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ORIGINAL PAPER

Inverse optimal value problem on minimum spanning tree under unit l_∞ norm

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Abstract

We consider the inverse optimal value problem on minimum spanning tree under unit l_∞ norm. Given an edge weighted connected undirected network $G = (V, E, \boldsymbol{w})$ and a spanning trees T^0, we aim to modify the weights of the edges such that T^0 is the minimum spanning tree under the new weight vector whose weight is equal to a given value K and the modification cost under unit l_∞ norm is minimized. We present a mathematical model of the problem. After analyzing the properties, we propose a sufficient and necessary condition for optimal solutions of the problem. Then we develop a strongly polynomial time algorithm with running time $O(|V||E|)$. Finally, we give an example to demonstrate the algorithm.

Keywords Minimum spanning tree · l_∞ norm · Inverse optimal value problem · Strongly polynomial time algorithm

1 Introduction

In recent years, spanning tree problems have become an important topic in the field of combinatorial optimization due to many applications in transportation networks, communication networks, etc. Let $G = (V, E, \boldsymbol{w})$ be a connected undirected network, where $V = \{1, 2, \ldots, n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. Each edge e_i is associated with a weight w_i. Let $\boldsymbol{w} = (w_1, w_2, \ldots, w_m)$ be the weight vector. Let Γ be the set of all spanning trees of G. The weight of a spanning tree T is defined as the sum of weights of edges in T, that is, $\boldsymbol{w}(T) = \sum_{e_i \in T} w_i$. The minimum spanning tree (**MST**) problem is to find a spanning tree $T \in \Gamma$ whose weight is minimum.

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We consider the inverse optimal value problems on minimum spanning trees under unit l_∞ norm (denoted by $\mathbf{IOVMST}_\infty$), which can be described as follows. Given a spanning tree T^0 of G, we aim to find a new edge weight vector $\bar{\boldsymbol{w}}$ such that

(1) The weight of T^0 under $\bar{\boldsymbol{w}}$ is equal to a given value K, where $K < \boldsymbol{w}(T^0)$;
(2) T^0 is the minimum spanning tree under $\bar{\boldsymbol{w}}$, that is, the weight of any spanning tree under $\bar{\boldsymbol{w}}$ is not less than K;
(3) The maximum modification $\max_{e_i \in E} |\bar{w}_i - w_i|$ is minimized.

The problem $\mathbf{IOVMST}_\infty$ can be formulated as follows:

$$\begin{aligned} \min_{\bar{\boldsymbol{w}}} \ & \max_{e_i \in E} |\bar{w}_i - w_i| \\ \text{s.t.} \ & \sum_{e_i \in T^0} \bar{w}_i = K, \\ & \sum_{e_j \in T} \bar{w}_j \geq K, \qquad \forall T \in \Gamma. \end{aligned} \tag{1.1}$$

The problem $\mathbf{IOVMST}_\infty$ has practical background. An enterprise has n agents in a region, each agent has connection with other agents [2]. Any commercial message passed between agents i and j will fall into hostile hands with certain probability p_{ij}. Define the edge weight of edge e_k between agents i and j as $w_k = p_{ij}$. The enterprise leader wants to transmit confidential commercial messages among all the agents through a spanning tree T^0 of a **safe network**. A **safe network** T^0 means the total probability of T^0 to be intercepted is minimum and the total probability is small enough to be a constant K. The leader aims to change edge weight $\boldsymbol{w}$ to $\bar{\boldsymbol{w}}$ such that the maximum deviation $|\bar{w}_k - w_k|$ of each edge e_k is minimum. To meet this requirement, the leader needs to arrange some training courses for some agents to decrease the probabilities of interception when transmitting commercial messages among some edges. Notice that more deviation $|\bar{w}_k - w_k|$ of edge e_k needs more cost c_k of training courses for agents i and j. For example, let $c_k = a^{|\bar{w}_k - w_k|}$ and $a = 10$. Then the cost c_k is 10US\$ when $|\bar{w}_k - w_k| = 1$, but c_k is 100 US\$ when $|\bar{w}_k - w_k| = 2$. This is just an inverse optimal value problem on minimum spanning tree.

Notice that there is another kind of inverse optimal value problems on minimum spanning trees under unit l_∞ norm, in which we do not need to preassign a spanning tree T^0, but to make the weight $\bar{\boldsymbol{w}}(T)$ of any **MST** under $\bar{\boldsymbol{w}}$ to be a given value K. The mathematical model is given below.

$$\begin{aligned} \min_{\bar{\boldsymbol{w}}} \ & \max_{e_i \in E} |\bar{w}_i - w_i| \\ \text{s.t.} \ & \min_{T \in \Gamma} \sum_{e_i \in T} \bar{w}_i = K. \end{aligned} \tag{1.2}$$

However, the problem (1.2) is trivial to solve. Firstly, we find a minimum spanning tree T^* under $\boldsymbol{w}$, then solve equation $\sum_{e \in T^*}(w_i + \lambda^*) = K$ to calculate $\lambda^* = \frac{K - \boldsymbol{w}(T^*)}{n-1}$. If $\lambda^* \geq 0$, then $w_i^* = w_i + \lambda^*$ for each $e_i \in E$. If $\lambda^* < 0$, then $w_i^* = w_i + \lambda^*$ for each $e_i \in T^*$, and $w_i^* = w_i - \lambda^*$ for each $e_i \notin T^*$. It is not difficult to prove that T^* is still a minimum spanning tree under $\boldsymbol{w}^*$ and $\boldsymbol{w}^*$ is an optimal solution of the problem (1.2) with objective value $|\lambda^*|$. Thus, in this paper, we consider the problem $\mathbf{IOVMST}_\infty$.

Related to the problem $\mathbf{IOVMST}_\infty$, the inverse minimum spanning tree problems (1.3) (denoted by **IMST**) are mostly studied under different norms including l_1 and l_∞ norms and Hamming distance [3–10,12–18].

$$\textbf{(IMST)}\quad \begin{array}{l} \min\ \|\bar{\boldsymbol{w}} - \boldsymbol{w}\| \\ \text{s.t.} \displaystyle\sum_{e_i \in T^0} \bar{w}_i \le \sum_{e_j \in T} \bar{w}_j, \quad \forall T \in \Gamma. \end{array} \tag{1.3}$$

Note that in the problem **IMST** we do not need to preassign a value K to the weight $\bar{\boldsymbol{w}}(T^0)$ for the given **MST** T^0 under the new weight vector $\bar{\boldsymbol{w}}$, hence the problem $\mathbf{IOVMST}_\infty$ is a subproblem of the problem $\mathbf{IMST}_\infty$ under unit l_∞ norm. For the problem $\mathbf{IMST}_\infty$, Sokkalingam et al. [13] showed that it can be solved in $O(n^2)$ time, and Hochbaum [8] proposed an $O(m \log n)$ algorithm, which is more efficient if the graph is not dense.

For the partial inverse minimum spanning tree problem (denoted by **PIMST**), in which not a full spanning tree but a part of it (a forest) is given, Lai and Orlin [9] showed that the problem **PIMST** under the weighted l_∞ norm can be solved in strongly polynomial time. Li et al. [10] showed that the capacitated **PIMST** under l_∞ norm can be solved in polynomial time, in which the deviation $|\bar{w}_i - w_i|$ is upper-bounded by given values for each $e_i \in E$. Cai et al. [4] considered the capacitated **PIMST** when weight increasing is forbidden and presented a strongly polynomial time algorithm for a general criterion objective function including l_1, l_2, l_∞ norms.

Ahmed and Guan [1] considered the inverse optimal value (**IOVLP**) problem on linear programming (**LP**) problem. Given an **LP** problem with a desired optimal objective value and a set of feasible cost vectors, they aim to determine a cost vector such that the optimal objective value of the **LP** is closest to the desired value. They first proved that the general problem **IOVLP** is **NP-hard**, then for a concave maximization/minimization problem, they provided sufficient conditions, under which the problem **IOVLP** is polynomially solvable. When the set of feasible cost vectors is polyhedral, they described an algorithm for the problem **IOVLP** by solving linear and bilinear programming problems. Lv et al. [11] transformed the problem **IOVLP** under some conditions into a nonlinear bilevel programming problem, which was transformed into a single-level nonlinear program using the Kuhn–Tucker optimality condition of the lower level problem. They proposed an algorithm based on exact penalty method and analysed its convergence.

The paper is organized as follows. In Sect. 2, we study some properties of optimal solutions of the problem $\mathbf{IOVMST}_\infty$ and propose a sufficient and necessary condition. In Sect. 3, we develop a strongly polynomial time algorithm with time complexity $O(mn)$. A computational example of the problem $\mathbf{IOVMST}_\infty$ is given in Sect. 4. Finally, conclusion and discussion are given in Sect. 5.

2 Properties of an optimal solution of the problem IOVMST$_\infty$

In this section, we analyze some properties of an optimal solution of the problem **IOVMST**$_\infty$. Most importantly, we obtain a sufficient and necessary condition for an optimal solution.

We first introduce some notations. For each $e_j \notin T^0$, $T^0 \cup \{e_j\}$ contains a fundamental cycle, and we denote by P_j all edges in this cycle except e_j and define $\Omega_i = \{e_j \notin T^0 | e_i \in P_j\}$. If both edge $e_j \notin T$ and edge $e_i \in T$ are in at least one cycle, then $e_j \in \Omega_i$. Let $\Omega^0 = \{e_i \in T^0 | \Omega_i = \emptyset\}$ be the set of isolated edges in T_0, which belongs to every spanning tree of G. If $\Omega^0 \neq \emptyset$, then Ω^0 is also the set of cut edges of G. If $\Omega^0 = \emptyset$, then each edge in T^0 belongs to at least one fundamental cycle. Let

$$\delta = \max_{e_i \in T^0} \max_{e_k \in \Omega_i} \{w_i - w_k\} \tag{2.1}$$

be the maximum deviation between w_i and w_k for each $e_i \in T^0$ and $e_k \in \Omega_i$. Denote $\Theta_i = \{e_{j_i} \in \Omega_i | w_i - w_{j_i} = \max_{e_k \in \Omega_i} \{w_i - w_k\} > 0\}$ as the set of edges $e_{j_i} \in \Omega_i$ achieving the maximum value for e_i. Let $\Theta^0 = \{e_i \in T^0 \backslash \Omega^0 | \Theta_i = \emptyset\}$. If $\Theta^0 \neq \emptyset$, then for each $e_i \in \Theta^0$, if $e_p \in \Omega_i$, then $w_p \geq w_i$. If $\Theta^0 = \emptyset$, then for each $e_i \in T^0 \backslash \Omega^0$, there exists an edge $e_p \in \Omega_i$ such that $w_p < w_i$. In order to explain the meaning of above notations, we give the next example.

Example 1 As shown in Fig. 1, let $V = \{v_1, v_2, \ldots, v_{11}\}$, $E = \{e_1, e_2, \ldots, e_{17}\}$, $\boldsymbol{w} = (3, 4, 3, 3, 4, 3, 1, 5, 5, 3, 4, 5, 5, 3, 4, 4, 1)$, $T^0 = \{e_1, e_2, e_3, e_4, e_8, e_9, e_{12}, e_{13}, e_{15}, e_{16}\}$ (T^0 is denoted by thick lines in Fig. 1).

In **Example** 1 shown in Fig. 1,

(1) for each $e_i \notin T^0$, $P_5 = \{e_3, e_4\}$, $P_6 = \{e_2, e_4\}$, $P_7 = \{e_2, e_4, e_8\}$, $P_{10} = \{e_8, e_9\}$, $P_{11} = \{e_8, e_9, e_{13}\}$, $P_{14} = \{e_8, e_9, e_{12}, e_{13}\}$, and $P_{17} = \{e_8, e_9, e_{12}, e_{13}, e_{15}\}$;

(2) for each $e_i \in T^0$, $\Omega_1 = \Omega_{16} = \emptyset$, $\Omega_2 = \{e_6, e_7\}$, $\Omega_3 = \{e_5\}$, $\Omega_4 = \{e_5, e_6, e_7\}$, $\Omega_8 = \{e_7, e_{10}, e_{11}, e_{14}, e_{17}\}$, $\Omega_9 = \{e_{10}, e_{11}, e_{14}, e_{17}\}$, $\Omega_{12} = \{e_{14}, e_{17}\}$, $\Omega_{13} = \{e_{11}, e_{14}, e_{17}\}$, $\Omega_{15} = \{e_{17}\}$;

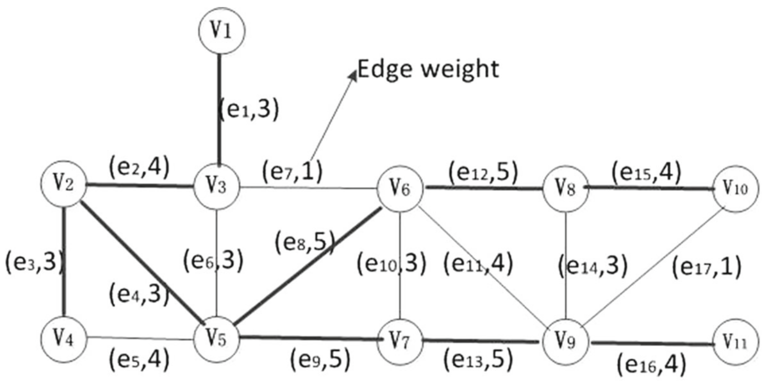

Fig. 1 An example to show the meanings of notations

(3) for each $e_i \in T^0$, $\Theta_1 = \Theta_{16} = \emptyset$, $\Theta_2 = \{e_7\}$, $\Theta_3 = \emptyset$, $\Theta_4 = \{e_7\}$, $\Theta_8 = \{e_7, e_{17}\}$, $\Theta_9 = \{e_{17}\}$, $\Theta_{12} = \{e_{17}\}$, $\Theta_{13} = \{e_{17}\}$, $\Theta_{15} = \{e_{17}\}$;

(4) $\Omega^0 = \{e_1, e_{16}\}$ and $\Theta^0 = \{e_3\}$;

(5) $e_{j_2} = e_7$, $e_{j_4} = e_7$, $e_{j_8} = e_7$ or $e_{j_8} = e_{17}$, $e_{j_9} = e_{17}$, $e_{j_{12}} = e_{17}$, $e_{j_{13}} = e_{17}$, $e_{j_{15}} = e_{17}$; $\delta = w_8 - w_{j_8} = w_8 - w_7 = 4$;

Next, we analyze some properties of a feasible solution of the problem **IOVMST**$_\infty$.

