \\ &= \|x_n - x^*\|^2 - \|w_n - y_n\|^2 - \|y_n - x_n\|^2 + 2\langle x_n - \lambda_n Ay_n - y_n, w_n - y_n\rangle. \end{aligned} \tag{11}$$

Since $y_n = P_{T_n}(x_n - \lambda_n Ax_n)$ and $w_n \in T_n$, we have

$$\begin{aligned} &2\langle x_n - \lambda_n Ay_n - y_n, w_n - y_n\rangle \\ &\quad = 2\langle x_n - \lambda_n Ax_n - y_n, w_n - y_n\rangle + 2\lambda_n\langle Ax_n - Ay_n, w_n - y_n\rangle \\ &\quad \leq 2\lambda_n\langle Ax_n - Ay_n, w_n - y_n\rangle, \end{aligned} \tag{12}$$

which, together with (8), implies that

$$2\langle x_n - \lambda_n Ay_n - y_n, w_n - y_n\rangle \leq \mu\frac{\lambda_n}{\lambda_{n+1}}\|x_n - y_n\|^2 + \mu\frac{\lambda}{\lambda_{n+1}}\|w_n - y_n\|^2$$

From (11) and (12), we get

$$\|w_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|y_n - x_n\|^2 - \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|w_n - y_n\|^2.$$

This completes the proof. □

Remark 3.3 Unlike the proof in [42], our Lemma 3.6 is proved when A is pseudomonotone instead of the fact that A is monotone.

We adapt the technique developed in [40] to obtain the following result.

Lemma 3.7 *Assume that Conditions 1–3 hold and $\{x_n\}$ is a sequence generated by Algorithm 3.1. If there exists a subsequence $\{x_{n_k}\}$ convergent weakly to $z \in H$ and*

$$\lim_{k\to\infty}\|x_{n_k} - y_{n_k}\| = 0,$$

then $z \in VI(C, A)$.

Proof We have

$$\langle x_{n_k} - \lambda_{n_k} Ax_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq 0, \quad \forall x \in C.$$

or, equivalently,

$$\frac{1}{\lambda_{n_k}}\langle x_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq \langle Ax_{n_k}, x - y_{n_k}\rangle, \quad \forall x \in C.$$

Consequently, we have

$$\frac{1}{\lambda_{n_k}}\langle x_{n_k} - y_{n_k}, x - y_{n_k}\rangle + \langle Ax_{n_k}, y_{n_k} - x_{n_k}\rangle \leq \langle Ax_{n_k}, x - x_{n_k}\rangle, \quad \forall x \in C. \tag{13}$$

Since $\{x_{n_k}\}$ is weakly convergent, $\{x_{n_k}\}$ is bounded. Then, by the Lipschitz continuity of A, $\{Ax_{n_k}\}$ is bounded. Since $\|x_{n_k} - y_{n_k}\| \to 0$, $\{y_{n_k}\}$ is also bounded and, according to Lemma 3.5, we have

$$\lambda_{n_k} \geq \min\left\{\lambda_0, \frac{\mu}{L}\right\}.$$

Passing (13) to limit as $k \to \infty$, we get

$$\liminf_{k\to\infty}\langle Ax_{n_k}, x - x_{n_k}\rangle \geq 0, \quad \forall x \in C. \tag{14}$$

Moreover, we have

$$\langle Ay_{n_k}, x - y_{n_k}\rangle = \langle Ay_{n_k} - Ax_{n_k}, x - x_{n_k}\rangle + \langle Ax_{n_k}, x - x_{n_k}\rangle + \langle Ay_{n_k}, x_{n_k} - y_{n_k}\rangle. \tag{15}$$

Since $\lim_{k\to\infty}\|x_{n_k} - y_{n_k}\| = 0$ and A is L-Lipschitz continuous on H, we get

$$\lim_{k\to\infty}\|Ax_{n_k} - Ay_{n_k}\| = 0,$$

which, together with (14) and (15), implies that

$$\liminf_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle \geq 0, \quad \forall x \in C.$$

