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ORIGINAL PAPER

Stochastic variational formulation for a general random time-dependent economic equilibrium problem

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Abstract

In the paper, in a Hilbert space setting, a random time-dependent oligopolistic market equilibrium problem in presence of both production and demand excesses is studied and the random time-dependent Cournot–Nash equilibrium principle by means of a stochastic variational inequality is characterized. Then, some existence results to such problem are established and the stochastic continuity of the equilibrium solution is proved. Moreover a simple numerical example illustrates the theoretical results.

Keywords Random time-dependent Cournot–Nash equilibrium principle · Oligopolistic market equilibrium problem · Stochastic variational inequalities

1 Introduction

The purpose of this note is to combine the new advances of the theory of stochastic and time-dependent variational inequalities with the Nash equilibrium game, and to propose an effective model of a oligopolistic market equilibrium problem. Taking into account these new tools, we are able to generalize results which have been obtained in the field of oligopolistic markets.

In the last years many authors (see [8–13]) developed the study of stochastic variational inequalities and random equilibrium problems. Recently, a comprehensive study on the stochastic variational inequalities with anticipativity in a dynamic multistage setting is done in [19].

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In the last decade, the time-dependent variational formulation of the oligopolistic market equilibrium problem was introduced and intensively studied starting by [1]. In [3,4] the authors observed that during an economic crisis period the presence of production excesses can be due to a demand decrease in demand markets and, on the other hand, the presence of demand excesses may occur when the supply cannot satisfy the demand, especially for fundamental goods. Moreover, the presence of both production and demand excesses is a consequence of the fact that the physical transportation of commodities between a firm and a demand market is evidently limited, therefore, it is more realistic that some firms produce more fundamental good than they can send to all the demand markets and, on the other hand, some of the demand markets require more goods.

The model in presence of uncertainty in which both production and demand excesses occur was analyzed in [2]. The development of the oligopolistic market equilibrium problem under conditions of uncertainty arises because the constraints or the data are often variable over time in a non-regular and unpredictable manner. It is sufficient to think about unpredictable events and sudden accidents. A suitable choice is one in which it is possible to handle random constraints. We consider for our model a Hilbert space setting, which allows us to obtain existence results and to perform a complete duality theory.

In this setting, we focus our attention on the study of a more general oligopolistic market equilibrium problem with uncertainty and time-dependence. The time-dependent formulation of equilibrium problems allows one to explore the dynamics of adjustment processes in which a delay on time response is operating (as Beckmann and Wallace stressed in [6]). In particular, we propose a time-dependent oligopolistic market equilibrium problem in presence of both production and demand excesses in condition of uncertainty. Recently, a new time-dependent weighted transportation model in conditions of uncertainty was introduced in [5].

The paper is structured as follows. After this introductory section, in Sect. 2 we introduce the random time-dependent oligopolistic market equilibrium problem. In Sect. 3, we show some existence results for the random time-dependent equilibrium distribution. In Sect. 4, we recall the Kuratowski's set convergence and some stochastic continuity definitions. After that, the Kuratowski's set convergence property for the feasible set of our model is obtained. Thank to this property, we are able to prove the stochastic continuity of the solution to the stochastic variational inequality which expresses the random time-dependent Cournot–Nash equilibrium principle. In Sect. 5 the theoretical results are illustrated with the help of a two-player example.

2 The model

In this section we extend the random model for the oligopolistic market to the time-dependent case. This generalization seems reasonable since the time-independent model is quite unrealistic.

The "natural" setting of the random time-dependent oligopolistic market equilibrium problem with excesses involving the time and random variable will be the Hilbert space $L^2([0,T]\times\Omega,\mathbb{R}^k,\mathbb{P})$, endowed with the inner product denoted by $\langle\langle\cdot,\cdot\rangle\rangle$.[1]

The model we will consider is the following: let P_i, $i=1,\ldots,m$, be m firms, that produce a homogeneous commodity and n demand markets Q_j, $j=1,\ldots,n$, that are generally spatially separated. Assume that the homogeneous commodity, produced by the m firms and consumed by the n markets, is considered depending by random variables. Let p_i, $i=1,\ldots,m$, denote the random variable expressing the nonnegative commodity output produced by firm P_i and suppose that $p_i=p_i(t,\omega)$, $(t,\omega)\in[0,T]\times\Omega$. Let q_j, $j=1,\ldots,n$, denote the random variable expressing the nonnegative demand for the commodity of demand market Q_j, namely $q_j=q_j(t,\omega)$, $(t,\omega)\in[0,T]\times\Omega$. Let x_{ij}, $i=1,\ldots,m$, $j=1,\ldots,n$, denote the random variable expressing the nonnegative commodity shipment between the supply producer P_i and the demand market Q_j, namely $x_{ij}=x_{ij}(t,\omega)$, $(t,\omega)\in[0,T]\times\Omega$. In particular, let us set the vector $x_i(t,\omega)=(x_{i1}(t,\omega),\ldots,x_{in}(t,\omega))$, $i=1,\ldots,m$, $(t,\omega)\in[0,T]\times\Omega$, as the strategy vector for the firm P_i. Let ε_i, $i=1,\ldots,m$, denote the random variable expressing the nonnegative production excess for the commodity of the firm P_i, namely $\varepsilon_i=\varepsilon_i(t,\omega)$, $(t,\omega)\in[0,T]\times\Omega$. Let δ_j, $j=1,\ldots,n$, denote the random variable expressing the nonnegative demand excess for the commodity of the demand market Q_j, namely $\delta_j=\delta_j(t,\omega)$, with $(t,\omega)\in[0,T]\times\Omega$. To be more precise the random variables $p_i, q_j, \varepsilon_i, \delta_j$ lie in the L^2 space as below the x_{ij}.