Lemma 1 [2] *T^0 is a minimum spanning tree with respect to the weight vector $\bar{\boldsymbol{w}}$ if and only if the following optimality conditions are satisfied:*

$$\bar{w}_i \leq \bar{w}_j \quad \text{for each } e_j \notin T^0 \text{ and } e_i \in P_j. \tag{2.2}$$

Lemma 2 *If $\bar{\boldsymbol{w}}$ is a feasible solution of problem* (1.1) *with objective value $\bar{\lambda}$, then $\hat{\boldsymbol{w}}$ is also a feasible solution of* (1.1) *with objective value $\hat{\lambda} = \bar{\lambda}$, where*

$$\hat{w}_k = \begin{cases} \bar{w}_k, & \text{if } e_k \in T^0, \\ w_k + \bar{\lambda}, & \text{if } e_k \notin T^0. \end{cases} \tag{2.3}$$

Proof We first prove the feasibility of $\hat{\boldsymbol{w}}$. It follows from (2.2) and (2.3) that $\hat{w}_j = w_j + \bar{\lambda} \geq \bar{w}_j \geq \bar{w}_i = \hat{w}_i$ for $e_i \in P_j$ and $e_j \notin T^0$, where the first inequality follows from the feasibility of $\bar{\boldsymbol{w}}$ and the definition of $\bar{\lambda}$, and the second inequality follows from the feasibility of $\bar{\boldsymbol{w}}$. Moreover, $\sum_{e_i \in T^0} \hat{w}_i = \sum_{e_i \in T^0} \bar{w}_i = K$. To calculate the objective value, we have $|\hat{w}_i - w_i| = |\bar{w}_i - w_i| \leq \bar{\lambda}$ for each $e_i \in T^0$, and $|\hat{w}_j - w_j| = |w_j + \bar{\lambda} - w_j| = \bar{\lambda}$ for $e_j \notin T^0$. So $\hat{\lambda} = \max_{e_k \in E} |\hat{w}_k - w_k| = \bar{\lambda}$. □

By Lemma 2, we know that the weight of edge $e_j \notin T^0$ in an optimal solution can be increased to the maximum weight $\hat{w}_j = w_j + \hat{\lambda}$ if the optimal objective value $\hat{\lambda}$ is obtained. In the following part of the paper, we find an optimal solution of the problem (1.1) satisfying (2.3).

By using the notation of Ω_i for $e_i \in T^0$, the problem (1.1) is equivalent to

$$\begin{aligned} &\min \max |\bar{w}_i - w_i| \\ &\text{s.t.} \;\; \bar{w}_i \leq \bar{w}_j, \qquad e_i \in T^0 \text{ and } e_j \in \Omega_i, \\ &\qquad \bar{w}_j \geq w_j, \qquad e_j \notin T^0, \\ &\qquad \sum_{e_i \in T^0} \bar{w}_i = K. \end{aligned} \tag{2.4}$$

Lemma 3 *If $\bar{\boldsymbol{w}}$ is a feasible solution of problem* (2.4) *with objective value $\bar{\lambda}$, then $\bar{\lambda} \geq \frac{\delta}{2}$, where δ is defined as in* (2.1).

Proof We only consider $\delta > 0$. In fact, if $\delta \leq 0$, then T^0 is a minimum spanning tree of G under weight $\boldsymbol{w}$, hence $\bar{\lambda} \geq 0 \geq \frac{\delta}{2}$. It follows from Lemma 2 that $\hat{\boldsymbol{w}}$ defined as in (2.3) is a feasible solution of problem (2.4). Let $\delta = \delta_p = w_p - w_{j_p}$ for some edge $e_p \in T^0$ and $e_{j_p} \in \Theta_p$. Then

$$\delta = w_p - w_{j_p} = w_p - \hat{w}_p + (\hat{w}_p - w_{j_p}) \leq (w_p - \hat{w}_p) + (\hat{w}_{j_p} - w_{j_p}) \leq 2\bar{\lambda},$$

where the first inequality follows from the feasibility of $\hat{\boldsymbol{w}}$ and the second inequality follows from the definition of $\hat{\boldsymbol{w}}$ and $\bar{\lambda}$. Hence $\bar{\lambda} \geq \frac{\delta}{2}$. □

Let $\hat{\boldsymbol{w}}$ be a feasible solution of problem (2.4) which satisfies (2.3) and its value is $\hat{\lambda}$. It follows from (2.3) that $\hat{w}_j = w_j + \hat{\lambda}$ for $e_j \notin T^0$. For edge $e_i \in T^0$, the weight w_i may be increased, decreased, or be unchanged to $\hat{w}_i$. If $\Omega^0 \neq \emptyset$ and $e_i \in \Omega^0$, then e_i is an isolated edge belonging to any tree including T^0 and thus w_i may be increased in an optimal solution in some cases (see Fig. 4b).

2.1 Properties of an optimal solution of the problem IOVMST$_\infty$ when $\Theta^0 = \emptyset$ and $\Omega^0 = \emptyset$

In this subsection, we analyze some properties of an optimal solution of the problem **IOVMST**$_\infty$ when $\Theta^0 = \emptyset$ and $\Omega^0 = \emptyset$, then we propose a sufficient and necessary condition for an optimal solution of the problem **IOVMST**$_\infty$.

If $\Theta^0 = \emptyset$, then $w_i - \hat{\lambda} \leq \hat{w}_i \leq w_{j_i} + \hat{\lambda}$ for $e_i \in T^0$, $e_{j_i} \in \Theta_i$. Define

$$\begin{cases} EU(\hat{\boldsymbol{w}}) = \{e_i \in T^0 |\ \hat{w}_i = w_{j_i} + \hat{\lambda}, e_{j_i} \in \Theta_i\}, \\ ED(\hat{\boldsymbol{w}}) = \{e_i \in T^0 |\ \hat{w}_i = w_i - \hat{\lambda}\}, \\ EM(\hat{\boldsymbol{w}}) = \{e_i \in T^0 |\ w_i - \hat{\lambda} < \hat{w}_i < w_{j_i} + \hat{\lambda}, e_{j_i} \in \Theta_i\}. \end{cases} \tag{2.5}$$

Obviously, $ED(\hat{\boldsymbol{w}}) \cup EM(\hat{\boldsymbol{w}}) \cup EU(\hat{\boldsymbol{w}}) = T^0$. Let $N_U = |EU(\hat{\boldsymbol{w}})|$, $N_D = |ED(\hat{\boldsymbol{w}})|$, $N_M = |EM(\hat{\boldsymbol{w}})|$. Define

$$\begin{cases} \gamma(\hat{\boldsymbol{w}}) = \min\{\hat{w}_i - (w_i - \hat{\lambda}) |\ e_i \in EM(\hat{\boldsymbol{w}})\}, \\ \eta(\hat{\boldsymbol{w}}) = \min\{(w_{j_i} + \hat{\lambda}) - \hat{w}_i |\ e_i \in EM(\hat{\boldsymbol{w}})\}. \end{cases} \tag{2.6}$$

In order to explain the meaning of above notations, we give the next example.

Example 2 As shown in Fig. 2, let $V = \{v_1, v_2, \ldots, v_9\}$, $E = \{e_1, e_2, \ldots, e_{15}\}$, $\boldsymbol{w} = (1, 4, 5, 3, 4, 3, 1, 5, 5, 3, 4, 5, 5, 3, 4)$, $T^0 = \{e_2, e_3, e_4, e_8, e_9, e_{12}, e_{13}, e_{15}\}$ (T^0 is denoted by thick lines in Fig. 2), $K = 22$, $\hat{\boldsymbol{w}} = \{3.5, \mathbf{3.5}, \mathbf{2.5}, \mathbf{2}, 6.5, 3.5, \mathbf{3.5}, \mathbf{3}, 5.5, 6.5, \mathbf{3.5}, \mathbf{2.5}, 5.5, \mathbf{1.5}\}$.

In **Example** 2 shown in Fig. 2, $\boldsymbol{w}(T^0) = 36$, and it is easy to check that $\hat{\boldsymbol{w}}$ is a feasible solution of problem (2.4) satisfying (2.3) with objective value $\hat{\lambda} = 2.5$.

(1) $EU(\hat{\boldsymbol{w}}) = \{e_2, e_8, e_{12}\}$. For example, $e_{j_2} = e_7$ and $\hat{w}_2 = w_7 + \hat{\lambda}$, so $e_2 \in EU(\hat{\boldsymbol{w}})$.
(2) $ED(\hat{\boldsymbol{w}}) = \{e_3, e_{13}, e_{15}\}$. For example, $e_{j_{13}} = e_1$ and $\hat{w}_{13} = w_{13} - \hat{\lambda}$, so $e_{13} \in ED(\hat{\boldsymbol{w}})$.
(3) $EM(\hat{\boldsymbol{w}}) = \{e_4, e_9\}$. For example, $e_{j_9} = e_1$ and $w_9 - \hat{\lambda} < \hat{w}_9 < w_1 + \hat{\lambda}$, so $e_9 \in EM(\hat{\boldsymbol{w}})$.
(4) $\gamma(\hat{\boldsymbol{w}}) = \min\{\hat{w}_4 - (w_4 - 2.5), \hat{w}_9 - (w_9 - 2.5)\} = \hat{w}_9 - (w_9 - 2.5) = 0.5$, $\eta(\hat{\boldsymbol{w}}) = \min\{(w_{j_4} + 2.5) - \hat{w}_4, (w_{j_9} + 2.5) - \hat{w}_9\} = (w_{j_9} + 2.5) - \hat{w}_9 = 0.5$.

It is not difficult to see that $\hat{\boldsymbol{w}}$ is not an optimal solution and $EM(\hat{\boldsymbol{w}}) = \{e_4, e_9\} \neq \emptyset$ in Example 2.

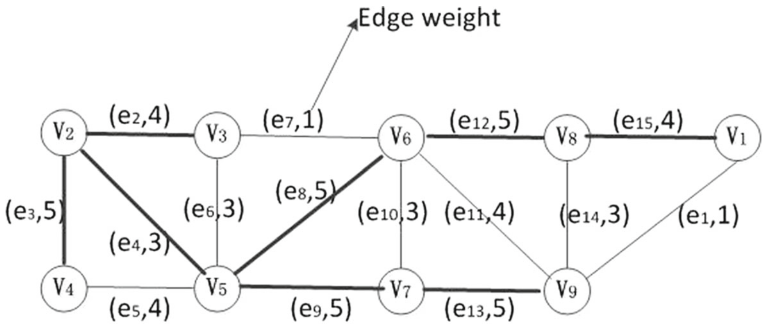

Fig. 2 An example to explain the meanings of $ED(\hat{\boldsymbol{w}})$, $EM(\hat{\boldsymbol{w}})$, $EU(\hat{\boldsymbol{w}})$, $\gamma(\hat{\boldsymbol{w}})$ and $\eta(\hat{\boldsymbol{w}})$

Next, we use the notations given above to analyze some properties of an optimal solution.

Theorem 4 *Suppose $\Omega^0 = \emptyset$ and $\Theta^0 = \emptyset$. Let $\hat{\boldsymbol{w}}$ be a feasible solution of problem (2.4) satisfying (2.3) and $\hat{\lambda} > \frac{\delta}{2}$. If $EM(\hat{\boldsymbol{w}}) \neq \emptyset$, then $\hat{\boldsymbol{w}}$ is not an optimal solution of (2.4), where $EM(\hat{\boldsymbol{w}})$ is defined as in (2.5).*

Proof We consider four cases, in which we find a better feasible solution $\boldsymbol{w^*}$ than $\hat{\boldsymbol{w}}$ satisfying $\lambda^* < \hat{\lambda}$. If $EM(\hat{\boldsymbol{w}}) \neq \emptyset$, then $\eta(\hat{\boldsymbol{w}}) > 0, \gamma(\hat{\boldsymbol{w}}) > 0$. Let $\alpha = \min\{\eta(\hat{\boldsymbol{w}}), \gamma(\hat{\boldsymbol{w}})\}$, then $\alpha > 0$.

