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A note on primal-dual stability in infinite linear programming

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Abstract

In this note we analyze the simultaneous preservation of the consistency (and of the inconsistency) of linear programming problems posed in infinite dimensional Banach spaces, and their corresponding dual problems, under sufficiently small perturbations of the data. We consider seven different scenarios associated with the different possibilities of perturbations of the data (the objective functional, the constraint functionals, and the right hand-side function), i.e., which of them are known, and remain fixed, and which ones can be perturbed because of their uncertainty. The obtained results allow us to give sufficient and necessary conditions for the coincidence of the optimal values of both problems and for the stability of the duality gap under the same type of perturbations. There appear substantial differences with the finite dimensional case due to the distinct topological properties of cones in finite and infinite dimensional Banach spaces.

Keywords Linear programming · Infinite dimensions · Primal-dual stability · Consistency · Inconsistency

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1 Introduction

In many practical situations, statisticians, operations researchers or engineers have to minimize a linear functional c, defined on some linear space X (the *decision space*), subject to a given set of linear constraints $\langle a_t, x\rangle \geq b_t$, where a_t is a linear functional on X and $b_t \in \mathbb{R}$ for all $t \in T$. Such a linear problem

$$P: \quad \operatorname*{Inf}_{x\in X} \{\langle c, x\rangle : \langle a_t, x\rangle \geq b_t, \quad \forall t \in T\}, \tag{1}$$

called *primal*, is said to be *infinite* when the dimension of X and the cardinality of the index set T are both infinite, *semi-infinite* when $X = \mathbb{R}^n$ and T is infinite, and *ordinary* (or *finite*) when $X = \mathbb{R}^n$ and T is finite. The subdisciplines of optimization dealing with such types of problems are called linear infinite programming (LIP in short), linear semi-infinite programming (LSIP), and (ordinary or finite) linear programming (LP). Classical monographs on the theory, methods and applications of LIP, LSIP and LP are [2,8,13], respectively. The recent literature on duality in LIP has been briefly reviewed in [23, Section 1] and the one of LSIP, in a more detailed way, in [15], which also contains information on LIP.

Let $\mathbb{R}_+^{(T)}$ be the positive cone in the space $\mathbb{R}^{(T)}$ of *generalized finite sequences* formed by all functions $\lambda \in \mathbb{R}^T$ with finite support. Denote by $\langle \lambda, b\rangle$ the duality product in the dual pair $\left(\mathbb{R}^T, \mathbb{R}^{(T)}\right)$, i.e., $\langle \lambda, b\rangle = \sum_{t\in T} \lambda_t b_t$. If $\lambda \in \mathbb{R}_+^{(T)}$ satisfies $\sum_{t\in T} \lambda_t a_t = c$ and $x \in X$ is a feasible solution of P, then $\langle \lambda, b\rangle \leq \langle c, x\rangle$. So, defining the (*Haar*) *dual problem* of P as

$$D: \quad \operatorname*{Sup}_{\lambda\in\mathbb{R}_+^{(T)}} \left\{\langle \lambda, b\rangle : \sum_{t\in T} \lambda_t a_t = c\right\}, \tag{2}$$

the *duality gap*, that is, the difference between the optimal values of P and D is either a non-negative real number when P and D are both consistent, or $+\infty$ else (once adopted the standard conventions for the optimal value of inconsistent mathematical programming problems). When P and D are simultaneously consistent, the duality gap is necessarily equal to 0 in LP while it may be positive in LIP and in LSIP (see [2, Sections 3.4.1 and 3.5.1] for the famous counterexamples of Gale, in LIP, and of Ben-Israel, Charnes and Kortanek, in LSIP).