Let us suppose that the following feasibility conditions hold:

$$p_i(t,\omega)=\sum_{j=1}^{n}x_{ij}(t,\omega)+\varepsilon_i(t,\omega),\quad i=1,\ldots,m,\ \text{a.e. in }[0,T],\ \mathbb{P}-\text{a.s.},$$
$$q_j(t,\omega)=\sum_{i=1}^{m}x_{ij}(t,\omega)+\delta_j(t,\omega),\quad j=1,\ldots,n,\ \text{a.e. in }[0,T],\ \mathbb{P}-\text{a.s.}$$

More precisely, the quantity produced by each firm P_i must be equal to the commodity shipments from that firm to all the demand markets plus the production excess. Moreover, the quantity demanded by each demand market Q_j must be equal to the commodity shipments from all the firms to that demand market plus the demand excess. Taking into account that the production and the demand excesses are non-

[1] In the Hilbert space $L^2([0,T],\mathbb{R}^k,\mathbb{P})$, we define the canonical bilinear form on $L^2([0,T],\mathbb{R}^k,\mathbb{P})^*\times L^2([0,T],\mathbb{R}^k,\mathbb{P})$, by

$$\langle\langle\phi,\mathrm{w}\rangle\rangle:=\int_0^T\int_\Omega\langle\phi(t,\omega),\mathrm{w}(t,\omega)\rangle\,dt\,d\mathbb{P},$$

where $\phi\in(L^2([0,T],\mathbb{R}^k,\mathbb{P}))^*=L^2([0,T],\mathbb{R}^k,\mathbb{P})$, $\mathrm{w}\in L^2([0,T],\mathbb{R}^k,\mathbb{P})$ and

$$\langle\phi(t),\mathrm{w}(t)\rangle=\sum_{l=1}^{k}\phi_l(t)\mathrm{w}_l(t).$$

negative random variables, we can rewrite the feasible conditions in the following equivalent way:

$$\sum_{j=1}^{n} x_{ij}(t,\omega) \leq p_i(t,\omega), \quad i=1,\ldots,m, \text{ a.e. in } [0,T], \ \mathbb{P}-\text{a.s.},$$
$$\sum_{i=1}^{m} x_{ij}(t,\omega) \leq q_j(t,\omega), \quad j=1,\ldots,n, \text{ a.e. in } [0,T], \ \mathbb{P}-\text{a.s.}$$

Furthermore, we assume that the nonnegative commodity shipment between the producer P_i and the demand market Q_j belongs to $L^2([0,T]\times\Omega,\mathbb{R}_+,\mathbb{P})$ and has to satisfy two capacity constraints, namely there exist two nonnegative random variables $\underline{x},\overline{x} \in L^2([0,T]\times\Omega,\mathbb{R}^{mn}_+,\mathbb{P})$ such that

$$0 \leq \underline{x}_{ij}(t,\omega) \leq x_{ij}(t,\omega) \leq \overline{x}_{ij}(t,\omega),$$
$$\forall i=1,\ldots,m, \ \forall j=1,\ldots,n, \text{ a.e. in } [0,T], \ \mathbb{P}-\text{a.s.}$$

As a consequence, the feasible set is given by:

$$\mathbb{K} = \Big\{ x \in L^2([0,T]\times\Omega,\mathbb{R}^{mn},\mathbb{P}) :$$
$$\underline{x}_{ij}(t,\omega) \leq x_{ij}(t,\omega) \leq \overline{x}_{ij}(t,\omega), \ \forall i=1,\ldots,m, \ \forall j=1,\ldots,n, \text{ a.e. in } [0,T], \ \mathbb{P}-\text{a.s.},$$
$$\sum_{j=1}^{n} x_{ij}(t,\omega) \leq p_i(t,\omega), \ \forall i=1,\ldots,m, \text{ a.e. in } [0,T], \ \mathbb{P}-\text{a.s.},$$
$$\sum_{i=1}^{m} x_{ij}(t,\omega) \leq q_j(t,\omega), \ \forall j=1,\ldots,n, \text{ a.e. in } [0,T], \ \mathbb{P}-\text{a.s.} \Big\}. \quad (1)$$

Let us note that $\mathbb{K}$ is a convex, closed and bounded of the Hilbert space $L^2([0,T]\times\Omega,\mathbb{R}^{mn}_+,\mathbb{P})$.

