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ORIGINAL PAPER

On the zeros of lacunary-type polynomials

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Abstract
Let $p \geq 2$ be an integer, $M > 0$ be a real number and

$$\mathcal{C}(p, M) = \Big\{ z^n + a_{n-p} z^{n-p} + \cdots + a_1 z + a_0 \,\Big|\, \max_{0 \leq j \leq n-p} |a_j| = M,\ n = p, p+1, \ldots \Big\},$$

where the coefficients a_j $(j = 0, 1, \ldots, n-p)$ are complex numbers. Guggenheimer (Am Math Mon 71:54–55, 1964) and Aziz and Zargar (Proc Indian Acad Sci 106:127–132, 1996) proved that if $P \in \mathcal{C}(p, M)$, then all zeros of P lie in the disk $|z| < \delta(p, M)$, where $\delta(p, M)$ is the only positive solution of $x^p - x^{p-1} = M$. We show that $\delta(p, M)$ is the best possible value. Moreover, we present some monotonicity/concavity/convexity properties and limit relations of $\delta(p, M)$.

Keywords Polynomials · Zeros · Optimal bound · Monotonic · Concave · Convex

1 Introduction

Finding bounds for the zeros of polynomials is a classical problem which attracted (and still attracts) the attention of numerous mathematicians. A well-known result

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published by Cauchy in 1829 states that if

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

is a complex polynomial of degree n, then all zeros of P lie in the disk

$$|z| \leq 1 + \max_{0 \leq j \leq n-1} |a_j|.$$

In the literature, various refinements of Cauchy's theorem and many related results on the location of the zeros of polynomials are given. For more information on this subject we refer to Milovanović et al. [4, Chapter 3] and Rahman and Schmeisser [5, Chapter 8], as well as the references cited therein.

Our work has been inspired by two interesting papers of Guggenheimer [2] and Aziz and Zargar [1], who studied the following class of lacunary-type polynomials,

$$\mathcal{C}(p, M) = \left\{ z^n + a_{n-p}z^{n-p} + \cdots + a_1 z + a_0 \,\middle|\, \max_{0 \leq j \leq n-p} |a_j| = M,\ n = p, p+1, \ldots \right\}.$$

Here, $p \geq 2$ is an integer, $M > 0$ is a real number and a_j $(j = 0, 1, \ldots, n-p)$ are complex numbers.

They proved the following result.

Proposition 1.1 *Let $P \in \mathcal{C}(p, M)$. All zeros of P lie in disk*

$$|z| < \delta(p, M), \tag{1.1}$$

where $\delta(p, M)$ is the only positive solution of

$$x^p - x^{p-1} = M. \tag{1.2}$$

Remark 1.2 All solutions of the algebraic equation (1.2) for $p \leq 4$ can be obtained in symbolic form using the well-known software packages MATHEMATICA, MAPLE or MATLAB. In this paper all computations were performed in MATHEMATICA, Ver. 12.1.1, on MacBook Pro (2017), OS Catalina Ver. 10.15.5 in both, symbolic and numerical mode.

The corresponding command in MATHEMATICA for $p = 2$ is

```
TeXForm[x/.Solve[x^2-x-M == 0, x]]
```

and then we obtain the two solutions in `TeXForm`

$$\left\{ \frac{1}{2}\left(1 - \sqrt{4M+1}\right), \frac{1}{2}\left(\sqrt{4M+1} + 1\right) \right\}.$$

In order to get only one solution, e.g. for $p = 3$ there is only one real positive solution, we use the following command

```
TeXForm[x/.Solve[x^3-x^2-M == 0, x][[1]]]
```

and we get

$$\frac{1}{3}\left(\frac{\sqrt[3]{3\sqrt{3}\sqrt{27M^2+4M}+27M+2}}{\sqrt[3]{2}}+\frac{\sqrt[3]{2}}{\sqrt[3]{3\sqrt{3}\sqrt{27M^2+4M}+27M+2}}+1\right).$$

This expression, as well as one for $p = 4$, can be written in a simpler form.