The authors of works on optimization whose titles include the term "primal-dual" (e.g., [1,3,19], etc.) try to emphasize that they are analyzing and/or solving, simultaneously, some given minimization problem called primal and its dual. When the data defining these two problems are uncertain due to perturbations, that can be caused by their intrinsic randomness, or by measurements and/or rounding errors, the researchers may use the expression "primal-dual stability" with different meanings: the maintaining of certain desirable characteristics of the dual pair $P - D$ (e.g., the consistency, the boundedness, the solvability or the well-posedness of both problems) under sufficiently small perturbations of the data, some type of continuity of the associated extended real-valued functions (e.g., upper and lower semicontinuity of the primal and the dual optimal value functions, or of the duality gap function) or set-valued mappings (e.g., Berge or Hausdorff, upper or lower, semicontinuity of the primal and dual feasible set mappings or the optimal set mappings), etc. For instance, regarding (ordinary) nonlinear programming, the monograph [4] provides results guaranteeing the continuous dependence of the primal and dual optimal sets with respect to perturbations of the data under the condition that the primal optimal set is nonempty and there exists a Slater point. Regarding particular classes of nonlinear optimization problems, [11] has analyzed the simultaneous preservation, under small perturbations of the data, of the primal and dual consistency in conic linear programming as well as the maintaining of the zero duality gap in semidefinite programming, while [22] is focused on the continuity of the set of saddle points of the Lagrangian function (whose infimum for the decision variable is the objective function of the dual problem) for convex separable optimization problems. Concerning LSIP problems, some papers deal with the simultaneous maintaining of desirable properties of primal and dual problems, as consistency, boundedness, etc. (see [14,15,21], and references therein) while the preservation of the zero duality gap under perturbations (i.e., the so-called 0-stability) has been analyzed in [17]. In [20] some tools from the generalized differentiation theory for set-valued mappings are applied to study the quantitative stability of the feasible sets of the dual pair associated with a LIP problem with infinitely many linear constraints and an additional conic constraint.

Some works consider the primal stability in LIP (see, e.g., those marked with an asterisk in Table 5.1 of [14]) but, to the best of our knowledge, the unique published work involving primal-dual stability in this setting is [23], which is devoted to the analysis of the continuity properties of the duality gap function and whose main limitation is that only perturbations of b and c are allowed.

Up to our knowledge, this is the first paper devoted to analyze the simultaneous preservation of the primal-dual consistency (inconsistency) under small perturbations of the data in LIP, so that we only provide characterizations for certain types of perturbations in the main results of the paper, Theorems 12 and 13. From these results we obtain sufficient as well as necessary conditions for 0-stability, i.e., the equality of the optimal values of the primal problem and its dual under small perturbations. So, this paper can be seen as the infinite dimensional counterpart of [17].

We assume that the problem P in (1) is posed on a Banach space X and all functionals a_t are continuous on X, that is, we deal with the LIP problems usually encountered in the literature, where the feasible set is frequently written as $\{x \in X : Ax \geq b\}$, with A being the linear operator $A : X \to \mathbb{R}^T$, defined by

$$Ax = (\langle a_t, x\rangle)_{t\in T}\ , \forall x \in X. \tag{3}$$

When $\{a_t\}_{t\in T}$ is bounded and $b \in \ell_\infty(T)$ (the linear space of real bounded functions on T), the linear operator $A : X \to \ell_\infty(T)$, $Ax := \{\langle a_t, x\rangle\}_{t\in T}$ is bounded, and so A is continuous, and we may apply to the following problem with equality constrains

$$\operatorname*{Inf}_{x\in X}\ \{\langle c, x\rangle\ :\ Ax\ = b\}$$

the results about strong duality gap in [2, Section 3.6] and about 0-stability in [23] in scenarios where b can be perturbed.

Our approach to primal-dual stability in LIP is inspired in [17,21], even though the finite dimension of the decision space in these two papers is essential for some results in LSIP which cannot be extended to LIP.

The paper is organized as follows: in Sect. 2 we present the necessary notation, definitions and already known properties. Section 3 shows the failure, in LIP, of a useful tool for the study of the primal-dual stability in LSIP. Sections 4 and 5 are devoted to the primal stability and to the dual stability, respectively. Section 6 provides conditions characterizing the primal-dual stability in some scenarios in LIP. It also includes some necessary, or sufficient, conditions in the other scenarios. Finally, Sect. 7 discusses the stability of the duality gap function in different scenarios.