Next, we show that $z \in VI(C, A)$. We choose a sequence $\{\epsilon_k\}$ of positive numbers such that $\{\epsilon_k\}$ is decreasing and convergent to 0. For each $k \geq 1$, we denote by n_{N_k} the smallest positive integer such that

$$\langle Ay_{n_j}, x - y_{n_j}\rangle + \epsilon_k \geq 0, \quad \forall j \geq n_{N_k}. \tag{16}$$

Since $\{\epsilon_k\}$ is decreasing, it is easy to see that the sequence $\{n_{N_k}\}$ is increasing. Furthermore, for each $k \geq 1$, since $\{y_{n_{N_k}}\} \subset C$, we have $Ay_{n_{N_k}} \neq 0$ and, setting

$$v_{n_{N_k}} = \frac{Ay_{n_{N_k}}}{\|Ay_{n_{N_k}}\|^2},$$

we have $\langle Ay_{n_{N_k}}, v_{n_{N_k}} \rangle = 1$ for each $k \geq 1$. Now, it follows from (16) that, for each $k \geq 1$,

$$\langle Ay_{n_{N_k}}, x + \epsilon_k v_{n_{N_k}} - y_{n_{N_k}} \rangle \geq 0.$$

Since A is pseudomonotone on H, we get

$$\langle A(x + \epsilon_k v_{n_{N_k}}), x + \epsilon_k v_{n_{N_k}} - y_{n_{N_k}} \rangle \geq 0.$$

This implies that

$$\langle Ax, x - y_{n_{N_k}} \rangle \geq \langle Ax - A(x + \epsilon_k v_{n_{N_k}}), x + \epsilon_k v_{n_{N_k}} - y_{n_{N_k}} \rangle - \epsilon_k \langle Ax, v_{n_{N_k}} \rangle. \quad (17)$$

Now, we show that $\lim_{k\to\infty} \epsilon_k v_{n_{N_k}} = 0$. Indeed, since $x_{n_k} \rightharpoonup z$ and $\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0$, we obtain $y_{N_k} \rightharpoonup z$ as $k \to \infty$. Since $\{y_n\} \subset C$, we have $z \in C$. We can suppose that $Az \neq 0$ (otherwise, z is a solution). Since the mapping A satisfies the condition (5), we obtain

$$0 < \|Az\| \leq \liminf_{k\to\infty} \|Ay_{n_k}\|.$$

Since $\{y_{n_{N_k}}\} \subset \{y_{n_k}\}$ and $\epsilon_k \to 0$ as $k \to \infty$, we obtain

$$0 \leq \limsup_{k\to\infty} \|\epsilon_k v_{n_{N_k}}\| = \limsup_{k\to\infty} \left(\frac{\epsilon_k}{\|Ay_{n_k}\|} \right) \leq \frac{\limsup_{k\to\infty} \epsilon_k}{\liminf_{k\to\infty} \|Ay_{n_k}\|} = 0,$$

which implies that $\lim_{k\to\infty} \epsilon_k v_{n_{N_k}} = 0$. Now, letting $k \to \infty$, the right hand side of (17) tends to zero since A is Lipschitz continuous, $\{x_{n_{N_k}}\}$, $\{v_{n_{N_k}}\}$ are bounded and $\lim_{k\to\infty} \epsilon_k v_{n_{N_k}} = 0$. Thus we get

$$\liminf_{k\to\infty} \langle Ax, x - y_{n_{N_k}} \rangle \geq 0.$$

Hence, for all $x \in C$, we have

$$\langle Ax, x - z \rangle = \lim_{k\to\infty} \langle Ax, x - y_{n_{N_k}} \rangle = \liminf_{k\to\infty} \langle Ax, x - y_{n_{N_k}} \rangle \geq 0.$$

Therefore, by Lemma 2.3, $z \in VI(C, A)$. This completes the proof. □

Remark 3.4 When A is monotone, it is not necessary to impose the sequential weak continuity of A, see [8].