Thus, using MATHEMATICA we easily determine $\delta(p, M)$ for $p = 2, 3, 4$, in symbolic form,

$$\delta(2, M) = \frac{1}{2}\Big(\sqrt{4M+1}+1\Big), \quad \delta(3, M) = \frac{1}{3}\Big(1+U+\frac{1}{U}\Big),$$
$$\delta(4, M) = \frac{1}{4}\Big(\sqrt{\frac{16M}{V}\Big(\frac{2}{3}\Big)^{1/3} - 2V\Big(\frac{2}{3}\Big)^{2/3}+\frac{2}{W}+2}+W+1\Big),$$

where

$$U = U(M) = \sqrt[3]{\frac{1}{2}\Big(3\sqrt{3}\sqrt{27M^2+4M}+27M+2\Big)},$$
$$V = V(M) = \sqrt[3]{\sqrt{3}\sqrt{256M^3+27M^2}-9M},$$
$$W = W(M) = \sqrt{2V\Big(\frac{2}{3}\Big)^{2/3} - \frac{16M}{V}\Big(\frac{2}{3}\Big)^{1/3}+1}.$$

The special case $M = 1$ leads to

$$\delta(2, 1) = \frac{1}{2}(\sqrt{5}+1) = 1.61803398\ldots,$$
$$\delta(3, 1) = \frac{1}{3}+\sqrt[3]{\frac{29}{54}+\frac{1}{18}\sqrt{93}}+\sqrt[3]{\frac{29}{54}-\frac{1}{18}\sqrt{93}} = 1.46557123\ldots,$$
$$\delta(4, 1) = \frac{1}{4}\left(1+W(1)+\sqrt{2+\frac{2}{W(1)}-2V(1)\left(\frac{2}{3}\right)^{2/3}+\frac{16}{V(1)}\left(\frac{2}{3}\right)^{1/3}}\right)$$
$$= 1.38027756\ldots,$$

because of

$$V(1) = \sqrt[3]{\sqrt{849}-9} = 2.72062866\ldots,$$
$$W(1) = \sqrt{2\left(\frac{2}{3}\right)^{2/3}V(1) - \frac{16}{V(1)}\sqrt[3]{\frac{2}{3}}+1} = 0.12221011\ldots .$$

For $p \geq 5$ we must use a numerical version of the command for solving equations `NSolve`, with an optional parameter `WorkingPrecision` (`WP`) that specifies how many digits of precision should be maintained in internal computations. Setting `WorkingPrecision->MachinePrecision` causes all internal computations to be done with machine numbers. In this case, this optional parameter can be omitted. Using

```
delta[p_,M_,WP_]:=
x/.NSolve[x^p-x^(p-1)==M,x,WorkingPrecision->WP][[p]]
```

we obtain $\delta(p, M)$, with a given `WP`. For example, `delta[7,10,20]` gives 1.5987802009531565756.

It is natural to ask whether the bound $\delta(p, M)$ given in (1.1) can be replaced by a smaller value. We show that the answer to this question is "no". This means that the only positive solution of (1.2) is the optimal bound for the zeros of the polynomials given in $\mathcal{C}(p, M)$.

Theorem 1.3 *Let $p \geq 2$ and $M > 0$ be fixed numbers. The value $\delta(p, M)$ given in Proposition 1.1 is best possible.*

In the next section, we offer a proof of Theorem 1.3 and in Sect. 3, we present several properties of $\delta(p, M)$.

2 Proof of the main result

Proof of Theorem 1.3 We assume (for a contradiction) that in (1.1) the value $\delta = \delta(p, M)$ can be replaced by a smaller expression, say $\delta^* = \delta^*(p, M)$. We define

$$F(z) = F_n(z) = z^n - M(z^{n-p} + \cdots + z + 1) \quad (n \geq p).$$

Then, $F \in \mathcal{C}(p, M)$. Let

$$f(x) = x^p - x^{p-1} - M.$$

We have $f(1) = -M < 0$ and $f(\delta) = 0$ which implies that $\delta > 1$. In what follows, let $n \geq [p + 1/M]$. (As usual, $[x]$ denotes the greatest integer not greater than x.) Since $F(0) = -M$ and $\lim_{z\to\infty} F(z) = \infty$, we conclude that F has precisely one positive zero, $r = r_n(p, M)$. We have $F(1) = 1 - (n + 1 - p)M < 0$ and $F(\delta) = M/(\delta - 1) > 0$. This gives