2 Preliminaries

We begin this section by introducing some necessary notation. Given a Banach space $(X, \|\cdot\|)$ and $Y \subset X$, $\operatorname{int} Y$ and $\operatorname{cl} Y$ denote the interior and the closure of Y, respectively, for the norm topology. Moreover, $\operatorname{conv} Y$ stands for the convex hull of Y, whereas $\operatorname{cone} Y := \mathbb{R}_+ \operatorname{conv} Y$ represents the convex conical hull of $Y \cup \{\mathbf{0}\}$, where $\mathbf{0}$ denotes the null vector of X. Moreover, $\dim X$ indicates the dimension of X.

Recall that for a given pair $x, y \in X$ with $x \neq y$ the *line segment* joining x and y is the set $[x, y] := \{\alpha x + (1-\alpha) y : 0 \leq \alpha \leq 1\}$. Given $Y \subset X$, we say that y is in the *core* of Y, denoted by $\operatorname{core} Y$, if for each $x \in X$ there exists $\varepsilon > 0$ for which $[y - \varepsilon x, y + \varepsilon x] \subset Y$. Clearly, $\operatorname{int} Y \subset \operatorname{core} Y$. Moreover, the identity

$$\operatorname{core} Y = \operatorname{int} Y \tag{4}$$

holds true whenever Y is a convex set such that $\operatorname{int} Y \neq \emptyset$ (see, e.g., [18, II, §11]).

We use w^* to indicate the weak* topology on the topological dual, X^*, of X, and we denote by $\operatorname{cl}^* \Phi$ the closure in the weak* topology of a given subset $\Phi \subset X^*$ (or $\Phi \subset X^* \times \mathbb{R}$, endowed with the product topology). The null vector in X^* and the dual norm are also denoted by $\mathbf{0}$ and $\|\cdot\|$, respectively. So, *the dual norm* of $c \in X^*$ is $\|c\| = \sup \{|\langle c, x\rangle| : x \in X, \|x\| \leq 1\}$.

We also use the letter w to indicate the weak topology, $\sigma\left(\mathbb{R}^T, \mathbb{R}^{(T)}\right)$, associated to the dual pair$\left(\mathbb{R}^T, \mathbb{R}^{(T)}\right)$ with duality product $\langle \lambda, b\rangle = \sum_{t \in T} \lambda_t b_t$. We denote by $\operatorname{cl}^w V$ the closure in this weak topology of any $V \subset \mathbb{R}^T$ (or $V \subset \mathbb{R}^T \times \mathbb{R}$, with the product topology).

In this paper we consider the problems P and D defined in (1) and (2), with X being an infinite dimensional Banach space, $a_t \in X^*$ for all $t \in T$, so that $a := (a_t)_{t \in T} \in (X^*)^T$, and $b := (b_t)_{t \in T} \in \mathbb{R}^T$. It is clear that the consistency of P and D depends on the data $(a, b, c) \in (X^*)^T \times \mathbb{R}^T \times X^*$ describing them. We are interested in the maintenance of the consistency (or inconsistency) under small perturbations of all, or some, of the data. To this aim, we identify the nominal problem P with the given triplet (a, b, c), and embed these data into a suitable topological space of admissible perturbed triplets, the so-called *space of parameters* Θ. Following [17], we consider different scenarios determined by the elements of the triplet (a, b, c) which

Table 1 Scenarios considered in the paper

Scenario	Perturbable data	Parameter space Θ
1	(a, b, c)	$(X^*)^T \times \mathbb{R}^T \times X^*$
2	(a, b)	$(X^*)^T \times \mathbb{R}^T$
3	(a, c)	$(X^*)^T \times X^*$
4	(b, c)	$\mathbb{R}^T \times X^*$
5	a	$(X^*)^T$
6	b	$\mathbb{R}^T$
7	c	X^*

Table 2 The basic primal-dual partition of Θ

	Δ_c	Δ_i
Π_c	Θ_{cc}	Θ_{ci}
Π_i	Θ_{ic}	Θ_{ii}

are allowed to present changes. We will consider the seven cases appearing in Table 1 above, using the same notation Θ for any of these parameter spaces.