Theorem 3.1 *Assume that Conditions 1–4 hold. Then the sequence* $\{x_n\}$ *generated by Algorithm 3.1 converges strongly to* $x^* \in VI(C, A)$, *where*

$$\|x^*\| = \min\{\|z\| : z \in VI(C, A)\}.$$

Proof Since $\lim_{n\to\infty}\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right) = 1-\mu > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$1-\mu\frac{\lambda_n}{\lambda_{n+1}} > 0, \quad \forall n \geq n_0. \tag{18}$$

Combining (7) and (18), we get

$$\|w_n - x^*\| \leq \|x_n - x^*\|, \quad \forall n \geq n_0. \tag{19}$$

Claim 1. The sequence $\{x_n\}$ is bounded. It follows from (19) that

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|(1-\gamma_n-\beta_n)x_n + \beta_n w_n - x^*\| \\
&= \|(1-\gamma_n-\beta_n)(x_n - x^*) + \beta_n(w_n - x^*) - \gamma_n x^*\| \\
&\leq \|(1-\gamma_n-\beta_n)(x_n - x^*) + \beta_n(w_n - x^*)\| + \gamma_n\|x^*\| \\
&\leq (1-\gamma_n-\beta_n)\|x_n - x^*\| + \beta_n\|w_n - x^*\| + \gamma_n\|x^*\| \\
&\leq (1-\gamma_n-\beta_n)\|x_n - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|x^*\| \;\; \forall n \geq n_0 \\
&= (1-\gamma_n)\|x_n - x^*\| + \gamma_n\|x^*\| \;\; \forall n \geq n_0 \\
&\leq \max\{\|x_n - x^*\|, \|x^*\|\}| \;\; \forall n \geq n_0 \\
&\leq \max\{\|x_{n_0} - x^*\|, \|x^*\|\}.
\end{aligned}$$

That is, the sequence $\{x_n\}$ is bounded and $\{w_n\}$ is also. **Claim 2.** Note that

$$\begin{aligned}
&a\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|x_n - y_n\|^2 \\
&\quad + a\left(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|y_n - w_n\|^2 \\
&\quad \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \gamma_n\|x^*\|^2.
\end{aligned}$$

Indeed, using (4), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\gamma_n-\beta_n)x_n + \beta_n w_n - x^*\|^2 \\
&= \|(1-\gamma_n-\beta_n)(x_n - x^*) + \beta_n(w_n - x^*) + \gamma_n(-x^*)\|^2 \\
&= (1-\gamma_n-\beta_n)\|x_n - x^*\|^2 + \beta_n\|w_n - x^*\|^2 \\
&\quad + \gamma_n\|x^*\|^2 - \beta_n(1-\gamma_n-\beta_n)\|x_n - w_n\|^2 \\
&\quad - \gamma_n(1-\gamma_n-\beta_n)\|x_n\|^2 - \gamma_n\beta_n\|w_n\|^2 \\
&\leq (1-\gamma_n-\beta_n)\|x_n - x^*\|^2 + \beta_n\|w_n - x^*\|^2 + \gamma_n\|x^*\|^2.
\end{aligned} \tag{20}$$

It follows from (7) and (20) that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le (1-\gamma_n-\beta_n)\|x_n-x^*\|^2+\beta_n\|x_n-x^*\|^2\\
&\quad -\beta_n\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|x_n-y_n\|^2\\
&\quad -\beta_n\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|y_n-w_n\|^2+\gamma_n\|x^*\|^2\\
&=(1-\gamma_n)\|x_n-x^*\|^2-\beta_n\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|x_n-y_n\|^2\\
&\quad -\beta_n\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|y_n-w_n\|^2+\gamma_n\|x^*\|^2\\
&\le \|x_n-x^*\|^2-\beta_n\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|x_n-y_n\|^2\\
&\quad -\beta_n\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|y_n-w_n\|^2+\gamma_n\|x^*\|^2.
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\beta_n\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|x_n-y_n\|^2\\
&\quad +\beta_n\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|y_n-w_n\|^2\\
&\quad \le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\gamma_n\|x^*\|^2.
\end{aligned}$$

Moreover, since $\beta_n \ge a$ for all $n \ge 1$, we obtain

$$\begin{aligned}
&a\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|x_n-y_n\|^2\\
&\quad +a\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|y_n-w_n\|^2\\
&\quad \le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\gamma_n\|x^*\|^2.
\end{aligned}$$

Claim 3. Note that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le (1-\gamma_n)\|x_n-x^*\|^2+\gamma_n[2\beta_n\|x_n-w_n\|\|x_{n+1}-x^*\|\\
&\quad +2\langle x^*, x^*-x_{n+1}\rangle],\quad \forall n\ge n_0.
\end{aligned}$$