$$1 < r_n(p, M) < \delta.$$

The function

$$G(z) = G_n(z) = (z - 1)F(z) = z^{n+1} - z^n - Mz^{n+1-p} + M$$

has precisely two positive zeros, 1 and r. Using

$$p + \frac{1}{M} < \left[p + \frac{1}{M}\right] + 1 \leq n + 1$$

gives

$$G'(1) = 1 - M(n + 1 - p) < 0.$$

This implies that there exists a number $\varepsilon = \varepsilon(p, M) > 0$ such that G is negative on $(1, 1 + \varepsilon)$. Let $x^* \in (1, 1 + \varepsilon)$. Since

$$G(x^*) < 0 \quad \text{and} \quad \lim_{z \to \infty} G(z) = \infty,$$

G has a zero on (x^*, ∞). It follows that $x^* < r$. Otherwise, G has three positive zeros which is not true. Thus,

$$1 < x^* < r_n(p, M) < \delta.$$

Since the sequence (r_n) is bounded, there exists a convergent subsequence (r_{j_n}). Let

$$\lim_{n \to \infty} r_{j_n} = s.$$

Then, we have $s \geq x^* > 1$. This implies that

$$\lim_{n \to \infty} (r_{j_n})^{j_n + p - 1} = \infty.$$

We have

$$(r_{j_n})^p - (r_{j_n})^{p-1} - M + \frac{M}{(r_{j_n})^{j_n + 1 - p}} = \frac{(r_{j_n} - 1) F_{j_n}(r_{j_n})}{(r_{j_n})^{j_n + 1 - p}} = 0.$$

Next, we let $n \to \infty$ and obtain

$$s^p - s^{p-1} - M = 0.$$

Hence, $s = \delta$.

By assumption, we have

$$r_n = |r_n| < \delta^* < \delta.$$

Thus,

$$r_{j_n} < \delta^* < \delta.$$

We let $n \to \infty$. Then,

$$\delta = s \leq \delta^* < \delta.$$

A contradiction. It follows that in (1.1) the value $\delta = \delta(p, M)$ is best possible. □

3 Properties of $\delta(p, M)$

First, we present two limit relations.

Theorem 3.1 *Let $p \geq 2$. Then,*

$$\lim_{M\to\infty} \frac{\delta(p, M)}{M^{1/p}} = 1 \tag{3.1}$$

and

$$\lim_{M\to\infty} \big(\delta(p, M) - M^{1/p}\big) = \frac{1}{p}.$$

Proof Let $\delta = \delta(p, M)$. Since

$$M^{1/p} = \delta\Big(1 - \frac{1}{\delta}\Big)^{1/p} < \delta,$$

we conclude that if $M \to \infty$, then $\delta \to \infty$. Thus,

$$\lim_{M\to\infty} \frac{\delta}{M^{1/p}} = \lim_{M\to\infty} \Big(1 - \frac{1}{\delta}\Big)^{-1/p} = 1.$$

We have

$$\delta - M^{1/p} = \frac{1 - t^{1/p}}{1 - t}$$

with $t = 1 - 1/\delta$. Thus, if $M \to \infty$, then $t \to 1$ and

$$\frac{1 - t^{1/p}}{1 - t} \to \frac{1}{p}.$$

□

We conclude the paper with some monotonicity and concavity/convexity properties of $\delta(p, M)$. The following preliminaries are helpful. We make use of the notations

$$h_x = \frac{\partial}{\partial x} h(x, y), \quad h_y = \frac{\partial}{\partial y} h(x, y),$$
$$h_{xx} = \frac{\partial^2}{\partial x^2} h(x, y), \quad h_x(r, s) = \frac{\partial}{\partial x} h(x, y)\Big|_{(x,y)=(r,s)}.$$

First, we show that $M \mapsto \delta(p, M)$ is differentiable on $(0, \infty)$. Let

$$H(x, M) = x^p - x^{p-1} - M.$$

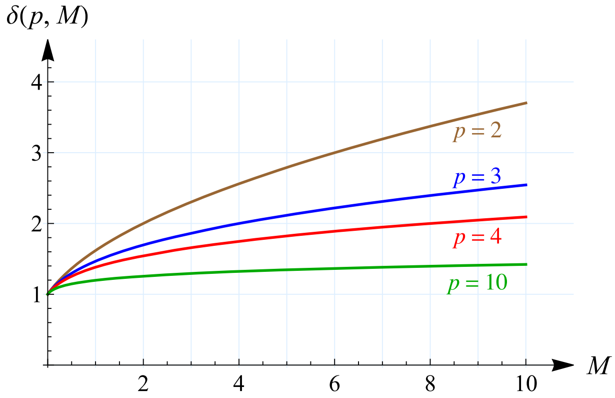

Fig. 1 Graphics of $M \mapsto \delta(p, M)$ for $p = 2, 3, 4, 10$

The function δ is determined implicitly by $H(\delta, M) = 0$. Since $\delta > 1$, we obtain