Notice that the seven parameter spaces are real linear spaces and we endow them with the uniform convergence topology given by the box *Chebyshev (pseudo) norm*, $\|\cdot\|_\infty$. For $\theta = (a, b, c) \in (X^*)^T \times \mathbb{R}^T \times X^*$ it is defined as

$$\|(a, b, c)\|_\infty := \max\left\{\|c\|,\ \sup_{t\in T}\|a_t\|,\ \sup_{t\in T}\|b_t\|\right\}. \tag{5}$$

All the different spaces of parameters in Table 1 are to be identified with subspaces of $(X^*)^T \times \mathbb{R}^T \times X^*$ equipped with the corresponding restriction of $\|\cdot\|_\infty$.

As in [17], we consider the basic primal-dual partition of Θ described in Table 2 above. Here, the basic primal (dual, respectively) partition is formed by the set Π_c (Δ_c) of parameters providing a consistent primal (dual) problem and the set Π_i (Δ_i) of parameters providing an inconsistent primal (dual) problem. Observe that the sets Π_c, Δ_c, Π_i, and Δ_i are cones. We denote $\Theta_{\alpha\beta} := \Pi_\alpha \cap \Delta_\beta$, for $\alpha, \beta \in \{c, i\}$. The cells in Table 2, which are the intersections of the corresponding entries, indicate all the different states in the basic primal-dual partition.

A given parameter $\theta \in \Theta$ is said to be *primal* (*dual*, respectively) *stable with respect to consistency* whenever $\theta \in \operatorname{int}\Pi_c$ ($\theta \in \operatorname{int}\Delta_c$), i.e., the consistency of the primal (dual) problem is preserved by sufficiently small perturbations of the data. Analogously, $\theta \in \Theta$ is said to be *primal* (*dual*, respectively) *stable with respect to inconsistency* whenever $\theta \in \operatorname{int}\Pi_i$ ($\theta \in \operatorname{int}\Delta_i$). Finally, $\theta \in \Theta$ is said to be *primal-dual stable with respect to consistency* (*inconsistency*, respectively) when $\theta \in \operatorname{int}\Theta_{cc}$ ($\theta \in \operatorname{int}\Theta_{ii}$).

Hence, in order to analyze the primal-dual stability of a particular parameter, we need to provide characterizations of $\operatorname{int}\Theta_{cc}$ and $\operatorname{int}\Theta_{ii}$. In doing this, the following sets associated with each triplet $(a, b, c) \in \Theta$ turn out to be very useful:

$$\begin{aligned} \operatorname{conv}(a) &:= \operatorname{conv}\{a_t, t \in T\} \subset X^*, \\ \operatorname{conv}(a, b) &:= \operatorname{conv}\{(a_t, b_t), t \in T\} \subset X^* \times \mathbb{R}, \\ \operatorname{cone}(a) &:= \operatorname{cone}\{a_t, t \in T\} \subset X^*, \end{aligned}$$

and

$$\operatorname{cone}(a, b) := \operatorname{cone}\{(a_t, b_t), t \in T\} \subset X^* \times \mathbb{R}.$$

The characterization of Π_c and Δ_c can be written with the aid of these cones. Indeed, from [12, Theorem 1] (whose simpler LSIP version can be found in [9, Lemma 2.4]), we have that

$$(a, b, c) \in \Pi_c \Leftrightarrow (\mathbf{0}, 1) \notin \operatorname{cl}^* \operatorname{cone}(a, b), \tag{6}$$

while it is clear that

$$(a, b, c) \in \Delta_c \Leftrightarrow c \in \operatorname{cone}(a). \tag{7}$$

Therefore,

$$(a, b, c) \in \Theta_{cc} \Leftrightarrow (\mathbf{0}, 1) \notin \operatorname{cl}^* \operatorname{cone}(a, b) \text{ and } c \in \operatorname{cone}(a).$$

We say that the *strong Slater condition holds at* $\theta = (a, b, c) \in \Theta$ whenever there exists some $\widehat{x} \in X$ with

$$\inf_{t \in T} (\langle a_t, \widehat{x}\rangle - b_t) > 0.$$

Such a point $\widehat{x}$ is called a *strong Slater point* for θ [actually, for the constraint system of problem P in (1)] in the LSIP literature, and a *Slater point* for θ in the LP literature.