Indeed, setting $t_n=(1-\beta_n)x_n+\beta_n w_n$ for each $n \ge 1$, we have

$$\begin{aligned}
\|t_n-x^*\| &= \|(1-\beta_n)(x_n-x^*)+\beta_n(w_n-x^*)\|\\
&\le (1-\beta_n)\|x_n-x^*\|+\beta_n\|w_n-x^*\|\\
&\le (1-\beta_n)\|x_n-x^*\|+\beta_n\|x_n-x^*\|=\|x_n-x^*\|,\quad \forall n\ge n_0, \qquad (21)
\end{aligned}$$

and

$$\|t_n - x_n\| = \beta_n \|x_n - w_n\|. \tag{22}$$

Using (21) and (22), we get

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1 - \gamma_n - \beta_n)x_n + \beta_n w_n - x^*\|^2 \\ &= \|(1 - \beta_n)x_n + \beta_n w_n - \gamma_n x_n - x^*\|^2 \\ &= \|(1 - \gamma_n)(t_n - x^*) - \gamma_n(x_n - t_n) - \gamma_n x^*\|^2. \end{aligned} \tag{23}$$

Now, using the inequality (3), we get

$$\begin{aligned} &\|(1 - \gamma_n)(t_n - x^*) - \gamma_n(x_n - t_n) - \gamma_n x^*\|^2 \\ &\quad \le (1 - \gamma_n)^2\|t_n - x^*\|^2 - 2\langle \gamma_n(x_n - t_n) + \gamma_n x^*, x_{n+1} - x^*\rangle. \end{aligned} \tag{24}$$

Combining (23) and (24), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le (1 - \gamma_n)^2\|t_n - x^*\|^2 \\ &\quad + 2\gamma_n\langle x_n - t_n, x^* - x_{n+1}\rangle + 2\gamma_n\langle x^*, x^* - x_{n+1}\rangle \\ &\le (1 - \gamma_n)\|t_n - x^*\|^2 + 2\gamma_n\|x_n - t_n\|\|x_{n+1} - x^*\| \\ &\quad + 2\gamma_n\langle x^*, x^* - x_{n+1}\rangle \\ &\le (1 - \gamma_n)\|x_n - x^*\|^2 + \gamma_n[2\beta_n\|x_n - w_n\|\|x_{n+1} - x^*\| \\ &\quad + 2\langle x^*, x^* - x_{n+1}\rangle], \quad \forall n \ge n_0. \end{aligned}$$

Claim 4. $\{\|x_n - x^*\|^2\}$ converges to zero. Indeed, for each $n \ge 0$, set

$$a_n := \|x_n - x^*\|^2 \quad \text{and} \quad b_n := 2\beta_n\|x_n - w_n\|\|x_{n+1} - x^*\| + 2\langle x^*, x^* - x_{n+1}\rangle.$$

Then, **Claim 3** can be rewritten as follows:

$$a_{n+1} \le (1 - \gamma_n)a_n + \gamma_n b_n.$$

By Lemma 2.4, it is sufficient to show that $\limsup_{k\to\infty} b_{n_k} \le 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying

$$\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \ge 0.$$

This is equivalently to that we need to show $\limsup_{k\to\infty}\langle x^*, x^* - x_{n_k+1}\rangle \le 0$ and $\limsup_{k\to\infty}\|x_{n_k} - w_{n_k}\| \le 0$ for every subsequence $\{\|x_{n_k} - x^*\|\}$ of $\{\|x_n - x^*\|\}$ satisfying

$$\liminf_{k\to\infty}(\|x_{n_k+1}-x^*\|-\|x_{n_k}-x^*\|)\geq 0.$$

Suppose that $\{\|x_{n_k}-x^*\|\}$ is a subsequence of $\{\|x_n-x^*\|\}$ such that

$$\liminf_{k\to\infty}(\|x_{n_k+1}-x^*\|-\|x_{n_k}-x^*\|)\geq 0.$$

Then, we have

$$\begin{aligned}\liminf_{k\to\infty}&(\|x_{n_k+1}-x^*\|^2-\|x_{n_k}-x^*\|^2)\\ &=\liminf_{k\to\infty}[(\|x_{n_k+1}-x^*\|-\|x_{n_k}-x^*\|)(\|x_{n_k+1}-x^*\|+\|x_{n_k}-x^*\|)]\\ &\geq 0.\end{aligned}$$