$$p\delta^{p-1} - (p-1)\delta^{p-2} > 0. \tag{3.2}$$

According to the classical Implicit Function Theorem (see, for example, [3, p. 8, Theorem 1.3.1]), a sufficient condition for δ to be well-defined as an implicit function of M and for $\delta(p, M)$ to be differentiable (with respect to M) is

$$H_x(\delta, M) = p\delta^{p-1} - (p-1)\delta^{p-2} \neq 0,$$

which is implied by (3.2).

Instead of considering only integer values for p, we extend the definition of $\delta(p, M)$ to real numbers $p \geq 1$. Knowing that $p \mapsto \delta(p, M)$ is differentiable, we are allowed to differentiate both sides of

$$\delta(p, M)^p - \delta(p, M)^{p-1} = M \tag{3.3}$$

with respect to p, leading to

$$\delta^p \log(\delta) + p\delta^{p-1}\delta_p - \delta^{p-1}\log(\delta) - (p-1)\delta^{p-2}\delta_p = 0. \tag{3.4}$$

Now, the pair of equations $H(\delta, M) = 0$ and (3.3) is a set of equations implicitly defining the pair of functions (δ, δ_p) in terms of p. The Implicit Function Theorem can be applied again to justify that (δ, δ_p) is well defined and differentiable, provided that the corresponding Jacobian determinant is nonzero. The latter fact is straightforward to verify and we omit the details.

We are now in a position to prove the following theorem.

Theorem 3.2 *(a) Let* $M > 0$. *The sequence* $(\delta(p, M))_{p \geq 2}$ *is strictly decreasing and strictly convex with* $\lim_{p\to\infty} \delta(p, M) = 1$.

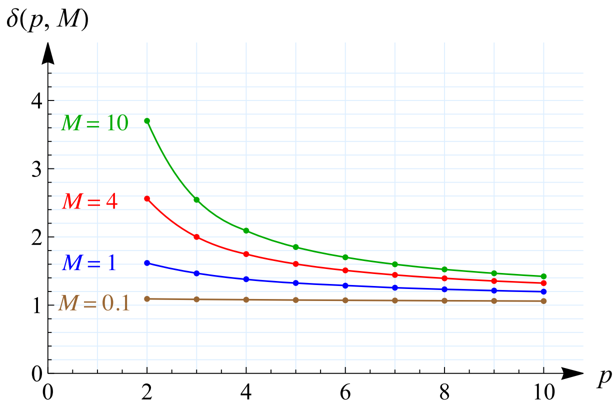

Fig. 2 Graphics of the sequences $(\delta(p, M))_{2\le p\le 10}$ for $M = 0.1, 1, 4, 10$

(b) *Let* $p \ge 2$. *The function* $M \mapsto \delta(p, M)$ *is strictly increasing and strictly concave on* $(0, \infty)$ *with* $\lim_{M\to\infty} \delta(p, M) = \infty$.

Proof (a) (i) We have

$$\delta(p, M)^p - \delta(p, M)^{p-1} = M = \delta(p+1, M)^{p+1} - \delta(p+1, M)^p.$$

Since $\delta(p, M) > 1$, we obtain

$$\begin{aligned}\delta(p+1, M)^p\big(\delta(p+1, M) - 1\big) &= \frac{1}{\delta(p, M)}\delta(p, M)^p\big(\delta(p, M) - 1\big)\\ &< \delta(p, M)^p\big(\delta(p, M) - 1\big).\end{aligned}$$

Thus,

$$\Theta\big(p, \delta(p+1, M)\big) < \Theta\big(p, \delta(p, M)\big),$$

with

$$\Theta(p, x) = x^p(x-1).$$

The function $x \mapsto \Theta(p, x)$ is strictly increasing on $[p/(p+1), \infty)$, so that we obtain $\delta(p+1, M) < \delta(p, M)$.