We denote by $\nu^P(\theta)$ and $\nu^D(\theta)$ the optimal values of the primal and dual problems associated to $\theta \in \Theta$, respectively, adopting the standard conventions for the optimal value of inconsistent mathematical programming problems.

The *duality gap* of a parameter $\theta \in \Theta$ is the extended real number given by

$$g(\theta) := \begin{cases} \nu^P(\theta) - \nu^D(\theta), & \text{for } \theta \in \Theta_{cc}, \\ +\infty, & \text{otherwise,} \end{cases}$$

where we adopt the convention that $(\pm\infty) - (\pm\infty) = +\infty$, so that $g(\theta) \in \mathbb{R}_+ \cup \{+\infty\}$.

We conclude this section recalling the weakest known conditions guaranteeing a zero duality gap (i.e., $g(\theta) = 0$) for $\theta = (a, b, c) \in \Theta$. These conditions involve the cones

$$K(\theta) := \operatorname{cone}(a, b) + \{\mathbf{0}\} \times \mathbb{R}_- \subset X^* \times \mathbb{R},$$

and

$$H(\theta) := \{(Ax, -\langle c, x\rangle) : x \in X\} - \mathbb{R}^T_+ \times \mathbb{R}_+ \subset \mathbb{R}^T \times \mathbb{R},$$

where A is the linear operator defined in (3), i.e., $A = (\langle a_t, \cdot\rangle)_{t\in T}$. The cone $H(\theta)$ was already used in [2] to study the duality gap in LIP.

We say that *condition* (KC) holds at $\theta = (a, b, c) \in \Theta$ whenever $K(\theta)$ is w^* -closed regarding to the set $\{c\} \times \mathbb{R}$, that is,

$$(KC) \quad K(\theta) \cap (\{c\} \times \mathbb{R}) = (\text{cl}^* K(\theta)) \cap (\{c\} \times \mathbb{R}).$$

Analogously, we say that *condition* (HC) holds at $\theta = (a, b, c) \in \Theta$ whenever $H(\theta)$ is w-closed regarding to the set $\left\{(b_t)_{t\in T}\right\} \times \mathbb{R}$, i.e.,

$$(HC) \quad H(\theta) \cap \left(\left\{(b_t)_{t\in T}\right\} \times \mathbb{R}\right) = (\text{cl}^w H(\theta)) \cap \left(\left\{(b_t)_{t\in T}\right\} \times \mathbb{R}\right).$$

Observe that $K(\theta)$ is independent of c and $H(\theta)$ is independent of b, while conditions (KC) and (HC) involve all the data, a, b, and c.

The zero duality gap with solvability (i.e., existence of optimal solutions) of one of the two problems of the dual pair has been characterized in [16, Corollaries 7 and 8] in terms of (KC) and (HC) as follows: if $\theta = (a, b, c) \in \Theta_{cc}$, then

$$[g(\theta) = 0 \text{ with } D \text{ solvable}] \Leftrightarrow (KC) \text{ holds at } \theta \tag{8}$$

and

$$[g(\theta) = 0 \text{ with } P \text{ solvable}] \Leftrightarrow (HC) \text{ holds at } \theta. \tag{9}$$

Of course, from (8) and (9), the closedness of at least one of the two cones is a sufficient condition for $g(\theta) = 0$ when $\theta = (a, b, c) \in \Theta_{cc}$. In the case of $H(\theta)$, this statement is [2, Theorem 3.9]. Regarding the other cone, if one assumes the w^*-closedness of $K(\theta)$, by the Farkas Lemma in LIP (see [9, Corollary 7]), every continuous linear consequence, say $\langle u, x\rangle \geq \alpha$, of the constraint system $\{\langle a_t, x\rangle \geq b_t : t \in T\}$ is a consequence of some finite subsystem, i.e., $(u, \alpha) \in K(\theta)$. Since $\theta \in \Theta_{cc}$ entails that $\nu^P(\theta) \in \mathbb{R}$, $\langle c, x\rangle \geq \nu^P(\theta)$ is a linear consequence of $\{\langle a_t, x\rangle \geq b_t : t \in T\}$ and, so, $\left(c, \nu^P(\theta)\right) \in K(\theta)$, which implies $\nu^P(\theta) = \nu^D(\theta)$, i.e., $g(\theta) = 0$.