By **Claim 2**, we obtain

$$\begin{aligned}&\limsup_{k\to\infty}\left[a\left(1-\mu\frac{\lambda_{n_k}}{\lambda_{n_k+1}}\right)\|x_{n_k}-y_{n_k}\|^2+a\left(1-\mu\frac{\lambda_{n_k}}{\lambda_{n_k+1}}\right)\|y_{n_k}-w_{n_k}\|^2\right]\\ &\leq\limsup_{k\to\infty}[\|x_{n_k}-x^*\|^2-\|x_{n_k+1}-x^*\|^2+\gamma_{n_k}\|x^*\|^2]\\ &\leq\limsup_{k\to\infty}[\|x_{n_k}-x^*\|^2-\|x_{n_k+1}-x^*\|^2]+\limsup_{k\to\infty}\gamma_{n_k}\|x^*\|^2\\ &=-\liminf_{k\to\infty}[\|x_{n_k+1}-x^*\|^2-\|x_{n_k}-x^*\|^2]\\ &\leq 0.\end{aligned}$$

This implies that

$$\lim_{k\to\infty}\|x_{n_k}-y_{n_k}\|=0,\quad \lim_{k\to\infty}\|y_{n_k}-w_{n_k}\|=0.$$

Thus we have

$$\|x_{n_k}-w_{n_k}\|\leq\|x_{n_k}-y_{n_k}\|+\|y_{n_k}-w_{n_k}\|\to 0\quad \text{as } k\to\infty.$$

On the other hand, we have

$$\|x_{n_k+1}-x_{n_k}\|\leq\gamma_{n_k}\|x_{n_k}\|+\beta_{n_k}\|x_{n_k}-w_{n_k}\|\to 0\quad \text{as } k\to\infty. \tag{25}$$

Since the sequence $\{x_{n_k}\}$ is bounded, it follows that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$, which converges weakly to some $z\in H$, such that

$$\limsup_{k\to\infty}\langle x^*,x^*-x_{n_k}\rangle=\lim_{j\to\infty}\langle x^*,x^*-x_{n_{k_j}}\rangle=\langle x^*,x^*-z\rangle. \tag{26}$$

From $\lim_{k\to\infty}\|x_{n_k}-y_{n_k}\|=0$ and Lemma 3.7, we have $z\in VI(C,A)$ and, from (26) and the definition of $x^*=P_{VI(C,A)}0$, we have

$$\limsup_{k\to\infty}\langle x^*,x^*-x_{n_k}\rangle=\langle x^*,x^*-z\rangle\leq 0. \tag{27}$$

Combining (25) and (27), we have

$$\limsup_{k\to\infty}\langle x^*, x^* - x_{n_k+1}\rangle \le \limsup_{k\to\infty}\langle x^*, x^* - x_{n_k}\rangle = \langle x^*, x^* - z\rangle \le 0. \tag{28}$$

Hence it follows from (28), $\lim_{k\to\infty}\|x_{n_k} - w_{n_k}\| = 0$, **Claim 3** and Lemma 2.4 that

$$\lim_{n\to\infty}\|x_n - x^*\| = 0.$$

This completes the proof. □

Remark 3.5 Our result generalizes some related results in the literature and hence might be applied to a wider class of nonlinear mappings. For example, in the next section, we presented the advantages of our method compared with the recent results [22, Theorem 3.1], [34, Theorem 3.3], [41, Theorem 3.1] and [42, Theorem 3.7] as follows:

(1) In Theorem 3.1, we replaced the monotonicity by the pseudomonotonicity and sequentially weakly continuity of A.
(2) We also obtained the strong convergence without using the viscosity technique.