Case 1 $EU(\hat{\boldsymbol{w}}) = \emptyset$, $ED(\hat{\boldsymbol{w}}) = \emptyset$.

Then $EM(\hat{\boldsymbol{w}}) = T^0$ and $\hat{\lambda} \geq \alpha$. Next we show that the weight vector $\boldsymbol{w^*}$ defined below is a feasible solution of problem (2.4) whose objective value is $\lambda^* = \hat{\lambda} - \alpha < \hat{\lambda}$.

$$w_k^* = \begin{cases} \hat{w}_k, & \text{if } e_k \in T^0, \\ \hat{w}_k - \alpha, & \text{if } e_k \notin T^0. \end{cases}$$

For each $e_i \in EM(\hat{\boldsymbol{w}})$, if $e_k \in \Omega_i$, and $e_{j_i} \in \Theta_i$, then $w_k \geq w_{j_i}$ and $e_k, e_{j_i} \notin T^0$.

$$\begin{aligned} w_k^* \geq w_{j_i}^* &= \hat{w}_{j_i} - \alpha = w_{j_i} + \hat{\lambda} - \alpha \\ &= (w_{j_i} + \hat{\lambda} - \hat{w}_i) + \hat{w}_i - \alpha \geq \eta(\hat{\boldsymbol{w}}) + \hat{w}_i - \alpha \geq \hat{w}_i = w_i^*, \end{aligned}$$

where the first inequality follows from the definition of w^*, $\hat{w}$ and e_{j_i}, the second inequality follows from the definition of $\eta(\hat{w})$ and the third inequality follows from the definition of α.

Furthermore, $w_k^* = \hat{w}_k - \alpha = w_k + \hat{\lambda} - \alpha \geq w_k$ for $e_k \notin T^0$ and $\sum_{e_i \in T^0} w_i^* = \sum_{e_i \in T^0} \hat{w}_i = K$. Hence, $\boldsymbol{w^*}$ is a feasible solution of problem (2.4).

Now we calculate the objective value λ^*. Firstly, for each $e_j \notin T^0$, $|w_j^* - w_j| = |w_j + \hat{\lambda} - \alpha - w_j| = |\hat{\lambda} - \alpha| = \lambda^*$. Secondly, we show that $|w_i^* - w_i| \leq \lambda^*$ for each $e_i \in T^0 = EM(\hat{\boldsymbol{w}})$. (a) $w_i^* - w_i \leq w_{j_i}^* - w_i = w_{j_i} + (\hat{\lambda} - \alpha) - w_i \leq \hat{\lambda} - \alpha = \lambda^*$; (b) Notice that $\hat{w}_i - w_i + \hat{\lambda} \geq \gamma(\hat{\boldsymbol{w}})$. Then $w_i^* = \hat{w}_i \geq w_i - \hat{\lambda} + \gamma(\hat{\boldsymbol{w}}) \geq w_i - \hat{\lambda} + \alpha$, hence

$w_i^* - w_i \geq -\hat{\lambda} + \alpha = -\lambda^*$. Thus $\boldsymbol{w^*}$ is a better feasible solution than $\hat{\boldsymbol{w}}$ satisfying $\lambda^* = \hat{\lambda} - \alpha < \hat{\lambda}$.

Case 2 $EU(\hat{\boldsymbol{w}}) = \emptyset$, $ED(\hat{\boldsymbol{w}}) \neq \emptyset$.

Then $0 < N_D < n-1$, $EM(\hat{\boldsymbol{w}}) = T^0 \backslash ED(\hat{\boldsymbol{w}})$ and $N_M = n - N_D - 1 > 0$. Let

$$w_k^* = \begin{cases} \hat{w}_k + \xi, & \text{if } e_k \in ED(\hat{\boldsymbol{w}}), \\ \hat{w}_k - \beta, & \text{if } e_k \in EM(\hat{\boldsymbol{w}}), \\ \hat{w}_k - \xi, & \text{if } e_k \notin T^0, \end{cases}$$

where ξ, β are solutions of the following equation system

$$\begin{cases} \xi \cdot N_D = (n - N_D - 1) \cdot \beta, \\ \xi + \beta = \mu(\hat{\boldsymbol{w}}) = \min\{\gamma(\hat{\boldsymbol{w}}), \eta(\hat{\boldsymbol{w}}), \hat{\lambda} - \frac{\delta}{2}\}. \end{cases}$$

Then $\xi = \frac{n-N_D-1}{n-1}\mu(\hat{\boldsymbol{w}})$, $\beta = \frac{N_D}{n-1}\mu(\hat{\boldsymbol{w}})$. Notice that $\beta > 0$ and $\xi > 0$ as $\mu(\hat{\boldsymbol{w}}) > 0$.

For each $e_i \in EM(\hat{\boldsymbol{w}})$, if $e_k \in \Omega_i$, $e_{j_i} \in \Theta_i$, then

$$w_k^* \geq w_{j_i}^* = \hat{w}_{j_i} - \xi \geq \hat{w}_{j_i} - \eta(\hat{\boldsymbol{w}}) \geq \hat{w}_{j_i} + \hat{w}_i - (w_{j_i} + \hat{\lambda}) = \hat{w}_i > \hat{w}_i - \beta = w_i^*,$$

where the second inequality follows from $0 < \xi \leq \mu(\hat{\boldsymbol{w}}) \leq \eta(\hat{\boldsymbol{w}})$, the third inequality follows from the definition of $\eta(\hat{\boldsymbol{w}})$ and the fourth inequality follows from $\beta > 0$.

For each $e_i \in ED(\hat{\boldsymbol{w}})$, if $e_k \in \Omega_i$, notice that $w_i - w_{j_i} \leq \delta$ and $\xi < \hat{\lambda} - \frac{\delta}{2}$, then

$$w_k^* \geq w_{j_i}^* = (w_{j_i} + \hat{\lambda}) - \xi \geq w_{j_i} + \frac{\delta}{2} \geq w_i - \frac{\delta}{2} \geq w_i - (\hat{\lambda} - \xi) = w_i^*.$$

Furthermore,

$$\begin{aligned} \sum_{e_i \in T^0} w_i^* &= \sum_{e_i \in ED(\hat{\boldsymbol{w}})} w_i^* + \sum_{e_i \in EM(\hat{\boldsymbol{w}})} w_i^* = \sum_{e_i \in ED(\hat{\boldsymbol{w}})} (\hat{w}_i + \xi) + \sum_{e_i \in EM(\hat{\boldsymbol{w}})} (\hat{w}_i - \beta) \\ &= \sum_{e_i \in ED(\hat{\boldsymbol{w}})} \hat{w}_i + \sum_{e_i \in EM(\hat{\boldsymbol{w}})} \hat{w}_i + \xi \cdot N_D - \beta \cdot (n - N_D - 1) = \sum_{e_i \in T^0} \hat{w}_i = K. \end{aligned}$$

Then $\boldsymbol{w^*}$ is a feasible solution of problem (2.4).

Now we show that the objective value $\lambda^* = \hat{\lambda} - \xi < \hat{\lambda}$. Firstly, we show that $|w_i^* - w_i| \leq \lambda^*$ for each $e_i \in EM(\hat{\boldsymbol{w}})$. (a) $w_i^* - w_i \leq w_{j_i}^* - w_i = w_{j_i} + (\hat{\lambda} - \xi) - w_i \leq \hat{\lambda} - \xi = \lambda^*$, (b) Notice that $\xi + \beta \leq \gamma(\hat{\boldsymbol{w}}) \leq \hat{w}_i + \hat{\lambda} - w_i$, then $\hat{w}_i - \beta - w_i \geq \xi - \hat{\lambda}$. Hence $w_i^* - w_i = \hat{w}_i - \beta - w_i \geq -\hat{\lambda} + \xi = -\lambda^*$. Secondly, for each $e_i \in ED(\hat{\boldsymbol{w}})$, $|w_i^* - w_i| = |w_i - \hat{\lambda} + \xi - w_i| = |-\hat{\lambda} + \xi| = \lambda^*$. Thirdly, for each $e_j \notin T^0$, $|w_j^* - w_j| = |w_j + \hat{\lambda} - \xi - w_j| = |\hat{\lambda} - \xi| = \lambda^*$.

Case 3 $EU(\hat{\boldsymbol{w}}) \neq \emptyset$, $ED(\hat{\boldsymbol{w}}) = \emptyset$.

Then $0 < N_U < n-1$, $EM(\hat{\boldsymbol{w}}) = T^0 \backslash EU(\hat{\boldsymbol{w}})$ and $N_M = n - N_U - 1 > 0$. Let

$$w_k^* = \begin{cases} \hat{w}_k - \xi, & \text{if } e_k \in EU(\hat{\boldsymbol{w}}), \\ \hat{w}_k + \beta, & \text{if } e_k \in EM(\hat{\boldsymbol{w}}), \\ \hat{w}_k - \xi, & \text{if } e_k \notin T^0, \end{cases}$$

where $\xi = \frac{n-N_U-1}{n-1}\mu(\hat{\boldsymbol{w}}) > 0$, $\beta = \frac{N_U}{n-1}\mu(\hat{\boldsymbol{w}}) > 0$ are defined the same as in **Case 2**.

For each $e_i \in EU(\hat{\boldsymbol{w}})$, if $e_k \in \Omega_i$, we have $w_k^* \geq w_{j_i}^* = \hat{w}_{j_i} - \xi = \hat{w}_i - \xi = w_i^*$. By using the same discussion as in **Case 2**, we can know that $\boldsymbol{w}^*$ is a better feasible solution than $\hat{\boldsymbol{w}}$ satisfying $\lambda^* = \hat{\lambda} - \xi < \hat{\lambda}$.

Case 4 $EU(\hat{\boldsymbol{w}}) \neq \emptyset$, $ED(\hat{\boldsymbol{w}}) \neq \emptyset$.

Then $0 < N_D < n-1, 0 < N_U < n-1-N_D, EM(\hat{\boldsymbol{w}}) = T^0 \backslash (ED(\hat{\boldsymbol{w}}) \cup EU(\hat{\boldsymbol{w}}))$, and $N_M = n - N_D - N_U - 1 > 0$. Let

$$w_k^* = \begin{cases} \hat{w}_k + \tau, & \text{if } e_k \in ED(\hat{\boldsymbol{w}}), \\ \hat{w}_k - \rho, & \text{if } e_k \in EU(\hat{\boldsymbol{w}}), \\ \hat{w}_k, & \text{if } e_k \in EM(\hat{\boldsymbol{w}}), \\ \hat{w}_k - \xi, & \text{if } e_k \notin T^0, \end{cases}$$

where $\xi = \min\{\tau, \rho\}$, and τ, ρ are solutions of the following equation system

$$\begin{cases} \rho \cdot N_U = \tau \cdot N_D, \\ \rho + \tau = \mu(\hat{\boldsymbol{w}}) = \min\{\gamma(\hat{\boldsymbol{w}}), \eta(\hat{\boldsymbol{w}}), \hat{\lambda} - \frac{\delta}{2}\}. \end{cases}$$

Then $\tau = \frac{N_U}{N_D+N_U}\mu(\hat{\boldsymbol{w}}) > 0$, $\rho = \frac{N_D}{N_D+N_U}\mu(\hat{\boldsymbol{w}}) > 0$.

For each $e_i \in EM(\hat{\boldsymbol{w}})$, if $e_k \in \Omega_i, e_{j_i} \in \Theta_i$, then

$$w_k^* \geq w_{j_i}^* = \hat{w}_{j_i} - \xi \geq \hat{w}_{j_i} - \eta(\hat{\boldsymbol{w}}) \geq \hat{w}_{j_i} + \hat{w}_i - (w_{j_i} + \hat{\lambda}) = \hat{w}_i = w_i^*,$$

where the second inequality follows from $\xi \leq \tau \leq \mu(\hat{\boldsymbol{w}}) \leq \eta(\hat{\boldsymbol{w}})$ and the third inequality follows from the definition of $\eta(\hat{\boldsymbol{w}})$.