3 The dimension of the decision space matters

A key property, in [17,21], that allows to obtain several crucial results about dual stability w.r.t. consistency and primal stability w.r.t. inconsistency in LP and LSIP, where the dimension of the decision space X is finite, establishes that the set of parameters $a \in (\mathbb{R}^n)^T$ such that $\text{int cone}(a) \neq \emptyset$ is an open set in $(\mathbb{R}^n)^T$, this space being equipped with the Chebyshev (pseudo) norm. Unfortunately, as shown next, this property does not hold true in the infinite dimensional setting.

Lemma 1 *Let* $(Y, \|\cdot\|)$ *be an infinite dimensional normed space. Then, there exist an infinite set* T *and a subset* $\{a_t, t \in T\} \subset Y$ *such that* $\text{int cone}(a) \neq \emptyset$ *and, for any* $\varepsilon > 0$, *there is a subset* $\left\{a_t^\varepsilon, t \in T\right\} \subset Y$ *such that* $\sup_{t\in T} \left\|a_t^\varepsilon - a_t\right\| < \varepsilon$ *and* $\text{int cone}(a^\varepsilon) = \emptyset$.

Proof Let $B = \{x_i, i \in I\}$ be a Hammel basis of Y with $\|x_i\| = 1$ for all $i \in I$. We associate with B the Yudin cone $K := \text{cone}\, B$; obviously $K - K = Y$.

Set $T := B \cup (-B)$ and select some infinite countable subset $M = \left\{x_{i_k},\ k \in \mathbb{N}\right\} \subset B$. Let $a \in Y^T$ be defined by

$$a_x := \begin{cases} \frac{1}{k}x, \text{ if } & x = \pm x_{i_k}, \quad k \in \mathbb{N}, \\ x, \quad \text{if } & x \in T \backslash (M \cup (-M)). \end{cases}$$

Then $\operatorname{cone}(a) \equiv \operatorname{cone}\{a_x, x \in T\} = \operatorname{cone}(B \cup (-B)) = Y$, so it has nonempty interior.

Given $\varepsilon > 0$, take a positive integer k_0 such that $\frac{1}{k_0} < \varepsilon$, and consider the following perturbation $a^\varepsilon \in Y^T$ of a :

$$a_x^\varepsilon := \begin{cases} \mathbf{0}, \quad \text{if } & x = \pm x_{i_k}, \quad k \geq k_0 + 1, \\ a_x, \text{ otherwise.} \end{cases}$$

Then one has

$$\left\|a^\varepsilon - a\right\|_\infty := \sup\{\left\|a_x - a_x^\varepsilon\right\| : x \in T\} \leq \frac{1}{k_0} < \varepsilon.$$

Observe that $\operatorname{cone}(a)$ is generated by the whole set T while $\operatorname{cone}(a^\varepsilon)$ is generated by $T \backslash \left\{\pm x_{i_1}, \ldots, \pm x_{i_{k_0}}\right\}$. Since both generator sets are symmetric, $\operatorname{cone}(-a) = \operatorname{cone}(a) = Y$ and $\operatorname{cone}(-a^\varepsilon) = \operatorname{cone}(a^\varepsilon)$.

We now prove that $\operatorname{int}\operatorname{cone}(a^\varepsilon) = \emptyset$ by contradiction. If $x \in \operatorname{int}\operatorname{cone}(a^\varepsilon)$, then there exists $\delta > 0$ such that $v := x - \delta x_{i_{k_0}} \in \operatorname{cone}(a^\varepsilon)$, which yields

$$x_{i_{k_0}} = -\frac{v}{\delta} + \frac{x}{\delta} \in \operatorname{cone}\left(-a^\varepsilon\right) + \operatorname{cone}\left(a^\varepsilon\right) = \operatorname{cone}\left(a^\varepsilon\right).$$