Let us associate a random variable denoting the production cost f_i, $i=1,\ldots,m$, with each firm P_i, and assume that the production cost of a firm P_i may depend upon the entire production pattern, namely, $f_i = f_i(t,x(t,\omega))$, $(t,\omega)\in[0,T]\times\Omega$. Analogously, let us associate a random variable denoting the demand price for unity of the commodity d_j, $j=1,\ldots,n$, with each demand market Q_j, and assume that the demand price of a demand market Q_j may depend upon the entire consumption pattern, namely, $d_j = d_j(t,x(t,\omega))$, $(t,\omega)\in[0,T]\times\Omega$. Since production excesses occur, we consider the random variable g_i, $i=1,\ldots,m$, expressing the storage cost of the commodity produced by the firm P_i and assume that this cost may depend upon the entire production pattern, namely, $g_i = g_i(t,x(t,\omega))$, $(t,\omega)\in[0,T]\times\Omega$. Finally, let c_{ij}, $i=1,\ldots,m$, $j=1,\ldots,n$, denote the random variable expressing the transaction cost, which includes the transportation cost associated with trading of commodities between firm P_i and demand market Q_j. In our model, we assume that the transaction cost depends upon the entire shipment pattern, namely, $c_{ij} =$

$c_{ij}(t, x(t, \omega))$, $(t, \omega) \in [0, T] \times \Omega$. As a consequence, the profit v_i of the firm P_i, $i = 1, \dots, m$, is

$$v_i(t, x(t, \omega)) = \sum_{j=1}^{n} d_j(t, x(t, \omega)) x_{ij}(t, \omega) - f_i(t, x(t, \omega)) + \\ -g_i(t, x(t, \omega)) - \sum_{j=1}^{n} c_{ij}(t, x(t, \omega)) x_{ij}(t, \omega), \quad \text{a.e. in } [0, T], \mathbb{P} - \text{a.s.}, \tag{2}$$

namely, it is equal to the price that the demand markets are disposed to pay minus the production cost, the storage cost and the transportation cost.

In our model the m firms supply the commodity in a noncooperative fashion, each one trying to maximize its own profit function considered the optimal distribution pattern for the other firms, in a nondeterministic framework. We shall make suitable assumptions (as in [2]) on the payoff functions $v_i(t, x(t, \omega))$ in order to determine a nonnegative commodity distribution matrix-function x for which the m firms and the n demand markets will be in a state of equilibrium as defined below with the random generalized Cournot–Nash principle.

Definition 1 A feasible matrix-function $x^* \in \mathbb{K}$ is a random time-dependent oligopolistic market equilibrium distribution if and only if, for each $i = 1, \dots, m$, a.e. in [0,T] and $\mathbb{P}$-a.s., it results

$$v_i(x^*(t, \omega)) \geq v_i(x_i(t, \omega), \hat{x}_i^*(t, \omega)), \tag{3}$$

where $\hat{x}_i^*(t, \omega) = (x_1^*(t, \omega), \dots, x_{i-1}^*(t, \omega), x_{i+1}^*(t, \omega), \dots, x_m^*(t, \omega))$.

Let us denote by $\nabla_D v = \left(\frac{\partial v_i}{\partial x_{ij}} \right)_{\substack{i = 1, \dots, m \\ j = 1, \dots, n}}$. Let us assume the following assumptions:

1. $v_i(\cdot)$ is continuously differentiable for each $i = 1, \dots, m$,,
2. $\nabla_D v(\cdot)$ is a Carathéodory function such that

$$\exists h \in L^2([0, T] \times \Omega, \mathbb{R}_+, \mathbb{P}) : \; \|\nabla_D v(x(t, \omega))\| \leq h(t, \omega) \, \|x(t, \omega)\|, \\ \text{a.e. in } [0, T], \mathbb{P} - \text{a.s.},$$

3. $v_i(\cdot)$ is pseudoconcave with respect to the variable x_i, $i = 1, \dots, m$, namely the following condition holds (see [16])

$$\left\langle \frac{\partial v_i}{\partial x_i}(x_1, \dots, x_i, \dots, x_m), x_i - y_i \right\rangle \geq 0 \\ \Rightarrow v_i(x_1, \dots, x_i, \dots, x_m) \geq v_i(x_1, \dots, y_i, \dots, x_m).$$

Under assumptions 1., 2., 3. on the profit function, it is not difficult to establish the equivalent variational formulation as in the next theorem. The claim follows using the same technique in [4, Theorem 2.2].

Theorem 1 *Let us assume that assumptions 1., 2., 3. are satisfied. $x^* \in \mathbb{K}$ is a random time-dependent oligopolistic market equilibrium if and only if it satisfies the stochastic variational inequality*

$$\begin{aligned} &\langle\!\langle -\nabla_D v(t, x^*), x - x^* \rangle\!\rangle \\ &\quad = \int_\Omega \int_0^T - \sum_{i=1}^m \sum_{j=1}^n \frac{\partial v_i(x^*(\xi,\omega))}{\partial x_{ij}} (x_{ij}(\xi,\omega) - x^*_{ij}(\xi,\omega))\, d\xi\, d\mathbb{P} \geq 0, \\ &\quad \forall x \in \mathbb{K}. \end{aligned} \tag{4}$$

3 Existence results

In this section we will provide existence results for the random time-dependent oligopolistic market equilibrium distribution. Firstly, we recall some definitions. Let K be a subset of a reflexive Banach space X which dual is X^*.