For each $e_i \in ED(\hat{\boldsymbol{w}})$, if $e_k \in \Omega_i$, notice that $w_i - w_{j_i} \leq \delta$, $\xi < \hat{\lambda} - \frac{\delta}{2}$ and $\tau < \hat{\lambda} - \frac{\delta}{2}$, then

$$w_k^* \geq w_{j_i}^* = w_{j_i} + (\hat{\lambda} - \xi) \geq w_{j_i} + \frac{\delta}{2} \geq w_i - \frac{\delta}{2} \geq w_i - (\hat{\lambda} - \tau) = \hat{w}_i + \tau = w_i^*.$$

For each $e_i \in EU(\hat{\boldsymbol{w}})$, if $e_k \in \Omega_i$, then $w_k^* \geq w_{j_i}^* = \hat{w}_{j_i} - \xi \geq \hat{w}_i - \rho = w_i^*$. Furthermore,

$$\begin{aligned} \sum_{e_i \in T^0} w_i^* &= \sum_{e_i \in ED(\hat{\boldsymbol{w}})} w_i^* + \sum_{e_i \in EU(\hat{\boldsymbol{w}})} w_i^* + \sum_{e_i \in EM(\hat{\boldsymbol{w}})} w_i^* \\ &= \sum_{e_i \in ED(\hat{\boldsymbol{w}})} (\hat{w}_i + \tau) + \sum_{e_i \in EU(\hat{\boldsymbol{w}})} (\hat{w}_i - \rho) + \sum_{e_i \in EM(\hat{\boldsymbol{w}})} \hat{w}_i \\ &= \sum_{e_i \in T^0} \hat{w}_i + \tau \cdot N_D - \rho \cdot N_U = \sum_{e_i \in T^0} \hat{w}_i = K. \end{aligned}$$

Now we show that the objective value $\lambda^* = \hat{\lambda} - \xi < \hat{\lambda}$. Firstly, we show that $|w_i^* - w_i| \leq \lambda^*$ for each $e_i \in EM(\hat{\boldsymbol{w}})$. (a) $w_i^* - w_i \leq w_{j_i}^* - w_i = (w_{j_i} + \hat{\lambda} - \xi) - w_i \leq \hat{\lambda} - \xi = \lambda^*$, (b) notice that $\xi \leq \gamma(\hat{\boldsymbol{w}}) \leq \hat{w}_i + \hat{\lambda} - w_i$ and $\hat{w}_i = w_i^*$, then $\hat{w}_i - w_i \geq \xi - \hat{\lambda}$, hence $w_i^* - w_i \geq -\hat{\lambda} + \xi = -\lambda^*$. Then $|w_i^* - w_i| \leq \lambda^*$ for each $e_i \in EM(\hat{\boldsymbol{w}})$. Secondly, for each $e_i \in ED(\hat{\boldsymbol{w}})$, $|w_i^* - w_i| = |w_i - \hat{\lambda} + \tau - w_i| = |\hat{\lambda} - \tau| \le |\hat{\lambda} - \xi| = \lambda^*$. Thirdly, for each $e_i \in EU(\hat{\boldsymbol{w}})$, $|w_i^* - w_i| = |w_i + \hat{\lambda} - \rho - w_i| = |\hat{\lambda} - \rho| \le |\hat{\lambda} - \xi| = \lambda^*$. Finally, for each $e_j \notin T^0$, $|w_j^* - w_j| = |w_j + \hat{\lambda} - \xi - w_j| = |\hat{\lambda} - \xi| = \lambda^*$.

Thus $\boldsymbol{w^*}$ is a better feasible solution than $\hat{\boldsymbol{w}}$ satisfying $\lambda^* = \hat{\lambda} - \xi < \hat{\lambda}$. □

By using the similar proof as in Case 4 of Theorem 4, we can prove the following corollary.

Corollary 5 *Suppose $\Omega^0 = \emptyset$ and $\Theta^0 = \emptyset$. Let $\hat{\boldsymbol{w}}$ be a feasible solution of problem (2.4) satisfying (2.3) and $\hat{\lambda} > \frac{\delta}{2}$. If $ED(\hat{\boldsymbol{w}}) \neq \emptyset$ and $EU(\hat{\boldsymbol{w}}) \neq \emptyset$, then $\hat{\boldsymbol{w}}$ is not an optimal solution of (2.4).*

Define

$$\lambda_1 = \frac{\boldsymbol{w}(T^0) - K}{n-1}, \quad \lambda_2 = \frac{K - \sum\limits_{e_i \in T^0 \backslash \Theta^0, e_{j_i} \in \Theta_i} w_{j_i} - \sum\limits_{e_i \in \Theta^0} w_i}{n-1}. \tag{2.7}$$

In order to reduce $\boldsymbol{w}(T^0)$ to K, we have two possible cases depending on the values of λ_1 and λ_2. The first case is to reduce each weight w_i of $e_i \in T^0$ to $w_i - \lambda_1$. The second case is to change weight w_i to $w_{j_i} + \lambda_2$ if $\Theta_i \neq \emptyset$ and to $w_i + \lambda_2$ if $e_i \in \Theta^0$, for each $e_i \in T^0$.

Lemma 6 *Suppose $\Omega^0 = \emptyset$. If λ_1 and λ_2 are defined as in (2.7), then $\lambda_1 + \lambda_2 \le \delta$.*

Proof

$$\begin{aligned}
\lambda_1 + \lambda_2 &= \frac{\boldsymbol{w}(T^0) - K}{n-1} + \frac{K - \sum\limits_{e_i \in T^0 \backslash \Theta^0, e_{j_i} \in \Theta_i} w_{j_i} - \sum\limits_{e_i \in \Theta^0} w_i}{n-1} \\
&= \frac{\sum\limits_{e_i \in T^0 \backslash \Theta^0, e_{j_i} \in \Theta_i} (w_i - w_{j_i})}{n-1} \\
&\le \frac{|T^0 \backslash \Theta^0|}{n-1} \cdot \max\{w_i - w_{j_i} | e_i \in T^0 \backslash \Theta^0, e_{j_i} \in \Theta_i\} \\
&\le \frac{|T^0 \backslash \Theta^0|}{n-1} \cdot \delta = \frac{(n - 1 - |\Theta^0|)}{n-1} \cdot \delta \le \delta
\end{aligned}$$

The lemma holds. □

Now we prove the sufficient and necessary condition of an optimal solution of problem (2.4).

Theorem 7 *Suppose $\Omega^0 = \emptyset$ and $\Theta^0 = \emptyset$. Let $\hat{\boldsymbol{w}}$ be a feasible solution of problem (2.4) which satisfies (2.3). If its objective value is $\hat{\lambda} > \frac{\delta}{2}$, then $\hat{\boldsymbol{w}}$ is an optimal solution of problem (2.4) if and only if either $ED(\hat{\boldsymbol{w}}) \cup EM(\hat{\boldsymbol{w}}) = \emptyset$ or $EU(\hat{\boldsymbol{w}}) \cup EM(\hat{\boldsymbol{w}}) = \emptyset$.*

Proof (Necessity) Let $\hat{\boldsymbol{w}}$ be an optimal solution of problem (2.4) satisfying (2.3) and $\hat{\lambda} > \frac{\delta}{2}$, then we know that $EM(\hat{\boldsymbol{w}}) = \emptyset$ by Theorem 4, and either $ED(\hat{\boldsymbol{w}}) = \emptyset$ or $EU(\hat{\boldsymbol{w}}) = \emptyset$ by Corollary 5. Hence, $ED(\hat{\boldsymbol{w}}) \cup EM(\hat{\boldsymbol{w}}) = \emptyset$ or $EU(\hat{\boldsymbol{w}}) \cup EM(\hat{\boldsymbol{w}}) = \emptyset$.

(Sufficiency) Firstly, at least one of $ED(\hat{\boldsymbol{w}}) \cup EM(\hat{\boldsymbol{w}})$ and $EU(\hat{\boldsymbol{w}}) \cup EM(\hat{\boldsymbol{w}})$ is nonempty. Otherwise, $T^0 = \emptyset$. Notice that $\Theta^0 = \emptyset$, then $\lambda_2 = \frac{K - \sum_{e_i \in T^0, e_{j_i} \in \Theta_i} w_{j_i}}{n-1}$ and $\sum_{e_i \in T^0, e_{j_i} \in \Theta_i} w_{j_i} + (n-1)\lambda_2 = K$.

Case 1 If $ED(\hat{\boldsymbol{w}}) \cup EM(\hat{\boldsymbol{w}}) = \emptyset$ and $EU(\hat{\boldsymbol{w}}) \cup EM(\hat{\boldsymbol{w}}) \neq \emptyset$, then $T^0 = EU(\hat{\boldsymbol{w}}) \cup EM(\hat{\boldsymbol{w}}) \cup ED(\hat{\boldsymbol{w}}) = EU(\hat{\boldsymbol{w}})$. Due to the feasibility of $\hat{\boldsymbol{w}}$ and $\hat{\lambda} > \frac{\delta}{2}$, we have

$$\sum_{e_i \in T^0} \hat{w}_i = \sum_{e_i \in T^0, e_{j_i} \in \Theta_i} (w_{j_i} + \hat{\lambda}) = \sum_{e_i \in T^0, e_{j_i} \in \Theta_i} w_{j_i} + (n-1)\hat{\lambda} = K. \tag{2.8}$$

Thus, $\lambda_2 = \hat{\lambda} > \frac{\delta}{2}$ and $\lambda_1 < \frac{\delta}{2}$ by definition of λ_1, λ_2 and Lemma 6.

If $\hat{\boldsymbol{w}}$ is not an optimal solution of problem (2.4), suppose $\boldsymbol{w}^*$ is an optimal solution with $\lambda^* < \hat{\lambda}$, then $ED(\boldsymbol{w}^*) \cup EM(\boldsymbol{w}^*) = \emptyset$ or $EU(\boldsymbol{w}^*) \cup EM(\boldsymbol{w}^*) = \emptyset$ by the necessity of Theorem 7.

(a) If $EU(\boldsymbol{w}^*) = EM(\boldsymbol{w}^*) = \emptyset$ and $ED(\boldsymbol{w}^*) = T^0$, then

$$\sum_{e_i \in T^0} w_i^* = \sum_{e_i \in T^0} (w_i - \lambda^*) = \boldsymbol{w}(T^0) - (n-1)\lambda^* = K = \boldsymbol{w}(T^0) - (n-1)\lambda_1.$$

Hence, $\lambda^* = \lambda_1$, which contradicts that $\lambda^* \geq \frac{\delta}{2} > \lambda_1$ by Lemma 3.

(b) If $ED(\boldsymbol{w}^*) = EM(\boldsymbol{w}^*) = \emptyset$ and $EU(\boldsymbol{w}^*) = T^0$, then by (2.8) we have

$$\begin{aligned}\sum_{e_i \in T^0} w_i^* = \sum_{e_i \in T^0} (w_{j_i} + \lambda^*) &= \sum_{e_i \in T^0, e_{j_i} \in \Theta_i} w_{j_i} + (n-1)\lambda^* = K \\ &= \sum_{e_i \in T^0, e_{j_i} \in \Theta_i} w_{j_i} + (n-1)\hat{\lambda}.\end{aligned}$$

Hence, $\lambda^* = \hat{\lambda}$, which contradicts $\lambda^* < \hat{\lambda}$.

(c) If $ED(\boldsymbol{w}^*) = EM(\boldsymbol{w}^*) = EU(\boldsymbol{w}^*) = \emptyset$, then $T^0 = \emptyset$, which is impossible.

Thus, $\hat{\boldsymbol{w}}$ is an optimal solution of problem (2.4).

Case 2 If $ED(\hat{\boldsymbol{w}}) \cup EM(\hat{\boldsymbol{w}}) \neq \emptyset$ and $EU(\hat{\boldsymbol{w}}) \cup EM(\hat{\boldsymbol{w}}) = \emptyset$, then $T^0 = ED(\hat{\boldsymbol{w}})$. Due to the feasibility of $\hat{\boldsymbol{w}}$ and $\hat{\lambda} > \frac{\delta}{2}$, we have

$$\sum_{e_i \in T^0} \hat{w}_i = \sum_{e_i \in T^0} (w_i - \hat{\lambda}) = \boldsymbol{w}(T^0) - (n-1)\hat{\lambda} = K = \boldsymbol{w}(T^0) - (n-1)\lambda_1. \tag{2.9}$$

Thus, $\lambda_1 = \hat{\lambda} > \frac{\delta}{2}$ and $\lambda_2 < \frac{\delta}{2}$ by definition of λ_1, λ_2 and Lemma 6.

Suppose $\boldsymbol{w}^*$ is an optimal solution of problem (2.4) with $\lambda^* < \hat{\lambda}$, then $ED(\boldsymbol{w}^*) \cup EM(\boldsymbol{w}^*) = \emptyset$ or $EU(\boldsymbol{w}^*) \cup EM(\boldsymbol{w}^*) = \emptyset$ by the necessity of Theorem 7.

(a) If $EU(\boldsymbol{w^*}) = EM(\boldsymbol{w^*}) = \emptyset$ and $ED(\boldsymbol{w^*}) = T^0$, then

$$\sum_{e_i \in T^0} w_i^* = \sum_{e_i \in T^0} (w_i - \lambda^*) = \boldsymbol{w}(T^0) - (n-1)\lambda^* = K = \boldsymbol{w}(T^0) - (n-1)\hat{\lambda}.$$

Hence, $\lambda^* = \hat{\lambda}$, which contradicts $\lambda^* < \hat{\lambda}$.