Thus $x_{i_{k_0}}$ is a linear combination of elements in $B \backslash \{x_{i_{k_0}}\}$, which is not possible considering that B is a linearly independent set. Hence $\operatorname{int}\operatorname{cone}(a^\varepsilon) = \emptyset$. □

Combining the previous lemma, applied to the Banach space $(X^*, \|\cdot\|)$, and [21, Lemma 3.4] one gets:

Proposition 2 (A key property in LSIP and LP which fails in LIP) *The set*

$$\left\{a \in \left(X^*\right)^T : \operatorname{int}\operatorname{cone}(a) \neq \emptyset\right\}$$

is open in $\left((X^*)^T, \|\cdot\|_\infty\right)$ *if and only if* $\dim X < \infty$.

4 Primal stability

From now on we will consider a fixed triple $(a, b, c) \in \Theta$, and will use the same notation θ for the corresponding parameter in the different scenarios appearing in Table 1.

Recall that we consider a parameter θ as being primal stable with respect to consistency when $\theta \in \operatorname{int} \Pi_c$. Similarly, θ is primal stable with respect to inconsistency when $\theta \in \operatorname{int} \Pi_i$. We will show that Proposition 2 affects the characterization of the primal stability w.r.t. inconsistency in LIP. Nonetheless, with respect to the characterization of the primal stability w.r.t. consistency, we find no differences between the semi-infinite case and the infinite one by virtue of the following known property, which provides an equivalent description of the strong Slater condition (see e.g. [6, Lemma 2.3, (i) and (ii)]).

Lemma 3 *Let $\theta = (a, b, c) \in \Pi_c$. Then the following statements are equivalent:*

(i) *The strong Slater condition holds at θ.*
(ii) $(\mathbf{0}, 0) \notin \mathrm{cl}^* \operatorname{conv}(a, b)$.

The next proposition is the infinite dimensional counterpart of Lemma 2 in [17], which has been established in a finite dimensional setting.

Proposition 4 (The interior of Π_c) *Let $\theta \in \Pi_c$. Then the following statements hold:*

(i)
$$\theta \in \operatorname{int} \Pi_c \Leftrightarrow (\mathbf{0}, 0) \notin \mathrm{cl}^* \operatorname{conv}(a, b), \tag{10}$$
in Scenarios 1, 2, 4, and 6.

(ii) *In Scenarios 3 and 5, the equivalence (10) holds true if* $\sup_{t\in T} b_t > 0$; *otherwise,* $\operatorname{int} \Pi_c = \Pi_c = \Theta$.

(iii) $\operatorname{int} \Pi_c = \Pi_c$ *in Scenario 7, so* $\theta \in \operatorname{int} \Pi_c$ *always.*

Proof (i) Observe that if $\widehat{x} \in X$ is a strong Slater point for $\theta = (a, b, c)$, then $\widehat{x} \in X$ is also a strong Slater point for $\theta^1 = (a^1, b^1, c^1)$ for all θ^1 close enough to θ. In fact, if $\inf_{t\in T} (\langle a_t, \widehat{x}\rangle - b_t) = \rho > 0$, it is easily proved that $\widehat{x}$ is still a strong Slater point for θ^1 if, for instance,

$$\left\| (a^1, b^1, c^1) - (a, b, c) \right\|_\infty \leq \frac{\rho}{2}(1 + \|\widehat{x}\|)^{-1}.$$

In this way we have proved that θ is an interior point of Π_c. Conversely, notice that whenever $\theta \in \operatorname{int} \Pi_c$ in any scenario where b is a perturbable data, then the strong Slater condition holds at θ because of the existence of some $\varepsilon > 0$ such that there is a solution to $\langle a_t, x\rangle \geq b_t + \varepsilon$, $t \in T$.

So, the equivalence (10) follows from Lemma 3 by taking into account that the perturbations of b are only allowed in Scenarios 1, 2, 4, and 6.

(ii) It is the same proof as in [17, Lemma 2 (ii)] by extending the result of [7, Theorem 4.2 $(iii) \Leftrightarrow (iv)$] to the infinite dimensional context, which follows in a straightforward manner, only taking care of considering w^*−convergence.