Definition 2 An operator $A : K \to X^*$ is said to be

- *pseudomonotone in the sense of Karamardian* (*K-pseudomonotone*) on K if for every $u, v \in K$

$$\langle\!\langle A(v), u - v \rangle\!\rangle \geq 0 \Longrightarrow \langle\!\langle A(u), u - v \rangle\!\rangle \geq 0;$$

- *strongly pseudomonotone with degree* $\alpha > 0$ on K, if there exists $\eta > 0$ such that, for every $u_1, u_2 \in K$,

$$\langle\!\langle A(v), u - v \rangle\!\rangle \geq 0 \;\Rightarrow\; \langle\!\langle A(u), u - v \rangle\!\rangle \geq \eta \|u - v\|^\alpha;$$

pseudomonotone in the sense of Brézis (B-pseudomonotone) if:

(a) for every sequence $\{u_r\}$ weakly converging to u (shortly, $u_r \rightharpoonup u$) in K and such that $\limsup_r \langle\!\langle A(u_r), u_r - u \rangle\!\rangle \leq 0$ it results that

$$\liminf_r \langle\!\langle A(u_r), u_r - v \rangle\!\rangle \geq \langle\!\langle A(u), u - v \rangle\!\rangle, \quad \forall v \in K,$$

(b) for every $v \in K$ the function $u \to \langle\!\langle A(u), u - v \rangle\!\rangle$ is lower bounded on the bounded subsets of K.

Let us recall also the following definition for a convex subset K of X.

Definition 3 An operator $A : K \to X^*$ is said to be

- *hemicontinuous in the sense of Fan* (*F-hemicontinuous*) if for all $v \in K$, the function

$$K \ni u \longmapsto \langle\langle A(u), u - v \rangle\rangle$$

is weakly lower semi-continuous on K;
- *lower hemicontinuous along line segments* if the function

$$K \ni p \longmapsto \langle\langle A(p), u - v \rangle\rangle$$

is lower semi-continuous for all $u, v \in K$ on the line segment $[u, v]$.

Now we are able to present some existence results for the stochastic variational inequality which expresses the random time-dependent market equilibrium principle.

Theorem 2 *Let K be as in (1). Let us assume that assumptions 1., 2., 3. are satisfied. Let $A : L^2([0, T] \times \Omega, \mathbb{R}^{mn}, \mathbb{P}) \to L^2([0, T] \times \Omega, \mathbb{R}^{mn}, \mathbb{P})$ be the operator defined by $A = \left(-\frac{\partial v_i}{\partial x_{ij}}(x^*)\right)_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$. If A is B-pseudomonotone or F-hemicontinuous, then the stochastic variational inequality:*

$$\left\langle\!\left\langle Ax^*, x - x^* \right\rangle\!\right\rangle \geq 0, \quad \forall x \in \mathbb{K}, \tag{5}$$

admits a solution $x^* \in \mathbb{K}$.

Proof $\mathbb{K}$ is a nonempty, closed, convex and bounded subset of $L^2([0, T] \times \Omega, \mathbb{R}^{mn}, \mathbb{P})$ and therefore it is a weakly compact subset of $L^2([0, T] \times \Omega, \mathbb{R}^{mn}, \mathbb{P})$. Then, the claim is achieved by applying [18, Theorems 2.6 & 2.7].

Moreover, the following result holds:

Theorem 3 *Let K be as in (1). Let us assume that assumptions 1., 2., 3. are satisfied. Let $A : L^2([0, T] \times \Omega, \mathbb{R}^{mn}, \mathbb{P}) \to L^2([0, T] \times \Omega, \mathbb{R}^{mn}, \mathbb{P})$ be the operator defined by $A = \left(-\frac{\partial v_i(x^*)}{\partial x_{ij}}\right)_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$. If A is K-pseudomonotone and lower hemicontinuous along line segments, then the stochastic variational inequality (5) admits a solution $x^* \in \mathbb{K}$.*

Proof Being $\mathbb{K}$ a weakly compact subset of $L^2([0, T] \times \Omega, \mathbb{R}^{mn}, \mathbb{P})$, the claim is achieved making use of [18, Corollary 3.7].

It is not surprising that requiring stronger hypothesis on the operator A, one can be obtained stronger results. In particular, if the operator A, in Theorem 3, is strongly pseudomonotone and lower hemicontinuous along line segments, the solution to (5) exists and is unique.

Let us remark that if we assume that the profit function v is continuously differentiable and verifies the condition:

$$\exists c \geq 0: \ \|\nabla_D v(x(t,\omega))\| \leq c\|x(t,\omega)\| \qquad \text{for} \mathbb{P}\text{-a.s.} \omega \in \Omega, \ \text{a.e.}\, t \in [0,T]$$

then $\nabla_D v$ belongs to the class of *Nemytskii operators* (see [18]) and is lower hemicontinuous along line segments.

4 Stochastic continuity result

In this section a stochastic regularity theorem will be proved. Indeed it will be shown that a solution of (4) is stochastic continuous on $[0,T]$, provided that the feasible set $\mathbb{K}$ defined in (1) verifies Kuratowski's set convergence property.