(b) If $ED(\boldsymbol{w^*}) = EM(\boldsymbol{w^*}) = \emptyset$ and $EU(\boldsymbol{w^*}) = T^0$, then

$$\begin{aligned}\sum_{e_i \in T^0} w_i^* = \sum_{e_i \in T^0} (w_{j_i} + \lambda^*) &= \sum_{e_i \in T^0, e_{j_i} \in \Theta_i} w_{j_i} + (n-1)\lambda^* = K \\ &= \sum_{e_i \in T^0, e_{j_i} \in \Theta_i} w_{j_i} + (n-1)\lambda_2.\end{aligned}$$

Hence, $\lambda^* = \lambda_2 < \frac{\delta}{2}$, which contradicts Lemma 3 that $\lambda^* \geq \frac{\delta}{2}$.

(c) If $ED(\boldsymbol{w^*}) = EM(\boldsymbol{w^*}) = EU(\boldsymbol{w^*}) = \emptyset$, then $T^0 = \emptyset$, which is impossible.

Thus, $\hat{\boldsymbol{w}}$ is an optimal solution of problem (2.4).

Hence, under the condition $\Omega^0 = \emptyset$, $\Theta^0 = \emptyset$, and $\hat{\lambda} > \frac{\delta}{2}$, if $ED(\hat{\boldsymbol{w}}) \cup EM(\hat{\boldsymbol{w}}) = \emptyset$ or $EU(\hat{\boldsymbol{w}}) \cup EM(\hat{\boldsymbol{w}}) = \emptyset$, then $\hat{\boldsymbol{w}}$ is an optimal solution of problem (2.4). □

2.2 Properties of an optimal solution of the problem IOVMST$_\infty$ when $\boldsymbol{\Theta^0 \neq \emptyset}$ and $\boldsymbol{\Omega^0 = \emptyset}$

For the feasible solution $\hat{\boldsymbol{w}}$ satisfying (2.3) of problem (2.4) with objective value $\hat{\lambda}$, if $\Theta^0 \neq \emptyset$, then we define a new weight vector $\tilde{\boldsymbol{w}}$ and three sets $EU(\hat{\boldsymbol{w}}), ED(\hat{\boldsymbol{w}}), EM(\hat{\boldsymbol{w}})$.

$$\tilde{w}_i = \begin{cases} w_{j_i} + \hat{\lambda}, & \text{if } e_i \in T^0 \backslash \Theta^0, e_{j_i} \in \Theta_i, \\ w_i + \hat{\lambda}, & \text{if } e_i \in \Theta^0, \\ \hat{w}_i = w_i + \hat{\lambda}, & \text{if } e_i \notin T^0, \end{cases} \tag{2.10}$$

$$\begin{cases} EU(\hat{\boldsymbol{w}}) = \{e_i \in T^0 | \hat{w}_i = \tilde{w}_i\}, \\ ED(\hat{\boldsymbol{w}}) = \{e_i \in T^0 | \hat{w}_i = w_i - \hat{\lambda}\}, \\ EM(\hat{\boldsymbol{w}}) = \{e_i \in T^0 | w_i - \hat{\lambda} < \hat{w}_i < \tilde{w}_i\}. \end{cases} \tag{2.11}$$

Obviously, if $\hat{\lambda} > \frac{\delta}{2}$, then $ED(\hat{\boldsymbol{w}}) \cap EU(\hat{\boldsymbol{w}}) = \emptyset$ and $T^0 = EM(\hat{\boldsymbol{w}})$.

Theorem 8 *Suppose $\Omega^0 = \emptyset$. Let $\hat{\boldsymbol{w}}$ be a feasible solution of problem (2.4) satisfying (2.3) with objective value $\hat{\lambda} > \frac{\delta}{2}$. If $EM(\hat{\boldsymbol{w}}) \neq \emptyset$, then $\hat{\boldsymbol{w}}$ is not an optimal solution of problem (2.4), where $EM(\hat{\boldsymbol{w}})$ is defined as in (2.11) and $\tilde{\boldsymbol{w}}$ used as in $EM(\hat{\boldsymbol{w}})$ is defined as in (2.10).*

Proof Note that when $\Theta^0 = \emptyset$, the sets $ED(\hat{\boldsymbol{w}}), EM(\hat{\boldsymbol{w}}), EU(\hat{\boldsymbol{w}})$ defined as in (2.11) are the same as in (2.5), then the result holds obviously due to Theorem 4. If $\Theta^0 \neq \emptyset$, we can prove similarly by considering four cases depending on the two sets $EU(\hat{\boldsymbol{w}}), ED(\hat{\boldsymbol{w}})$ empty or not. In each case, we can find a better feasible solution

$\boldsymbol{w^*}$ than $\hat{\boldsymbol{w}}$ with $\lambda^* < \hat{\lambda}$. Notice that the only difference in the proof of two theorems is the result related to edges $e_i \in EU(\hat{\boldsymbol{w}})$, for which we should study two different cases: (i) $e_i \in T^0 \backslash \Theta^0$ and (ii) $e_i \in \Theta^0$. Next we only take the proof of Case 4 as an example and omit the similar proof in the other three cases.

Case 4 $EU(\hat{\boldsymbol{w}}) \neq \emptyset$, $ED(\hat{\boldsymbol{w}}) \neq \emptyset$, we can get a better feasible solution $\boldsymbol{w^*}$ with $\lambda^* < \hat{\lambda}$.

Let $|ED(\hat{\boldsymbol{w}})| = N_D < n-1$, $|EU(\hat{\boldsymbol{w}})| = N_U < n-1-N_D$, $EM(\hat{\boldsymbol{w}}) = T^0 \backslash (ED(\hat{\boldsymbol{w}}) \cup EU(\hat{\boldsymbol{w}}))$, thus $|EM(\hat{\boldsymbol{w}})| = n_M = n - N_D - N_U - 1 > 0$. Let

$$w_k^* = \begin{cases} \hat{w}_k + \tau, & \text{if } e_k \in ED(\hat{\boldsymbol{w}}), \\ \hat{w}_k - \rho, & \text{if } e_k \in EU(\hat{\boldsymbol{w}}), \\ \hat{w}_k, & \text{if } e_k \in EM(\hat{\boldsymbol{w}}), \\ \hat{w}_k - \xi, & \text{if } e_k \notin T^0, \end{cases}$$

where $\xi = \min\{\tau, \rho\}$, and $\tau = \frac{N_U}{N_D+N_U}\mu(\hat{\boldsymbol{w}}) > 0$, $\rho = \frac{N_D}{N_D+N_U}\mu(\hat{\boldsymbol{w}}) > 0$.

For each $e_i \in EM(\hat{\boldsymbol{w}})$, if $e_k \in \Omega_i$, $e_{j_i} \in \Theta_i$, then

$$w_k^* \geq w_{j_i}^* = \hat{w}_{j_i} - \xi \geq \hat{w}_{j_i} - \eta(\hat{\boldsymbol{w}}) \geq \hat{w}_{j_i} + \hat{w}_i - (w_{j_i} + \hat{\lambda}) = \hat{w}_i = w_i^*.$$

For each $e_i \in ED(\hat{\boldsymbol{w}})$, if $e_k \in \Omega_i$, notice that $w_i - w_{j_i} \leq \delta$, $\xi < \hat{\lambda} - \frac{\delta}{2}$ and $\tau < \hat{\lambda} - \frac{\delta}{2}$, then

$$w_k^* \geq w_{j_i}^* = w_{j_i} + (\hat{\lambda} - \xi) \geq w_{j_i} + \frac{\delta}{2} \geq w_i - \frac{\delta}{2} \geq w_i - (\hat{\lambda} - \tau) = \hat{w}_i + \tau = w_i^*.$$

For each $e_i \in EU(\hat{\boldsymbol{w}})$, (1) if $e_i \in T^0 \backslash \Theta^0$, $e_k \in \Omega_i$ and $e_{j_i} \in \Theta_i$, then $w_k^* \geq w_{j_i}^* = \hat{w}_{j_i} - \xi \geq \hat{w}_i - \xi \geq \hat{w}_i - \rho = w_i^*$. (2) If $e_i \in \Theta^0$, $e_k \in \Omega_i$, then $w_k^* = \hat{w}_k - \xi \geq \hat{w}_i - \xi \geq \hat{w}_i - \rho = w_i^*$. Furthermore,

$$\begin{aligned} \sum_{e_i \in T^0} w_i^* &= \sum_{e_i \in ED(\hat{\boldsymbol{w}})} (\hat{w}_i + \tau) + \sum_{e_i \in EU(\hat{\boldsymbol{w}})} (\hat{w}_i - \rho) + \sum_{e_i \in EM(\hat{\boldsymbol{w}})} \hat{w}_i \\ &= \hat{\boldsymbol{w}}(T^0) + \tau \cdot N_D - \rho \cdot N_U = \hat{\boldsymbol{w}}(T^0) = K. \end{aligned}$$

Now we show that the objective value $\lambda^* = \hat{\lambda} - \xi < \hat{\lambda}$. Firstly, we show that $|w_i^* - w_i| \leq \lambda^*$ for each $e_i \in EM(\hat{\boldsymbol{w}})$. (a) $w_i^* - w_i \leq w_{j_i}^* - w_i = (w_{j_i} + \hat{\lambda} - \xi) - w_i \leq \hat{\lambda} - \xi = \lambda^*$, (b) Notice that $\xi \leq \gamma(\hat{\boldsymbol{w}}) \leq \hat{w}_i + \hat{\lambda} - w_i$ and $\hat{w}_i = w_i^*$, then $\hat{w}_i - w_i \geq \xi - \hat{\lambda}$, hence $w_i^* - w_i \geq -\hat{\lambda} + \xi = -\lambda^*$. Secondly, for each $e_i \in ED(\hat{\boldsymbol{w}})$, $|w_i^* - w_i| = |w_i - \hat{\lambda} + \tau - w_i| = |\hat{\lambda} - \tau| \leq |\hat{\lambda} - \xi| = \lambda^*$. Thirdly, for each $e_i \in EU(\hat{\boldsymbol{w}})$, (1) if $e_i \in T^0 \backslash \Theta^0$, then $|w_i^* - w_i| = |\hat{w}_i - \rho - w_i| = |w_{j_i} + \hat{\lambda} - \rho - w_i| \leq |\hat{\lambda} - \rho| \leq |\hat{\lambda} - \xi| = \lambda^*$. (2) If $e_i \in \Theta^0$, then $|w_i^* - w_i| = |w_i + \hat{\lambda} - \rho - w_i| = |\hat{\lambda} - \rho| \leq |\hat{\lambda} - \xi| = \lambda^*$. Finally, for each $e_j \notin T^0$, $|w_j^* - w_j| = |w_j + \hat{\lambda} - \xi - w_j| = |\hat{\lambda} - \xi| = \lambda^*$. □

By using the similar proof as in Theorem 7, we can prove the following theorem.

Theorem 9 *Suppose $\Omega^0 = \emptyset$. Let $\hat{\boldsymbol{w}}$ be a feasible solution of problem (2.4) satisfying (2.3) and $\hat{\lambda} > \frac{\delta}{2}$. $\hat{\boldsymbol{w}}$ is an optimal solution of problem (2.4) if and only if either $ED(\hat{\boldsymbol{w}}) \cup EM(\hat{\boldsymbol{w}}) = \emptyset$ or $EU(\hat{\boldsymbol{w}}) \cup EM(\hat{\boldsymbol{w}}) = \emptyset$, where $EM(\hat{\boldsymbol{w}})$, $EU(\hat{\boldsymbol{w}})$ and $ED(\hat{\boldsymbol{w}})$ are defined as in (2.11) and $\tilde{\boldsymbol{w}}$ used in $EM(\hat{\boldsymbol{w}})$ and $EU(\hat{\boldsymbol{w}})$ is defined as in (2.10).*

In fact, the results in Theorems 8 and 9 are more general than that in Theorems 4 and 7, since they include the case of $\Theta^0 = \emptyset$. So we omit the condition $\Theta^0 = \emptyset$ in Theorems 8 and 9.

3 An algorithm for the problem IOVMST$_\infty$

In this section, we first propose an algorithm to solve the problem **IOVMST**$_\infty$ when $\Omega^0 = \emptyset$ and then we propose an algorithm to solve the problem **IOVMST**$_\infty$ when $\Omega^0 \neq \emptyset$.

3.1 An algorithm for the problem IOVMST$_\infty$ when $\boldsymbol{\Omega^0 = \emptyset}$

In this subsection, we first propose an algorithm to solve the problem **IOVMST**$_\infty$ when $\Omega^0 = \emptyset$, then prove its optimality and analyze its time complexity.