4 Numerical illustrations

In this section, we present numerical experiments relative to the problem **(VI)**. The first example, we compare Algorithm 3.1 with Algorithm 2 of Yang et al. in [42]. The second example, we illustrate the convergence of Algorithms 3.1 and compare them with three well-known algorithms including Algorithm 2 of Yang et al. in [42], Algorithm 1 of Censor et al. in [10] and Algorithm 3.1 of Kraikaew et al. in [22]. All the numerical experiments are performed on a HP laptop with Intel(R) Core(TM)i5-6200U CPU 2.3GHz with 4 GB RAM. All the programs are written in Matlab2015a.

Problem 1 The first problem is the Example 2.1 in [4]. Assume that $A : \mathbb{R}^m \to \mathbb{R}^m$ is defined by

$$Ax = (e^{-x^T Qx} + \beta)(Px + q)$$

where Q is a positive definite matrix, P is a positive semidefinite matrix, $q \in \mathbb{R}^m$ and $\beta > 0$. Observe that A is differentiable and there exists $M > 0$ such that $\|\nabla Ax\| \le M$, $x \in \mathbb{R}^m$. Therefore, by the Mean Value Theorem A is Lipschitz continuous. Also, A is pseudo-monotone but not monotone.

Let $C := \{x \in \mathbb{R}^m | Bx \le b\}$, where B is a matrix of size $l \times m$ and $b \in \mathbb{R}^l_+$ with $l = 10$.

For all tests, we take $\beta = 0.01$, $P = R^T R$, $Q = U^T U$ with all entries of matrices $R, U \in \mathbb{R}^{m\times m}$ and vector $q \in \mathbb{R}^m$ are generated randomly from a normal distribution with mean zero and unit variance. B is a random matrix and random vector $b \in \mathbb{R}^l$ with non-negative entries. The starting points are $x_0 = (1, 1, \ldots, 1) \in \mathbb{R}^m$ and $\lambda_0 = 0.5$

Table 1 Numerical results obtained by other algorithms

Methods	m = 100		m = 150		m = 200	
	Sec.	Iter.	Sec.	Iter.	Sec.	Iter.
Proposed Alg. 3.1	2.4327	24	4.1211	27	5.5196	40
Yang et al. Alg. 2	5.0123	116	9.9642	156	11.6552	216

Table 2 Numerical results of all algorithms with different x_0

Methods	$x_0 = \frac{(sin(-3*t)+cos(-10*t))}{300}$			$x_0 = \frac{(t^2-\exp(-t))}{200}$		
	Sec.	Iter.	Error.	Sec.	Iter.	Error.
Proposed Alg. 3.1	0.1	100	9.9065e-05	0.0694	77	9.9488e-05
Yang et al. Alg. 2	0.45	500	0.0034	0.45	500	0.0026
Censor et al. Alg. 1	0.3	500	0.0034	0.3	500	0.0026
Kraikaew et al. Alg. 3.1	0.32	500	0.0034	0.32	500	0.0026

for Algorithms. To terminate Algorithms, we use the condition $\|x_n - y_n\| \le \epsilon$ with $\epsilon = 10^{-3}$. We choose $\gamma_n = \frac{1}{(n+3)}$ and $\mu = 0.05$ for all algorithms and $\beta_n = \frac{1}{10}(1-\gamma_n)$ for Algorithm 3.1. The numerical results are described in Table 1

Problem 2 Let $H = L^2([0, 1])$ with the norm $\|\cdot\|$ and the inner product $\langle x, y\rangle$ defined by

$$\|x\| = (\int_0^1 |x(t)|^2 dt)^{\frac{1}{2}}, \quad \langle x, y\rangle = \int_0^1 x(t)y(t)dt, \quad \forall x, y \in H,$$

respectively. The operator $A : H \to H$ is defined by

$$Ax(t) = \max\{0, x(t)\}, \quad \forall x \in H,\ t \in [0, 1].$$

It can be easily verified that A is 1-Lipschitz-continuous and monotone. The feasible set $C := \{x \in H : \|x\| \le 1\}$ be the unit ball. Observe that $0 \in VI(C, A)$ and so $VI(C, A) \neq \emptyset$. For all tests, we take $\lambda = \lambda_0 = 0.5$ for all algorithms. We choose $\gamma_n = \frac{1}{(n+3)}$, $\mu = 0.05$ for Algorithm 2 of Yang et al. Algorithms 3.1 and $\gamma_n = \frac{1}{(n+3)}$ for Algorithm 3.1 of Kraikaew et al. $\beta_n = \frac{1}{10}(1-\gamma_n)$ for Algorithms 3.1. To terminate Algorithms, we use the condition $\|x_n - 0\| \le \epsilon$ with $\epsilon = 10^{-4}$ or iterations ≥ 500. The numerical results are described in Table 2.