4.1 Set convergence and stochastic processes

We recall, very briefly, the classical notion of set convergence for a given metric space (X,d), introduced in the 50's by Kuratowski (see [15]).

Let $\{K_r\}_{r\in\mathbb{N}}$ be a sequence of subsets of X. Recall that

$$d - \underline{\lim}_r K_r = \{x \in X: \ \exists \{x_r\}_{r\in\mathbb{N}} \text{ eventually in } K_r \text{ such that } x_r \stackrel{d}{\to} x\},$$

and

$$d - \overline{\lim}_r K_r = \{x \in X: \ \exists \{x_r\}_{r\in\mathbb{N}} \text{ frequently in } K_r \text{ such that } x_r \stackrel{d}{\to} x\},$$

where *eventually* means that there exists $\delta \in \mathbb{N}$ such that $x_r \in K_r$ for any $r \geq \delta$, and *frequently* means that there exists an infinite subset $N \subseteq \mathbb{N}$ such that $x_r \in K_r$ for any $r \in N$ (in this last case, according to the notation given above, we also write that there exists a subsequence $\{x_{k_r}\}_{r\in\mathbb{N}} \subseteq \{x_r\}_{r\in\mathbb{N}}$ such that $x_{k_r} \in K_{k_r}$).

Finally we are now able to recall the Kuratowski's convergence of sets.

Definition 4 We say that $\{K_r\}_{r\in\mathbb{N}}$ converges to some subset $K \subseteq X$ in Kuratowski's sense, and we briefly write $K_r \to K$, if $d - \underline{\lim}_r K_r = d - \overline{\lim}_r K_r = K$. Thus, in order to verify that $K_r \to K$, it suffices to check that

- $d - \overline{\lim}_r K_r \subseteq K$, i.e. for any sequence $\{x_r\}_{r\in\mathbb{N}}$ frequently in K_r such that $x_r \stackrel{d}{\to} x$ for some $x \in S$, then $x \in K$;
- $K \subset d - \underline{\lim}_r K_r$, i.e. for any $x \in K$ there exists a sequence $\{x_r\}_{r\in\mathbb{N}}$ eventually in K_r such that $x_r \stackrel{d}{\to} x$.

Remark 1 We observe that the set convergence in Kuratowski's sense can also be expressed as follows:

let (X, d) be a metric space and K a nonempty, closed and convex subset of X. A sequence of nonempty, closed and convex sets $\{K_r\}_{r\in\mathbb{N}}$ of X converges to K in Kuratowski's sense, as $r \to +\infty$, i.e. $K_r \to K$, if and only if

(K1) for any $x \in K$, there exists a sequence $\{x_r\}_{r\in\mathbb{N}}$ converging to $x \in X$ such that x_r lies in K_r for all $r \in \mathbb{N}$,

(K2) for any subsequence $\{x_r\}_{r\in\mathbb{N}}$ converging to $x \in X$ such that x_r lies in K_r, for all $r \in \mathbb{N}$, then the limit x belongs to K.

Since we will show, under suitable hypothesis, that the solution to a stochastic variational inequality is a continuous stochastic process (see e.g. [7] for a general reference), it is useful to recall some of the most common definitions of continuity for a stochastic process present in literature (see e.g. [7,14]).

Definition 5 A stochastic process $X_t(\omega) = X(t, \omega)$ is said to be

(i) *mean-square continuous* at t if $\mathbb{E}(X_t^2) < \infty$ and

$$\lim_{s\to t} \mathbb{E}(|X_s - X_t|^2) = 0; \tag{6}$$

(ii) *stochastic continuous* at t if, for every $\epsilon > 0$, it holds

$$\lim_{s\to t} \mathbb{P}\Big(\{\omega \in \Omega : |X(s, \omega) - X(t, \omega)| \geq \epsilon\}\Big) = 0; \tag{7}$$

(iii) *sample-path continuous* if, for $\mathbb{P}$-almost every $\omega \in \Omega$, $s \to t$ implies $X(s, \omega) \to X(t, \omega)$. It is said to be continuous on $[0, T]$ if $X(t, \omega)$ is continuous at t, for every $t \in [0, T]$.

Remark 2 Let us note that the mean-square continuity implies the stochastic continuity. Moreover, if $X(t, \omega)$, $t \in [0, T]$ is a sample-path continuous stochastic process, then it is mean-square continuous.

4.2 Stochastic continuity

Now, we are able to prove the stochastic continuity result for the solution of the random time-dependent oligopolistic market equilibrium problem with excesses described in Sect. 2. As before let us consider the set $X = L^2([0, T] \times \Omega, \mathbb{R}^{mn}_+, \mathbb{P})$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Notice that in the following theorem and proposition it is required the additional hypothesis of continuity for the capacity constraints $\overline{x}, \underline{x}$ and for the output and demand p, q, in order to ensure Kuratowski's convergence of $\mathbb{K}$.