The main idea of the algorithm is as follows. Firstly, compute $\lambda_1 = \frac{\boldsymbol{w}(T^0)-K}{n-1}$. If $\lambda_1 > \frac{\delta}{2}$, then let $\lambda^* = \lambda_1$, let $w_i^* = w_i - \lambda^*$ for $e_i \in T^0$ and $w_i^* = w_i + \lambda^*$ for $e_i \notin T^0$, and then $EU(\boldsymbol{w^*}) \cup EM(\boldsymbol{w^*}) = \emptyset$, which means w^* is an optimal solution of problem (2.4) with $\lambda^* = \lambda_1$ by Theorem 9. Otherwise, compute λ_2 by (2.7). If $\lambda_2 > \frac{\delta}{2}$, then let $\lambda^* = \lambda_2$, let $w_i^* = w_i + \lambda^*$ for $e_i \in \Theta^0$, $w_i^* = w_{j_i} + \lambda^*$ for $e_i \in T^0 \backslash \Theta^0$, and $w_i^* = w_i + \lambda^*$ for $e_i \notin T^0$, and then $ED(\boldsymbol{w^*}) \cup EM(\boldsymbol{w^*}) = \emptyset$, hence w^* is an optimal solution of problem (2.4) with $\lambda^* = \lambda_2$ by Theorem 9. Secondly, if $\max\{\lambda_1, \lambda_2\} \leq \frac{\delta}{2}$, we will consider five cases to present an optimal solution w^* with optimal objective value $\lambda^* = \frac{\delta}{2}$.

Next we present Algorithm 1 to solve the problem **IOVMST**$_\infty$ when $\Omega^0 = \emptyset$.

Next we prove $\boldsymbol{w^*}$ obtained in Algorithm 1 is an optimal solution of problem (2.4).

Theorem 10 *Suppose $\Omega^0 = \emptyset$. If $\max\{\lambda_1, \lambda_2\} > \frac{\delta}{2}$, then the optimal objective value is $\max\{\lambda_1, \lambda_2\}$, and $\boldsymbol{w^*}$ defined by (3.12) in Algorithm 1 is an optimal solution of problem (2.4).*

Proof If $\max\{\lambda_1, \lambda_2\} > \frac{\delta}{2}$, then at most one of λ_1 and λ_2 is not less than $\frac{\delta}{2}$ as $\lambda_1 + \lambda_2 \leq \delta$ by Lemma 6.

(1) If $\lambda_1 = \lambda^* > \frac{\delta}{2}$, then $w_i^* = w_i - \lambda^*$ for $e_i \in T^0$ and $w_i^* = w_i + \lambda^*$ for $e_i \notin T^0$ according to (3.12). Hence, $EU(\boldsymbol{w^*}) \cup EM(\boldsymbol{w^*}) = \emptyset$, then $\boldsymbol{w^*}$ is an optimal solution of problem (2.4) with optimal objective value λ_1 by Theorem 9.

(2) If $\lambda_2 = \lambda^* > \frac{\delta}{2}$, then $w_i^* = w_i + \lambda^*$ for $e_i \in \Theta^0$, $w_i^* = w_{j_i} + \lambda^*$ for $e_i \in T^0 \backslash \Theta^0$, and $w_i^* = w_i + \lambda^*$ for $e_i \notin T^0$ according to (3.12). Hence $ED(\boldsymbol{w^*}) \cup EM(\boldsymbol{w^*}) = \emptyset$, then $\boldsymbol{w^*}$ is an optimal solution of problem (2.4) with optimal objective value λ_2 by Theorem 9. □

Algorithm 1 Solve the problem **IOVMST**$_\infty$ when $\Omega^0 = \emptyset$.

Input: A network $G = (V, E, \boldsymbol{w})$, a given spanning tree T^0 of G, a given number K.
Output: The optimal value λ^*, an optimal solution $\boldsymbol{w}^*$.

Let $\boldsymbol{w}(T^0) := \sum_{e_i \in T^0} w_i$, determine the set P_j for each $e_j \notin T^0$, and the sets Ω_i and Θ_i for each $e_i \in T^0$. Let $\Theta^0 := \{e_i \in T^0 | \Theta_i = \emptyset\}$, and $\delta := \max_{e_i \in T^0} \max_{e_j \in \Omega_i} \{w_i - w_j\}$. Let $\lambda_1 := \frac{\boldsymbol{w}(T^0) - K}{n-1}$, and $\lambda_2 := \frac{1}{n-1}\left(K - \sum_{e_i \in T^0 \setminus \Theta^0, e_{j_i} \in \Theta_i} w_{j_i} - \sum_{e_i \in \Theta^0} w_i\right)$.

if $\max\{\lambda_1, \lambda_2\} > \frac{\delta}{2}$ **then**
  $\lambda^* := \max\{\lambda_1, \lambda_2\}$ is the optimal objective value. Output an optimal solution $\boldsymbol{w}^*$ of problem (2.4).

$$w_i^* := \begin{cases} w_i - \lambda^*, & \text{if } e_i \in T^0,\ e_{j_i} \in \Theta_i, \lambda^* = \lambda_1 \\ w_{j_i} + \lambda^*, & \text{if } e_i \in T^0,\ e_{j_i} \in \Theta_i, \lambda^* = \lambda_2, \\ w_i + \lambda^*, & \text{if } e_i \in \Theta^0, \lambda^* = \lambda_2. \\ w_i + \lambda^*, & \text{if } e_i \notin T^0. \end{cases} \tag{3.12}$$

else
  Let $\lambda^* := \frac{\delta}{2}$. Let $EU^s := \{e_i \in T^0 | w_{j_i} + \frac{\delta}{2} < w_i, e_{j_i} \in \Theta_i\}$, $EU^b := \{e_i \in T^0 | w_{j_i} + \frac{\delta}{2} \geq w_i, e_{j_i} \in \Theta_i\}$, and $U^s := \sum_{e_i \in EU^s}(w_i - w_{j_i} - \frac{\delta}{2})$.
  if $U^s = \boldsymbol{w}(T^0) - K$ **then**
    output an optimal solution w^* defined as in (3.13).

$$w_i^* := \begin{cases} w_{j_i} + \frac{\delta}{2}, & \text{if } e_i \in EU^s,\ e_{j_i} \in \Theta_i, \\ w_i, & \text{if } e_i \in EU^b \cup \Theta^0, \\ w_i + \frac{\delta}{2}, & \text{if } e_i \notin T^0. \end{cases} \tag{3.13}$$

  else if $U^s > \boldsymbol{w}(T^0) - K$ **then**
    let $Rest^0 := U^s - (\boldsymbol{w}(T^0) - K)$, $D_\delta := \emptyset$, $D_d := \emptyset$, $EE := EU^b \cup \Theta^0$, $Rest := Rest^0$.
    while $EE \neq \emptyset$ and $Rest > 0$ **do**
      take an edge $e_c \in EE$.
      if $e_c \in EU^b$ **then**
        if $Rest \leq w_{jc} + \frac{\delta}{2} - w_c$ **then**
          output the **critical edge** e_c, the critical value $Rest$ and w^* defined as in (3.14).

$$w_i^* := \begin{cases} w_i + Rest, & \text{if } e_i = e_c, \\ w_i + \frac{\delta}{2}, & \text{if } e_i \in D_\delta, \\ w_{j_i} + \frac{\delta}{2}, & \text{if } e_i \in EU^s \cup D_d, \\ w_i, & \text{if } e_i \in (EU^b \cup \Theta^0) \setminus (D_d \cup D_\delta \cup \{e_c\}) \\ w_i + \frac{\delta}{2}, & \text{if } e_i \notin T^0. \end{cases} \tag{3.14}$$

        else
          let $Rest := Rest - (w_{jc} + \frac{\delta}{2} - w_c)$, $D_d := D_d \cup \{e_c\}$ and $EE := EE \setminus \{e_c\}$.
        end if
      else if $Rest \leq \frac{\delta}{2}$ **then**
        output the **critical edge** e_c, the critical value $Rest$ and w^* defined as in (3.14).
      else
        let $Rest := Rest - \frac{\delta}{2}$, $D_\delta := D_\delta \cup \{e_c\}$ and $EE := EE \setminus \{e_c\}$.
      end if
    end while
  else
    let $D^s := \frac{\delta}{2} \cdot |EU^s|$.
    if $D^s = \boldsymbol{w}(T^0) - K$ **then**
      output an optimal solution $\boldsymbol{w}^*$ defined below.

$$w_i^* := \begin{cases} w_i - \frac{\delta}{2}, & \text{if } e_i \in EU^s, \\ w_i, & \text{if } e_i \in EU^b \cup \Theta^0, \\ w_i + \frac{\delta}{2}, & \text{if } e_i \notin T^0. \end{cases} \tag{3.15}$$

    else if $D^s < \boldsymbol{w}(T^0) - K$ **then**
      let $\chi := \frac{\boldsymbol{w}(T^0) - K - D^s}{|EU^b \cup \Theta^0|}$, then output $\boldsymbol{w}^*$ defined below.

$$w_i^* := \begin{cases} w_i - \frac{\delta}{2}, & \text{if } e_i \in EU^s, \\ w_i - \chi, & \text{if } e_i \in EU^b \cup \Theta^0, \\ w_i + \frac{\delta}{2}, & \text{if } e_i \notin T^0. \end{cases} \tag{3.16}$$

    else
      let $Rest^0 := \boldsymbol{w}(T^0) - K - U^s$, $D_\delta := \emptyset$, $EE := EU^s$, $Rest := Rest^0$.
      while $EE \neq \emptyset$ **do**
        take an edge $e_p \in EE$.
        if $Rest \leq \delta - (w_p - w_{jp})$ **then**
          break and let $EE := EE \setminus \{e_p\}$ and output the **critical edge** e_p, the **critical value** $Rest$ and $\boldsymbol{w}^*$ defined as in (3.17).

$$w_i^* := \begin{cases} w_{j_i} + \frac{\delta}{2} - Rest, & \text{if } e_i = e_p,\ e_{j_i} \in \Theta_i, \\ w_i - \frac{\delta}{2}, & \text{if } e_i \in D_\delta, \\ w_{j_i} + \frac{\delta}{2}, & \text{if } e_{j_i} \in \Theta_i,\ e_i \in EU^s \setminus (D_\delta \cup \{e_p\}), \\ w_i, & \text{if } e_i \in EU^b \cup \Theta^0, \\ w_i + \frac{\delta}{2}, & \text{if } e_i \notin T^0. \end{cases} \tag{3.17}$$

        else
          let $D_\delta := D_\delta \cup \{e_p\}$ and $Rest := Rest - (\delta - w_p + w_{jp})$.
        end if
      end while
    end if
  end if
end if

Theorem 11 *Suppose $\Omega^0 = \emptyset$. If $\max\{\lambda_1, \lambda_2\} \leq \frac{\delta}{2}$, then the optimal objective value of problem (2.4) is $\lambda^* = \frac{\delta}{2}$ and $\boldsymbol{w}^*$ outputted by Algorithm 1 is an optimal solution.*

Proof We consider five cases according to lines 4-40 of Algorithm 1. In each case, we show that $\boldsymbol{w}^*$ obtained in the algorithm is an optimal solution of problem (2.4) with objective value $\frac{\delta}{2}$.

Case 1 If $U^s = \boldsymbol{w}(T^0) - K$, then we first show that $\boldsymbol{w}^*$ defined as in (3.13) satisfies $w_k^* \geq w_i^*$ for each $e_i \in T^0$ and $e_k \in \Omega_i$.

(1) If $e_i \in EU^s$, then $w_k^* = w_k + \frac{\delta}{2} \geq w_{j_i} + \frac{\delta}{2} = w_i^*$.

(2) If $e_i \in EU^b$, then $w_k^* = w_k + \frac{\delta}{2} \geq w_{j_i} + \frac{\delta}{2} \geq w_i = w_i^*$.

(3) If $e_i \in \Theta^0$, then $w_k^* = w_k + \frac{\delta}{2} \geq w_i + \frac{\delta}{2} > w_i = w_i^*$.

Notice that $\sum_{e_i \in EU^s}(w_{j_i} + \frac{\delta}{2}) = \sum_{e_i \in EU^s} w_i - U^s$, then $\sum_{e_i \in T^0} w_i^* = \sum_{e_i \in EU^s}(w_{j_i} + \frac{\delta}{2}) + \sum_{e_i \in EU^b \cup \Theta^0} w_i = \sum_{e_i \in EU^s} w_i + \sum_{e_i \in EU^b \cup \Theta^0} w_i - U^s = \boldsymbol{w}(T^0) - U^s = K$. Then $\boldsymbol{w}^*$ is a feasible solution of problem (2.4).

Furthermore, for each $e_i \in EU^s$, $|w_i^* - w_i| = |w_i - w_{j_i} - \frac{\delta}{2}| \leq \frac{\delta}{2}$, and for each $e_k \notin T^0$, $|w_k^* - w_k| = \frac{\delta}{2}$.

Hence $\boldsymbol{w}^*$ is an optimal solution of problem (2.4) with $\lambda^* = \frac{\delta}{2}$.

Case 2 If $U^s > \boldsymbol{w}(T^0) - K$, we first show that $\boldsymbol{w}^*$ obtained by (3.14) satisfies $w_k^* \geq w_i^*$ for each $e_i \in T^0$ and $e_k \in \Omega_i$.

(1) If $e_i = e_c$. (a) If $e_c \in \Theta^0$, then $w_k^* = w_k + \frac{\delta}{2} \geq w_c + \frac{\delta}{2} \geq w_c + Rest = w_c^* = w_i^*$. (b) If $e_c \in EU^b$, notice that $Rest \leq w_{j_c} + \frac{\delta}{2} - w_c$, then $w_k^* = w_k + \frac{\delta}{2} \geq w_{j_c} + \frac{\delta}{2} = (w_{j_c} + \frac{\delta}{2} - Rest) + Rest \geq w_c + Rest = w_c^* = w_i^*$. Thus, if $e_i = e_c$, then $w_k^* \geq w_i^*$.