(ii) Let $\delta = \delta(p, M)$. We claim that $\delta_{pp} > 0$. The discrete version then follows as a corollary.

From (3.4) we obtain

$$-\delta_p = \frac{\delta(\delta-1)\log(\delta)}{p\delta - (p-1)} = \frac{u}{v}, \quad \text{say.} \tag{3.5}$$

Since

$$u_p = \big((2\delta - 1)\log(\delta) + \delta - 1\big)\delta_p \quad \text{and} \quad v_p = p\delta_p + \delta - 1,$$

we get

$$\begin{aligned} v^2\delta_{pp} &= -u_p v + u v_p \\ &= -\big((2\delta - 1)\log(\delta) + \delta - 1\big)\big(p\delta - (p-1)\big)\delta_p \\ &\quad + \delta(\delta - 1)\log(\delta)\big(p\delta_p + \delta - 1\big). \end{aligned}$$

Using (3.5) gives

$$\begin{aligned} v^2\delta_{pp} =& \big((2\delta - 1)\log(\delta) + \delta - 1\big)\big(\delta(\delta - 1)\log(\delta)\big) \\ &+ \delta(\delta - 1)\log(\delta)\Big(-\frac{p\delta(\delta - 1)\log(\delta)}{p\delta - (p-1)} + \delta - 1\Big). \end{aligned}$$

It follows that

$$\frac{v^2}{\delta(\delta - 1)\log(\delta)}\delta_{pp} = (2\delta - 1)\log(\delta) + 2(\delta - 1) - \frac{p\delta(\delta - 1)}{p\delta - (p-1)}\log(\delta).$$

Since

$$\frac{p}{p\delta - (p-1)} < \frac{1}{\delta - 1},$$

we obtain

$$\begin{aligned} \frac{v^2}{\delta(\delta - 1)\log(\delta)}\delta_{pp} &> (2\delta - 1)\log(\delta) + 2(\delta - 1) - \delta\log(\delta) \\ &= (\delta - 1)\big(2 + \log(\delta)\big) > 0. \end{aligned}$$

Thus, $\delta_{pp} > 0$.

(iii) Let $\lim_{p\to\infty}\delta(p, M) = a$. Then, $a \geq 1$. We assume that $a > 1$. Then, $\lim_{p\to\infty}\delta(p, M)^p = \infty$, so that

$$\frac{M}{\delta(p, M)^p} = 1 - \frac{1}{\delta(p, M)} \tag{3.6}$$

leads to $0 = 1 - 1/a > 0$. This contradiction yields $a = 1$.

(b) (i) Let $0 < M_1 < M_2$. Then,

$$\delta(p, M_1)^p - \delta(p, M_1)^{p-1} = M_1 < M_2 = \delta(p, M_2)^p - \delta(p, M_2)^{p-1}.$$

This gives

$$\Theta\big(p-1, \delta(p, M_1)\big) < \Theta\big(p-1, \delta(p, M_2)\big).$$

Since $\delta(p, M) > 1$ and $x \mapsto \Theta(p-1, x)$ is strictly increasing on $(1, \infty)$, we obtain $\delta(p, M_1) < \delta(p, M_2)$.

(ii) Let $M > 0$. From (3.3) we obtain by differentiation with respect to M,

$$p\delta^{p-1}\delta_M - (p-1)\delta^{p-2}\delta_M = 1.$$

This leads to

$$\delta_M = \delta^{2-p} \cdot \big(p\delta - (p-1)\big)^{-1}.$$

It follows that δ_M is the product of a decreasing function and a strictly decreasing function. Since both factors are positive, we conclude that δ_M is strictly decreasing on $(0, \infty)$. This means that $M \mapsto \delta(p, M)$ is strictly concave on $(0, \infty)$.

(iii) Applying (3.1) and (3.6) gives

$$\lim_{M\to\infty} \frac{1}{\delta(p, M)} = \lim_{M\to\infty} \Big(1 - \frac{M}{\delta(p, M)^p}\Big) = 0.$$

Thus, if $M \to \infty$, then $\delta(p, M) \to \infty$. □

Remark 3.3 In Figures 1 and 2, we illustrate properties of the function $M \mapsto \delta(p, M)$ and the sequence $(\delta(p, M))_{p\geq 2}$, which are proved in Theorem 3.2.

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