Theorem 4 *Let $\mathbb{K}$ be as in (1) in which we suppose that $\overline{x}, \underline{x}, p, q$ are continuous functions. Let us assume that assumptions 1., 2., 3. are satisfied. Let $A : L^2([0, T] \times \Omega, \mathbb{R}^{mn}, \mathbb{P}) \to L^2([0, T] \times \Omega, \mathbb{R}^{mn}, \mathbb{P})$ be the operator defined by $A = \left(-\frac{\partial v_i(x^*)}{\partial x_{ij}}\right)_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$. Let us suppose that A is strongly pseudo-monotone with degree $\alpha > 1$. Then the solution $x^* = x^*(t, \omega)$ to (4) is stochastic continuous on $[0, T]$.*

In order to prove Theorem 4 we need the following

Proposition 1 *Let $\mathbb{K}$ be the feasible set defined in (1) in which we suppose that $\overline{x}, \underline{x}, p, q$ are continuous functions. Then $\mathbb{K}$ verifies Kuratowski's convergence with respect to t, namely for any fixed $\omega \in \Omega$ and $t_r \to t$ as $r \to +\infty$ then $\mathbb{K}(t_r, \omega) \to \mathbb{K}(t, \omega)$ in Kuratowski's sense, where*

$$\mathbb{K}(t,\omega) = \Big\{ x(t,\omega) \in \mathbb{R}^{mn} : \underline{x}_{ij}(t,\omega) \le x_{ij}(t,\omega) \le \overline{x}_{ij}(t,\omega), \ \ \forall i = 1,\ldots,m, \ \forall j = 1,\ldots,n, \ \sum_{j=1}^{n} x_{ij}(t,\omega) \le p_i(t,\omega), \ \ \forall i = 1,\ldots,m, \ \sum_{i=1}^{m} x_{ij}(t,\omega) \le q_j(t,\omega), \ \ \forall j = 1,\ldots,n \Big\}.$$

Proof It is sufficient to show that conditions $(K1)$ and $(K2)$ of Remark 1 hold true for

$$\mathbb{K}(t_r, \omega) \to \mathbb{K}(t, \omega).$$

This follows from similar computations contained in [4, Lemma 6], for ω fixed.

Now we are able to prove the stochastic continuity of the solution to (4).

Proof (of Theorem 4) By Remark 2, it will be sufficient to show the sample-path continuity of u. Fix $(t, \omega) \in [0, T] \times \Omega$, and a sequence $\{t_r\}$ in $[0, T]$ converging to t as $r \to +\infty$ and let $x^*(t_r, \omega)$ be the unique solution of the stochastic variational inequality

$$\langle\!\langle A(t_r, x^*(t_r,\omega)), x(t_r,\omega) - x^*(t_r,\omega) \rangle\!\rangle \ge 0, \quad \forall\, x(t_r,\omega) \in \mathbb{K}(t_r,\omega), \ \forall\, r \in \mathbb{N}. \tag{8}$$

For a fixed $(t, \omega) \in [0, T] \times \Omega$, it suffices to show that, for any sequence of times $\{t_r\}$ in $[0, T]$ converging to t as $r \to +\infty$, we have $x^*(t_r, \omega) \to x^*(t, \omega)$ as $r \to +\infty$.

Minty-Browder's lemma in its generalized form ensures that for every $(t, \omega) \in [0, T] \times \Omega$ we have

$$\langle\!\langle A(t, x(t,\omega)), x(t,\omega) - x^*(t,\omega) \rangle\!\rangle \ge 0, \quad \forall\, x(t,\omega) \in \mathbb{K}(t,\omega).$$

It follows by Proposition 1, that for any $x^*(t, \omega) \in \mathbb{K}(t, \omega)$, there exists a sequence $\{y(t_r, \omega)\}_{r\in\mathbb{N}}$ such that $y(t_r, \omega) \in \mathbb{K}(t_r, \omega)$ for r large enough and $y(t_r, \omega) \to x^*(t, \omega)$. The continuity of function A implies that $A(t_r, y(t_r, \omega)) \to A(t, x^*(t, \omega))$. For r large enough, setting $x(t_r, \omega) = y(t_r, \omega)$ in (8), it holds

$$\langle\!\langle A(t_r, x^*(t_r,\omega)), y(t_r,\omega) - x^*(t_r,\omega) \rangle\!\rangle \ge 0.$$

Since $A(t,\cdot)$ is strongly pseudo-monotone with degree $\alpha > 1$, we have

$$\begin{aligned}\eta\|y(t_r,\omega)-x^*(t_r,\omega)\|^\alpha &\le \left\langle\!\!\left\langle A(t_r,y(t_r,\omega)), y(t_r,\omega)-x^*(t_r,\omega)\right\rangle\!\!\right\rangle\\ &\le \|A(t_r,y(t_r,\omega))\|\|y(t_r,\omega)-x^*(t_r,\omega)\|\end{aligned}$$

and

$$\|y(t_r,\omega)-x^*(t_r,\omega)\| \le \eta^{\frac{1}{1-\alpha}}\|A(t_r,y(t_r,\omega))\|^{\frac{1}{\alpha-1}}.$$

Finally, we have

$$\begin{aligned}\|x^*(t_r,\omega)\| &\le \|x^*(t_r,\omega)-y(t_r,\omega)\|+\|y(t_r,\omega)\|\\ &\le \eta^{\frac{1}{1-\alpha}}\|A(t_r,y(t_r,\omega))\|^{\frac{1}{\alpha-1}}+\|y(t_r,\omega)\|,\end{aligned}$$

showing that $\{x^*(t_r,\omega)\}_{r\in\mathbb{N}}$ is a bounded sequence. Thus, there exist $y\in\mathbb{R}^d$ and a subsequence not relabeled and still denoted by $\{x^*(t_r,\omega)\}_{r\in\mathbb{N}}$ such that $x^*(t_r,\omega)\in\mathbb{K}(t_r,\omega)$, $\forall\, r\in\mathbb{N}$ and $x^*(t_r,\omega)\to y$. The Kuratowski's set convergence assumption ensures that $y\in\mathbb{K}(t,\omega)$.