(2) If $e_i \in D_\delta$, we know that $e_i \in \Theta^0$, then $w_k^* = w_k + \frac{\delta}{2} \geq w_i + \frac{\delta}{2} = w_i^*$.

(3) If $e_i \in EU^s \cup D_d$, then $e_i \in T^0 \backslash \Theta^0$, and $w_k^* = w_k + \frac{\delta}{2} \geq w_{j_i} + \frac{\delta}{2} = w_i^*$.

(4) If $e_i \in (EU^b \cup \Theta^0) \backslash (D_\delta \cup D_d \cup \{e_c\})$. (a) If $e_i \in \Theta^0$, then $w_k^* = w_k + \frac{\delta}{2} \geq w_i + \frac{\delta}{2} > w_i = w_i^*$. (b) If $e_i \in EU^b$, then $w_k^* = w_k + \frac{\delta}{2} \geq w_{j_i} + \frac{\delta}{2} \geq w_i = w_i^*$.

Now we show that $\sum_{e_i \in T^0} w_i^* = K$. Let e_c be the critical edge and $Rest$ be the critical value outputted by lines 9-23 of Algorithm 1. Then $Rest^0 = Rest + \frac{\delta}{2} \cdot |D_\delta| + \sum\limits_{e_i \in D_d} [(w_{j_i} + \frac{\delta}{2}) - w_i] = U^s - (\boldsymbol{w}(T^0) - K)$. Notice that $U^s = \sum_{e_i \in EU^s}(w_i - (w_{j_i} + \frac{\delta}{2}))$, then we have

$$\sum_{e_i \in EU^s} \left[w_i - \left(w_{j_i} + \frac{\delta}{2} \right) \right] - Rest - \frac{\delta}{2} \cdot |D_\delta| - \sum_{e_i \in D_d} \left[\left(w_{j_i} + \frac{\delta}{2} \right) - w_i \right]$$
$$= \boldsymbol{w}(T^0) - K.$$

Now we can calculate the value

$$\sum_{e_i \in T^0} w_i^* = (w_c + Rest) + \sum_{e_i \in D_\delta} \left(w_i + \frac{\delta}{2} \right) + \sum_{e_i \in EU^s \cup D_d} \left(w_{j_i} + \frac{\delta}{2} \right)$$
$$+ \sum_{e_i \in (EU^b \cup \Theta^0) \backslash ((D_\delta \cup D_d \cup \{e_c\})} w_i$$

$$= (w_c + Rest) + \sum_{e_i \in D_\delta} w_i + \sum_{e_i \in D_\delta} \frac{\delta}{2} + \sum_{e_i \in EU^s \cup D_d} w_i$$
$$+ \sum_{e_i \in EU^s \cup D_d} (w_{j_i} + \frac{\delta}{2} - w_i) + \sum_{e_i \in (EU^b \cup \Theta^0) \backslash ((D_\delta \cup D_d \cup \{e_c\})} w_i$$
$$= \boldsymbol{w}(T^0) + \{Rest + \frac{\delta}{2} \cdot |D_\delta| + \sum_{e_i \in D_d} [(w_{j_i} + \frac{\delta}{2}) - w_i]$$
$$- \sum_{e_i \in EU^s} [w_i - (w_{j_i} + \frac{\delta}{2})]\}$$
$$= \boldsymbol{w}(T^0) + (-\boldsymbol{w}(T^0) + K) = K.$$

Hence $\boldsymbol{w^*}$ obtained in Algorithm 1 is a feasible solution of problem (2.4).

Furthermore, (1) If $e_i = e_c$, then $|w_i - w_i^*| = |w_c - w_c^*| = Rest \leq \frac{\delta}{2}$. (2) If $e_i \in D_\delta$, $|w_i - w_i^*| = \frac{\delta}{2}$. (3) $e_i \in EU^s \cup D_d$. If $e_i \in EU^s$ then $|w_i - w_i^*| = w_i - w_{j_i} - \frac{\delta}{2} \leq \delta - \frac{\delta}{2} = \frac{\delta}{2}$. If $e_i \in D_d$, $|w_i - w_i^*| = w_{j_i} + \frac{\delta}{2} - w_i \leq \frac{\delta}{2}$ (4) If $e_i \in (EU^b \cup \Theta^0) \backslash (D_\delta \cup D_d \cup \{e_c\})$, then $|w_i - w_i^*| = w_i - w_i = 0 < \frac{\delta}{2}$. (5) If $e_i \notin T^0$, then $|w_i - w_i^*| = \frac{\delta}{2}$.

Hence $\boldsymbol{w^*}$ is an optimal solution of problem (2.4) with $\lambda^* = \frac{\delta}{2}$.

Case 3 If $U^s < \boldsymbol{w}(T^0) - K$ and $D^s = \boldsymbol{w}(T^0) - K$, then we first show that $\boldsymbol{w^*}$ defined as in (3.15) satisfies $w_k^* \geq w_i^*$ for each $e_i \in T^0$, $e_k \in \Omega_i$. (1) If $e_i \in EU^s$, then $w_k^* = w_k + \frac{\delta}{2} \geq w_{j_i} + \frac{\delta}{2} \geq w_i - \frac{\delta}{2} = w_i^*$. (2) If $e_i \in EU^b$, then $w_k^* = w_k + \frac{\delta}{2} \geq w_{j_i} + \frac{\delta}{2} \geq w_i = w_i^*$. (3) If $e_i \in \Theta^0$, then $w_k^* = w_k + \frac{\delta}{2} \geq w_i + \frac{\delta}{2} > w_i = w_i^*$. Notice that $D^s = \frac{\delta}{2} \cdot |EU^s| = \boldsymbol{w}(T^0) - K$, then $\sum\limits_{e_i \in T^0} w_i^* = \sum_{e_i \in EU^s} (w_i - \frac{\delta}{2}) + \sum_{e_i \in EU^b \cup \Theta^0} w_i = \boldsymbol{w}(T^0) - \frac{\delta}{2} \cdot |EU^s| = K$. Then $\boldsymbol{w^*}$ is a feasible solution of problem (2.4). Furthermore, $|w_i^* - w_i| = \frac{\delta}{2}$ for each $e_i \in EU^s$, and $|w_k^* - w_k| = \frac{\delta}{2}$ for each $e_k \notin T^0$. Hence $\boldsymbol{w^*}$ is an optimal solution of problem (2.4) with $\lambda^* = \frac{\delta}{2}$.

Case 4 If $U^s < \boldsymbol{w}(T^0) - K$ and $D^s < \boldsymbol{w}(T^0) - K$. Using the same analyses as in Case 1, we can prove that $\boldsymbol{w^*}$ defined as in (3.16) satisfies $w_k^* \geq w_i^*$ for each $e_i \in T^0$ and $e_k \in \Omega_i$. Notice that $\chi = \frac{\boldsymbol{w}(T^0) - K - D^s}{|EU^b \cup \Theta^0|}$ and $D^s = \frac{\delta}{2} \cdot |EU^s|$. If $\chi > \frac{\delta}{2}$, then $\boldsymbol{w}(T^0) - K - D^s > \frac{\delta}{2} \cdot |EU^b \cup \Theta^0|$, and $\sum_{e_i \in T^0} (w_i - \frac{\delta}{2}) > K = \sum_{e_i \in T^0} (w_i - \lambda_1)$, which means $\lambda_1 > \frac{\delta}{2}$ and contradicts $\lambda_1 \leq \frac{\delta}{2}$. Thus, $\chi \leq \frac{\delta}{2}$. Moreover,

$$\sum_{e_i \in T^0} w_i^* = \sum_{e_i \in EU^s} \left(w_i - \frac{\delta}{2}\right) + \sum_{e_i \in EU^b \cup \Theta^0} (w_i - \chi)$$
$$= \boldsymbol{w}(T^0) - \left(\frac{\delta}{2} \cdot |EU^s| + \chi \cdot |EU^b \cup \Theta^0|\right) = K.$$

Furthermore, it is obvious that $|w_i^* - w_i| \leq \frac{\delta}{2}$ for each $e_i \in T^0$ and $|w_k^* - w_k| = \frac{\delta}{2}$ for each $e_k \notin T^0$. Hence $\boldsymbol{w^*}$ is an optimal solution of problem (2.4) with $\lambda^* = \frac{\delta}{2}$.

Case 5 If $U^s < \boldsymbol{w}(T^0) - K$, and $D^s > \boldsymbol{w}(T^0) - K$, then we can similarly prove that $\boldsymbol{w}^*$ defined as in (3.17) satisfies $w_k^* \geq w_i^*$ for each $e_i \in T^0$ and $e_k \in \Omega_i$. Let e_p be the critical edge and $Rest$ be the critical value outputted by Line 35. Then $Rest^0 = Rest + \sum_{e_k \in D_\delta} (\delta - w_k + w_{j_k}) = Rest - \sum_{e_k \in D_\delta} (w_k - (w_{j_k} + \frac{\delta}{2}) - \frac{\delta}{2}) = \boldsymbol{w}(T^0) - K - U^s$. Notice that $U^s = \sum_{e_i \in EU^s} (w_i - (w_{j_i} + \frac{\delta}{2}))$, then we have

$$Rest + \frac{\delta}{2} \cdot |D_\delta| + \left[w_p - \left(w_{j_p} + \frac{\delta}{2} \right) \right] + \sum_{e_i \in EU^s \setminus (\{e_p\} \cup D_\delta)} \left[w_i - \left(w_{j_i} + \frac{\delta}{2} \right) \right]$$
$$= \boldsymbol{w}(T^0) - K.$$

Now we can calculate the value

$$\begin{aligned}
\sum_{e_i \in T^0} w_i^* &= \left(w_{j_p} + \tfrac{\delta}{2} - Rest \right) + \sum_{e_i \in D_\delta} \left(w_i - \tfrac{\delta}{2} \right) + \sum_{e_i \in EU^s \setminus (\{e_p\} \cup D_\delta)} \left(w_{j_i} \right. \\
&\quad \left. + \tfrac{\delta}{2} \right) + \sum_{e_i \in EU^b \cup \Theta^0} w_i \\
&= \{ w_p - [w_p - (w_{j_p} + \tfrac{\delta}{2})] - Rest \} + \sum_{e_i \in D_\delta} \left(w_i - \tfrac{\delta}{2} \right) \\
&\quad + \sum_{e_i \in EU^s \setminus (\{e_p\} \cup D_\delta)} \left(w_i - \left[w_i - \left(w_{j_i} + \tfrac{\delta}{2} \right) \right] \right) + \sum_{e_i \in EU^b \cup \Theta^0} w_i \\
&= \boldsymbol{w}(T^0) - \Big\{ Rest + \tfrac{\delta}{2} \cdot |D_\delta| + \left[w_p - \left(w_{j_p} + \tfrac{\delta}{2} \right) \right] \\
&\quad + \sum_{e_i \in EU^s \setminus (\{e_p\} \cup D_\delta)} \left[w_i - \left(w_{j_i} + \tfrac{\delta}{2} \right) \right] \Big\} \\
&= \boldsymbol{w}(T^0) - \boldsymbol{w}(T^0) + K = K.
\end{aligned}$$

Furthermore, it is obvious that $|w_i^* - w_i| \leq \frac{\delta}{2}$ for each $e_i \in T^0$ and $|w_k^* - w_k| = \frac{\delta}{2}$ for each $e_k \notin T^0$. Hence $\boldsymbol{w}^*$ is an optimal solution of problem (2.4) with $\lambda^* = \frac{\delta}{2}$. □

Finally, we analyze the time complexity of Algorithm 1.

Theorem 12 *Suppose $\Omega^0 = \emptyset$, the problem (2.4) can be solved in $O(mn)$ time by Algorithm 1.*

Proof In Algorithm 1, it is clear that line 1 takes $O(mn)$ time. Lines 3–41 take $O(m)$ time. Thus, Algorithm 1 runs in $O(mn)$ in the worst-case and hence it is a strongly polynomial time algorithm. □

3.2 An algorithm for IOVMST$_\infty$ problem when $\boldsymbol{\Omega^0 \neq \emptyset}$

In this subsection, we consider the case when $\Omega^0 \neq \emptyset$, which means that there is at least one edge belonging to every spanning tree of G. In this case, the algorithm to solve problem (2.4) is similar to Algorithm 1 and we only replace Θ^0 in Algorithm 1 with $\Phi^0 = \Omega^0 \cup \Theta^0$. Hence, we have the following corollary.