5 Conclusions

We proposed a new modified subgradient extragradient method for solving the pseudomonotone variational inequality problem in real Hilbert spaces. To obtain the strong convergence theorem, we combined by the subgradient extragradient method and the Mann type method [31]. The advantages of the proposed algorithm don't need any requirement of additional projections and the knowledge of the Lipschitz constant of the mapping. Further, we gave several numerical experiments to illustrate the performance of the proposed algorithm with the known algorithms.

Acknowledgements The authors would like to thank two anonymous reviewers for their comments on the manuscript which helped us very much in improving and presenting the original version of this paper.

References

1. Anh, P.K., Thong, D.V., Vinh, N.T.: Improved inertial extragradient methods for solving pseudo-monotone variational inequalities. Optimization (2020). https://doi.org/10.1080/02331934.2020.1808644
2. Bauschke, H.H., Combettes, P.L.: A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces. Mathem. Oper. Res. **26**, 248–264 (2001)
3. Bauschke, H.H., Combettes, P.L.: Construction of best Bregman approximations in reflexive Banach spaces. Proc. Am. Mathem. Soc. **131**, 3757–3766 (2003)
4. Bot, R.I., Csetnek, E.R., Vuong, P.T.: The forward-backward-forward method from discrete and continuous perspective for pseudo-monotone variational inequalities in Hilbert spaces. Eur. J. Oper. Res. **287**, 49–60 (2020)
5. Burachik, R.S., Lopes, J.O., Svaiter, B.F.: An outer approximation method for the variational inequality problem. SIAM J. Control Optim. **43**, 2071–2088 (2005)
6. Cai, G., Gibali, A., Iyiola, O.S., Shehu, Y.: A new double projection method for solving variational inequalities in Banach space. J. Optim. Theory Appl. **78**, 219–239 (2018)
7. Cegielski, A.: Iterative Methods for Fixed Point Problems in Hilbert Spaces. Lecture Notes in Mathematics. Springer, Berlin (2012)
8. Denisov, S.V., Semenov, V.V., Chabak, L.M.: Convergence of the modified extragradient method for variational inequalities with non-Lipschitz operators. Cybern. Syst. Anal. **51**, 757–765 (2015)
9. Censor, Y., Gibali, A., Reich, S.: Algorithms for the split variational inequality problem. Numer. Algorithms **56**, 301–323 (2012)
10. Censor, Y., Gibali, A., Reich, S.: The subgradient extragradient method for solving variational inequalities in Hilbert space. J. Optim. Theory Appl. **148**, 318–335 (2011)
11. Censor, Y., Gibali, A., Reich, S.: Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space. Optim. Meth. Softw. **26**, 827–845 (2011)
12. Censor, Y., Gibali, A., Reich, S.: Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space. Optimization **61**, 1119–1132 (2011)
13. Cottle, R.W., Yao, J.C.: Pseudo-monotone complementarity problems in Hilbert space. J. Optim. Theory Appl. **75**, 281–295 (1992)
14. Facchinei, F., Pang, J.S.: Finite-Dimensional Variational Inequalities and Complementarity Problems. Springer Series in Operations Research. Springer, New York (2003)
15. Fichera, G.: Sul problema elastostatico di Signorini con ambigue condizioni al contorno. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. 8 (34): 138-142 (1963)
16. Fichera, G.: Problemi elastostatici con vincoli unilaterali: il problema di Signorini con ambigue condizioni al contorno. Atti Accad. Naz. Lincei, Mem., Cl. Sci. Fis. Mat. Nat., Sez. I, VIII. Ser. **7**, 91-140 (1964)
17. Goebel, K., Reich, S.: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Marcel Dekker, New York (1984)