Now, we are left to prove that $y = x^*(t,\omega)$. Again applying the generalized version of Minty-Browder's Lemma to any $x^*(t_r,\omega)$ gives

$$\left\langle\!\!\left\langle A(t_r,x(t_r,\omega)), x(t_r,\omega)-x^*(t_r,\omega)\right\rangle\!\!\right\rangle \ge 0, \quad \forall\, x(t_r,\omega)\in\mathbb{K}(t_r,\omega).$$

By Proposition 1, for any $x(t,\omega)\in\mathbb{K}(t,\omega)$, we can find $\{x(t_r,\omega)\}_{r\in\mathbb{N}}$ such that $x(t_r,\omega)\in\mathbb{K}(t_r,\omega)$ for r large enough and $x(t_r,\omega)\to x(t,\omega)$. It follows

$$\left\langle\!\!\left\langle A(t_r,x(t_r,\omega)), x(t_r,\omega)-x^*(t_r,\omega)\right\rangle\!\!\right\rangle \ge 0, \quad \forall\, x(t_r,\omega)\in\mathbb{K}(t_r,\omega),$$

As $r\to+\infty$, we obtain

$$\langle\!\langle A(t,x(t,\omega)), x(t,\omega)-y\rangle\!\rangle \ge 0, \quad \forall\, x(t,\omega)\in\mathbb{K}(t,\omega).$$

Taking into account the generalized version of Minty-Browder's Lemma, it results

$$\langle\!\langle A(t,y), x(t,\omega)-y\rangle\!\rangle \ge 0, \quad \forall\, x(t,\omega)\in\mathbb{K}(t,\omega).$$

Finally, since the solution to (4) is unique, $y = x^*(t,\omega)$ and $x^*(t_r,\omega)\to x^*(t,\omega)$, completing the proof.

5 Example

Let us describe in this section a numerical example for the time-dependent random oligopolistic market equilibrium problem with excesses. Let us consider the market network constituted by three firms P_1, P_2 and P_3 which compete with two markets Q_1

and Q_2. Let $x_{ij}(t,\omega)$ be the commodity shipment from P_i to Q_j, $(i=1,2,3,\ j=1,2)$ and assume, for fixed $T>0$, that $\underline{x}_{ij}(t,\omega)\le x_{ij}(t,\omega)\le \overline{x}_{ij}(t,\omega)$ holds, where $\underline{x}_{ij}(t,\omega)$ and $\overline{x}_{ij}(t,\omega)$ are function on $[0,T]\times\Omega$ representing the capacity constraints. Assume that $\underline{x}_{ij}(t,\omega)=t^2\check{x}_{ij}(\omega)$ and $\overline{x}_{ij}(t,\omega)=t^2\hat{x}_{ij}(\omega)$ where $\hat{x}_{ij}(\omega)$ and $\check{x}_{ij}(\omega)$ are uniformly distributed random variables with probability density functions given by:

$$f_{\hat{x}_{i1}}(z)=\begin{cases}\frac{1}{2}, & \text{if } 0\le z\le 2,\\ 0 & \text{elsewhere,}\end{cases}$$

$$f_{\hat{x}_{i2}}(z)=\begin{cases}\frac{1}{5}, & \text{if } 0\le z\le 5,\\ 0 & \text{elsewhere,}\end{cases}$$

$$f_{\check{x}_{i1}}(z)=\begin{cases}\frac{1}{25}, & \text{if } 75\le z\le 100,\\ 0 & \text{elsewhere,}\end{cases}$$

$$f_{\check{x}_{i2}}(z)=\begin{cases}\frac{1}{20}, & \text{if } 80\le z\le 100,\\ 0 & \text{elsewhere.}\end{cases}$$

Set now the maximal commodity production of P_i $(i=1,2,3)$ and the maximal commodity demand of Q_j $(j=1,2)$. Let us define $p_i(t,\omega)=t^2\overline{p}_i(\omega)$ and $q_j(t,\omega)=t^2\overline{q}_j(\omega)$ where the density function of $\overline{p}_i(\omega)$ and $\overline{q}_j(\omega)$ are defined by

$$f_{\overline{p}_1}(z)=\begin{cases}\frac{1}{20}, & \text{if } 80\le z\le 100,\\ 0, & \text{elsewhere}\end{cases}$$

$$f_{\overline{p}_2}(z)=\begin{cases}\frac{1}{10}, & \text{if } 90\le z\le 100,\\ 0, & \text{elsewhere}\end{cases}$$

$$f_{\overline{p}_3}(z)=\begin{cases}\frac{1}{10}, & \text{if } 230\le z\le 240,\\ 0, & \text{elsewhere}\end{cases}$$

$$f_{\overline{q}_1}(z)=\begin{cases}\frac{1}{30}, & \text{if } 240\le z\le 270,\\ 0, & \text{elsewhere}\end{cases}$$

$$f_{\overline{q}_2}(z)=\begin{cases}\frac{1}{40}, & \text{if } 150\le z\le 190,\\ 0, & \text{elsewhere}\end{cases}$$

The feasible set $\mathbb{K}$ is then as in (1) with the above definition of $\underline{x}_{ij}(t,\omega)$, $\overline{x}_{ij}(t,\omega)$, $\overline{p}_i(t,\omega)$, $\overline{q}_j(t,\omega)$.