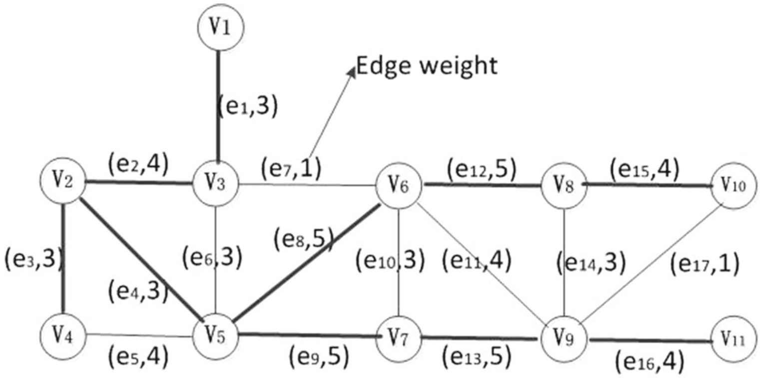

Fig. 3 An example of the $\mathbf{IOVMST}_\infty$ problem

Corollary 13 *If* $\Omega^0 = \{e_i | e_i \in T^0, \Omega_i = \emptyset\} \neq \emptyset$, $\boldsymbol{w}^*$ *obtained in Algorithm 1 by replacing* Θ^0 *with* $\Phi^0 = \Omega^0 \cup \Theta^0$ *is an optimal solution of problem* (2.4). *Furthermore, the time complexity is still* $O(mn)$.

4 A computational example of $\mathbf{IOVMST}_\infty$ problem

In this section, we use Example 1 to demonstrate the computing process of seven cases in Algorithm 1. Notice that in Example 1, $\Omega^0 \neq \emptyset$ and $\Theta^0 \neq \emptyset$.

Example 3 As shown in Fig. 3, let $V = \{v_1, v_2, \ldots, v_{11}\}$, $E = \{e_1, e_2, \ldots, e_{17}\}$, $\boldsymbol{w} = (3, 4, 3, 3, 4, 3, 1, 5, 5, 3, 4, 5, 5, 3, 4, 4, 1)$, $T^0 = \{e_1, e_2, e_3, e_4, e_8, e_9, e_{12}, e_{13}, e_{15}, e_{16}\}$ (T^0 is denoted by thick lines in Fig. 3).

After running Step 1 of Algorithm 1, we have

(I) $\boldsymbol{w}(T^0) = 41$; $\Omega_1 = \Omega_{16} = \emptyset$, $\Omega_2 = \{e_6, e_7\}$, $\Omega_3 = \{e_5\}$, $\Omega_4 = \{e_5, e_6, e_7\}$, $\Omega_8 = \{e_7, e_{10}, e_{11}, e_{14}, e_{17}\}$, $\Omega_9 = \{e_{10}, e_{11}, e_{14}, e_{17}\}$, $\Omega_{12} = \{e_{14}, e_{17}\}$, $\Omega_{13} = \{e_{11}, e_{14}, e_{17}\}$, $\Omega_{15} = \{e_{17}\}$;

(II) $\Omega^0 = \{e_1, e_{16}\}$ and $\Theta^0 = \{e_3\}$; $\Theta_1 = \Theta_{16} = \emptyset$, $\Theta_2 = \{e_7\}$, $\Theta_3 = \emptyset$, $\Theta_4 = \{e_7\}$, $\Theta_8 = \{e_7, e_{17}\}$, $\Theta_9 = \{e_{17}\}$, $\Theta_{12} = \{e_{17}\}$, $\Theta_{13} = \{e_{17}\}$, $\Theta_{15} = \{e_{17}\}$;

(III) $w_{j_2} = w_7 = 1$, $w_{j_4} = w_7 = 1$, $w_{j_8} = w_7 = 1$, $w_{j_9} = w_{17} = 1$, $w_{j_{12}} = w_{17} = 1$, $w_{j_{13}} = w_{17} = 1$, $w_{j_{15}} = w_{17} = 1$; and $\delta = w_8 - w_{j_8} = w_8 - w_7 = 4$;

(IV) $EU^s = \{e_2, e_8, e_9, e_{12}, e_{13}, e_{15}\}$, $EU^b = \{e_4\}$.

Seven cases are considered based on K.

$$\lambda_1 = \frac{\boldsymbol{w}(T^0) - K}{n-1} = \frac{41 - K}{10},$$

$$\lambda_2 = \frac{K - \sum\limits_{e_i \in T^0 \setminus (\Theta^0 \cup \Omega^0), e_{j_i} \in \Theta_i} w_{j_i} - \sum\limits_{e_i \in \Theta^0 \cup \Omega^0} w_i}{n-1} = \frac{K - 17}{10}.$$

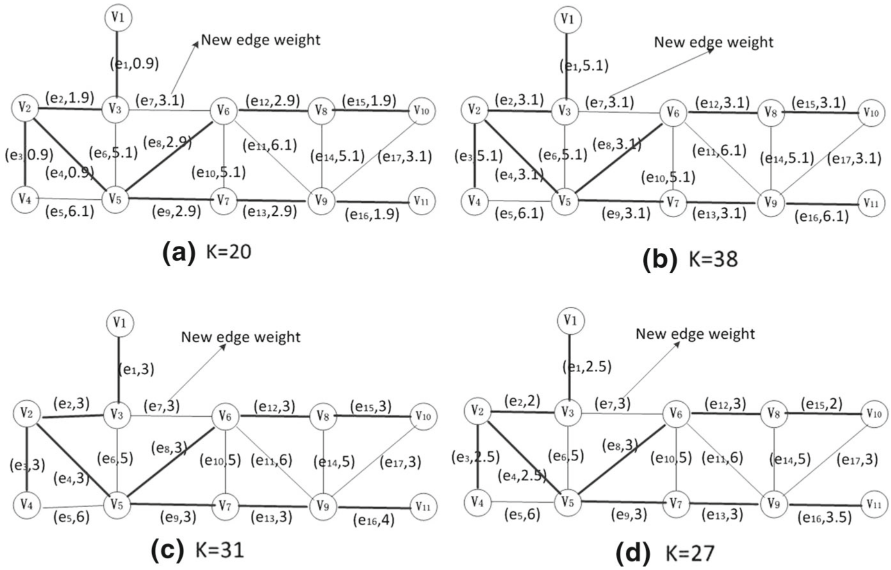

Fig. 4 An optimal solution of Example 1 when $K = 20, 38, 31, 27$

(1) If $K = 20$. $\lambda_1 = 2.1 > \frac{\delta}{2} = 2$, $\lambda_2 = 0.3 < \frac{\delta}{2} = 2$. Then run line 3 of Algorithm 1, obtain $\boldsymbol{w^*} = (0.9, 1.9, 0.9, 0.9, 6.1, 5.1, 3.1, 2.9, 2.9, 5.1, 6.1, 2.9, 2.9, 5.1, 1.9, 1.9, 3.1)$ and optimal objective value $\lambda^* = 2.1$ (see Fig. 4a).

(2) If $K = 38$. $\lambda_1 = 0.3 < \frac{\delta}{2} = 2$, $\lambda_2 = 2.1 > \frac{\delta}{2} = 2$. Then run line 3 of Algorithm 1, obtain $\boldsymbol{w^*} = (5.1, 3.1, 5.1, 3.1, 6.1, 5.1, 3.1, 3.1, 3.1, 5.1, 6.1, 3.1, 3.1, 5.1, 3.1, 6.1, 3.1)$ and $\lambda^* = 2.1$ (see Fig. 4b).

(3) If $K = 31$. $\lambda_1 = 1 < \frac{\delta}{2} = 2$, $\lambda_2 = 1.4 < \frac{\delta}{2} = 2$. Then $\max\{\lambda_1, \lambda_2\} = 1.4 < 2$, and $U^s = 10 = \boldsymbol{w}(T^0) - K$. Run line 7 of Algorithm 1, obtain $\boldsymbol{w^*} = (3, 3, 3, 3, 6, 5, 3, 3, 3, 5, 6, 3, 3, 5, 3, 4, 3)$ and $\lambda^* = 2$ (see Fig. 4c).

(4) If $K = 27$. $\lambda_1 = 1.4 < \frac{\delta}{2} = 2$, $\lambda_2 = 1 < \frac{\delta}{2} = 2$, $\chi = 0.5$. Then $\max\{\lambda_1, \lambda_2\} = 1.4 < 2$, $U^s = 10 < \boldsymbol{w}(T^0) - K$ and $D^s = 2 \cdot |EU^s| = 12 < \boldsymbol{w}(T^0) - K$. Run line 27 of Algorithm 1, obtain $\boldsymbol{w^*} = (2.5, 2, 2.5, 2.5, 6, 5, 3, 3, 3, 5, 6, 3, 3, 5, 2, 3.5, 3)$ and $\lambda^* = 2$ (see Fig. 4d).

(5) If $K = 35$. $\lambda_1 = 0.6 < \frac{\delta}{2} = 2$, $\lambda_2 = 1.8 < \frac{\delta}{2} = 2$. Then $\max\{\lambda_1, \lambda_2\} = 1.8 < 2$, $U^s = 10 > \boldsymbol{w}(T^0) - K = 6$. Run line 9-23 of Algorithm 1, obtain $Rest = 2$, e_3 is **critical edge**, $D_\delta = \{e_1\}$, $D_d = \emptyset$, $\boldsymbol{w^*} = (5, 3, 5, 3, 6, 5, 3, 3, 3, 5, 6, 3, 3, 5, 3, 4, 3)$ and $\lambda^* = 2$ (see Fig. 5a).

(6) If $K = 29$. $\lambda_1 = 1.2 < \frac{\delta}{2} = 2$, $\lambda_2 = 1.2 < \frac{\delta}{2} = 2$. Then $\max\{\lambda_1, \lambda_2\} = 1.2 < 2$, $U^s = 10 < \boldsymbol{w}(T^0) - K$ and $D^s = 2 \cdot |EU^s| = 12 = \boldsymbol{w}(T^0) - K$. Run line 27 of Algorithm 1, obtain $\boldsymbol{w^*} = (3, 2, 3, 3, 6, 5, 3, 3, 3, 5, 6, 3, 3, 5, 2, 4, 3)$ and $\lambda^* = 2$ (see Fig. 5b).

(7) If $K = 30$. $\lambda_1 = 1.1 < \frac{\delta}{2} = 2$, $\lambda_2 = 1.3 < \frac{\delta}{2} = 2$. Then $\max\{\lambda_1, \lambda_2\} = 1.3 < 2$, $U^s = 10 < \boldsymbol{w}(T^0) - K$ and $D^s = 2 \cdot |EU^s| = 12 > \boldsymbol{w}(T^0) - K$. Run line 31-39 of Algorithm 1, obtain $Rest = 1$, e_2 is **critical edge**, $EE = \{e_8, e_9, e_{12}, e_{13}, e_{15}\}$, $D_\delta = \emptyset$, $\boldsymbol{w^*} = (3, 2, 3, 3, 6, 5, 3, 3, 3, 5, 6, 3, 3, 5, 3, 4, 3)$ and $\lambda^* = 2$ (see Fig. 5c).

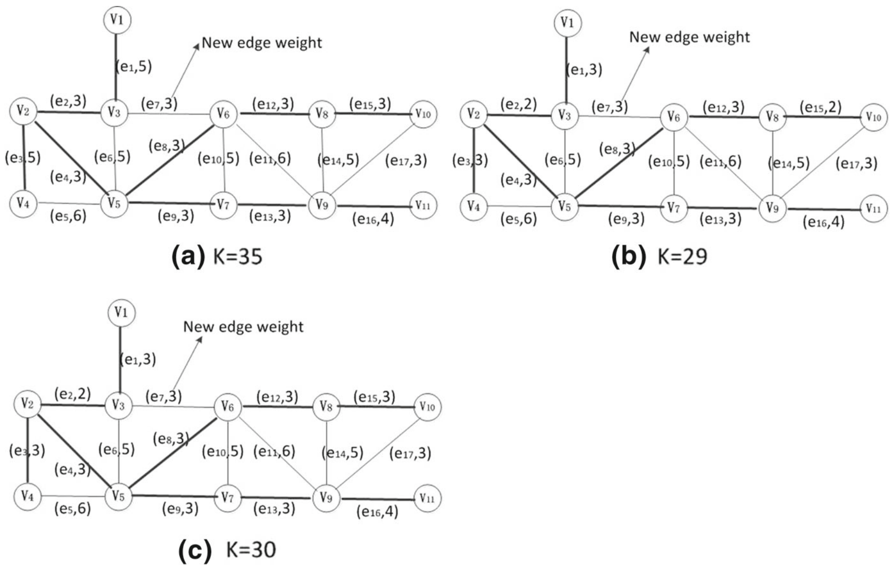

Fig. 5 An optimal solution of Example 1 when $K = 35, 29, 30$

5 Conclusions and further research

In this paper, we consider the inverse optimal value problem on minimum spanning tree under unit l_∞ norm. A sufficient and necessary condition of optimal solutions of the problem is first obtained, and then a strongly polynomial time algorithm with running time $O(mn)$ is proposed.

As a future research topic, we will study the inverse optimal value problem on minimum spanning tree under weighted l_∞ norm, l_1 norm and Hamming distance. It is also meaningful to consider inverse optimal value problems on some other network problems, such as inverse optimal value problems on shortest path, inverse optimal value problems on network flow, inverse optimal value problems on matching, and inverse optimal value problems on center location problems under l_1, l_∞ norm and Hamming distance.

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