18. Khanh, P.D., Vuong, P.T.: Modified projection method for strongly pseudomonotone variational inequalities. J. Glob. Optim. **58**, 341–350 (2014)
19. Khanh, P.Q., Thong, D.V., Vinh, N.T.: Versions of the subgradient extragradient method for pseudomonotone variational inequalities. Acta. Appl. Math. **170**, 319–345 (2020)
20. Karamardian, S., Schaible, S.: Seven kinds of monotone maps. J. Optim. Theory Appl. **66**, 37–46 (1990)
21. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and their Applications. Academic, New York (1980)
22. Kraikaew, R., Saejung, S.: Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces. J. Optim. Theory Appl. **163**, 399–412 (2014)
23. Korpelevich, G.M.: The extragradient method for finding saddle points and other problems. Ekon. Matem. Metody. **12**, 747–756 (1976)
24. Konnov, I.V.: Combined Relaxation Methods for Variational Inequalities. Springer, Berlin (2001)
25. Dong, Q.L., Cho, Y.J., Zhong, L.L., Rassias, MTh: Inertial projection and contraction algorithms for variational inequalities. J. Glob. Optim. **70**, 687–704 (2018)
26. Dong, Q.L., Cho, Y.J., Rassias, ThM: The projection and contraction methods for finding common solutions to variational inequality problems. Optim. Lett. **12**, 1871–1896 (2018)
27. Dong, Q.L., Tang, Y.C., Cho, Y.J., Rassias, ThM: Optimal choice of the step length of the projection and contraction methods for solving the split feasibility problem. J. Glob. Optim. **71**, 341–360 (2018)
28. Hieu, D.V., Cho, Y.J., Xiao, Y.: Modified extragradient algorithms for solving equilibrium problems. Optimization **67**, 2003–2029 (2018)
29. Malitsky, Y.V., Semenov, V.V.: A hybrid method without extrapolation step for solving variational inequality problems. J. Glob. Optim. **61**, 193–202 (2015)
30. Malitsky, Y.V.: Projected reflected gradient methods for monotone variational inequalities. SIAM J. Optim. **25**, 502–520 (2015)
31. Mann, W.R.: Mean value methods in iteration. Proc. Am. Math. Soc. **4**, 506–510 (1953)
32. Reich, S., Thong, D.V., Dong, Q.L., Xiao, H.L., Dung, V.T.: New algorithms and convergence theorems for solving variational inequalities with non-Lipschitz mappings. Numer. Algorithms (2020). https://doi.org/10.1007/s11075-020-00977-8
33. Saejung, S., Yotkaew, P.: Approximation of zeros of inverse strongly monotone operators in Banach spaces. Nonlinear Anal. **75**, 742–750 (2012)
34. Shehu, Y., Iyiola, O.S.: Strong convergence result for monotone variational inequalities. Numer. Algorithms **76**, 259–282 (2017)
35. Shehu, Y., Iyiola, O.S.: Projection methods with alternating inertial steps for variational inequalities: weak and linear convergence. Appl. Numer. Math. **157**, 315–337 (2020)
36. Solodov, M.V., Svaiter, B.F.: A new projection method for variational inequality problems. SIAM J. Control Optim. **37**, 765–776 (1999)
37. Stampacchia, G.: Formes bilineaires coercitives sur les ensembles convexes. C. R. Acad. Sci. **258**, 4413–4416 (1964)
38. Thong, D.V., Hieu, D.V.: Weak and strong convergence theorems for variational inequality problems. Numer. Algorithms **78**, 1045–1060 (2018)
39. Thong, D.V., Hieu, D.V.: Modified subgradient extragradient method for variational inequality problems. Numer. Algorithms **79**, 597–610 (2018)
40. Vuong, P.T.: On the weak convergence of the extragradient method for solving pseudomonotone variational inequalities. J. Optim. Theory Appl. **176**, 399–409 (2018)
41. Yang, J., Liu, H.: Strong convergence result for solving monotone variational inequalities in Hilbert space. Numer. Algorithms **80**, 741–752 (2019)
42. Yang, J., Liu, H., Liu, Z.: Modified subgradient extragradient algorithms for solving monotone variational inequalities. Optimization **67**, 2247–2258 (2018)
43. Yao, Y., Marino, G., Muglia, L.: A modified Korpelevich's method convergent to the minimum-norm solution of a variational inequality. Optimization **63**, 559–569 (2014)