Then we left to define the profit function $v_i(x(t,\omega))$ for the firms P_i, for $i=1,2,3$. We set

$$\begin{aligned}v_1(x(t,\omega)) &= -3x_{11}^2(t,\omega)-x_{12}^2(t,\omega)-4x_{31}^2(t,\omega)+x_{12}(t,\omega)x_{21}(t,\omega)\\ &\quad +a_1(t,\omega)x_{11}(t,\omega)+b_1(t,\omega)x_{12}(t,\omega)\\ v_2(x(t,\omega)) &= -2x_{21}^2(t,\omega)-3x_{22}^2(t,\omega)-x_{32}^2(t,\omega)+x_{11}(t,\omega)x_{22}(t,\omega)\\ &\quad +a_2(t,\omega)x_{21}(t,\omega)+b_2(t,\omega)x_{22}(t,\omega)\end{aligned}$$

$$v_3(x(t,\omega)) = -x_{31}^2(t,\omega) - 2x_{32}^2(t,\omega) - x_{21}^2(t,\omega) + x_{12}(t,\omega)x_{31}(t,\omega)$$
$$+a_3(t,\omega)x_{31}(t,\omega) + b_3(t,\omega)x_{32}(t,\omega)$$

where a_i, b_i $(i = 1, 2, 3)$ are uniformly distributed random variables with supports:

$$\begin{aligned} \operatorname{spt} a_1 &= [36, 108] \quad \operatorname{spt} b_1 = [10, 40] \\ \operatorname{spt} a_2 &= [40, 120] \quad \operatorname{spt} b_2 = [10, 40] \\ \operatorname{spt} a_3 &= [10, 40] \quad \operatorname{spt} b_3 = [40, 120] \end{aligned}$$

Let us compute the operator $\nabla_D v$

$$-\nabla_D v(x) = \begin{pmatrix} 6x_{11} - a_1 & 2x_{12} - x_{21} - b_1 \\ 4x_{21} - a_2 & 6x_{22} - x_{11} - b_2 \\ 2x_{31} - x_{12} - a_3 & 4x_{32} - b_3 \end{pmatrix},$$

where here, and in the following, we omit the arguments of the variables, simply writing x_{ij} instead of $x_{ij}(t,\omega)$.

The equilibrium condition is expressed by the following variational inequality problem: find $x^* \in L^2([0,T] \times \Omega, \mathbb{R}^6_+, \mathbb{P})$ such that

$$\langle\!\langle -\nabla_D v(x^*), x - x^* \rangle\!\rangle \geq 0, \quad \forall x \in \mathbb{K}.$$

First observe that existence and uniqueness of the solution to problem above is guaranteed by the theoretical result of Sect. 3. Then the solution is computed applying the direct methods as described in [17]. We solve the system

$$\begin{cases} 6x_{11}^* - a_1 = 0 \\ 2x_{12}^* - x_{21}^* - b_1 = 0 \\ 4x_{21}^* - a_2 = 0 \\ 6x_{22}^* - x_{11}^* - b_2 = 0 \\ 2x_{31}^* - x_{12}^* - a_3 = 0 \\ 4x_{32}^* - b_3 = 0 \end{cases}$$

which gives

$$\begin{cases} x_{11}^* = \frac{1}{6}a_1 \\ x_{12}^* = \frac{1}{4}a_2 + b_1 \\ x_{21}^* = \frac{1}{4}a_2 \\ x_{22}^* = \frac{1}{6}a_1 + b_2 \\ x_{31}^* = \frac{1}{4}a_2 + b_1 + b_3 \\ x_{32}^* = \frac{1}{4}b_3 \end{cases}$$

For the solution $x^* = x^*_{ij}$ are verified the following

$$\begin{array}{ll} \operatorname{spt} x^*_{11} = [6, 18] & \operatorname{spt} x^*_{12} = [20, 60] \\ \operatorname{spt} x^*_{21} = [10, 30] & \operatorname{spt} x^*_{22} = [16, 58] \\ \operatorname{spt} x^*_{31} = [60, 90] & \operatorname{spt} x^*_{32} = [10, 30] \end{array}$$

It is easy to check that the random vector x^* belongs to $\mathbb{K}$ proving that x^* is the solution of the random time-dependent oligopolistic market equilibrium problem with excesses. Again with simple computations it is possible to obtain the production and demand excesses described by the random time-dependent function $\varepsilon_i(t, \omega)$ and $\delta_j(t, \omega)$.

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