(iii) In Scenario 7, where (a, b) remains fixed, we always have $\Pi_c = \emptyset$ or $\Pi_c = \Theta$, so Π_c is open and then $\operatorname{int} \Pi_c = \Pi_c$ in any case. □

In the same vein, taking into account (6), one may ask whether or not $\theta \in \operatorname{int} \Pi_i$ is equivalent to $(\mathbf{0}, 1)$ being an interior point of $\mathrm{cl}^* \operatorname{cone}(a, b)$. Unfortunately, it is clear, from Proposition 2, that this relationship between the primal stability w.r.t.

inconsistency of $\theta = (a, b, c)$ and the property $(\mathbf{0}, 1) \in \operatorname{int}\operatorname{cone}(a, b)$ does not need to hold true. The next example shows that it is not necessary that $(\mathbf{0}, 1) \in \operatorname{int}\operatorname{cone}(a, b)$ in case of $\theta \in \operatorname{int}\Pi_i$.

Example 5 Consider $X = \ell_1$, $X^* = \ell_\infty$, $T = \mathbb{N}$, $\theta = (a, b, c) \in \Theta$ with c in X^*, $b_k = k$ and $a_k = e^k = (0, \ldots, 0, 1, 0, \ldots)$ being a canonical vector, i.e. $e^k_k = 1$ and all other $e^k_j = 0$, for $k \neq j$, k and j positive integers. So $\theta \in \operatorname{int}\Pi_i$ in any scenario. Nonetheless, $(\mathbf{0}, 1) \notin \operatorname{int}\operatorname{cone}(a, b)$ because $\operatorname{int}\operatorname{cone}(a, b) = \emptyset$ since any point (z, r) in $\operatorname{cone}(a, b)$ has only finitely many non zero components, while any open neighborhood of (z, r) contains points with infinitely many non zero components, e.g. $(z, r) + \varepsilon\mathbf{1}$, with $\mathbf{1} = (1, 1, 1, \ldots)$ for $\varepsilon > 0$ small enough. Recall that we are considering Π_i endowed with the norm topology obtained from the restriction of the box Chebyshev (pseudo) norm defined in (5).

The next proposition provides sufficient conditions for θ to be an interior point of Π_i, in different scenarios.

Proposition 6 (The interior of Π_i) *Let $\theta \in \Pi_i$. Then the following statements hold:*

(i) *If* $\sup_{t\in T} b_t = +\infty$ *and* $\sup_{t\in T} \|a_t\| < +\infty$, *then* $\theta \in \operatorname{int}\Pi_i$ *in all the Scenarios from 1 to 7.*
(ii) *If* $\inf_{t\in T} b_t > 0$ *then* $\theta \in \operatorname{int}\Pi_i$ *in Scenarios 4 and 6.*
(iii) $\operatorname{int}\Pi_i = \Pi_i$ *in Scenario 7, so* $\theta \in \operatorname{int}\Pi_i$ *always.*

Proof (i) There is no $x \in X$ such that $\langle a_t, x\rangle \geq b_t$, for all $t \in T$ because $\sup_{t\in T}\langle a_t, x\rangle \leq \sup_{t\in T}\|a_t\|\,\|x\| < +\infty$, while $\sup_{t\in T} b_t = +\infty$. The same reasoning applies for any $\left(a^1, b^1\right)$ at a finite distance to (a, b) in Θ. Thus, $\theta \in \operatorname{int}\Pi_i$ in any scenario.

(ii) and (iii) It is the same proof as in [17, Lemma 4 (iii) and (iv)] because a is fixed in Scenarios 4, 6 and 7. □

5 Dual stability

Recall that the dual stability of $\theta \in \Theta$ w.r.t. consistency (respectively, inconsistency) has been defined as the property of $\theta \in \operatorname{int}\Delta_c$ (respectively, $\theta \in \operatorname{int}\Delta_i$). In the following discussion related to the interior of Δ_c we will use the concept of core.

The following properties give, for each of the different scenarios, necessary conditions for dual stability w.r.t. consistency. Moreover, in the cases where the data a remains fixed, it is provided a characterization of $\operatorname{int}\Delta_c$, which coincides with the one in the finite dimensional setting. In Scenarios 1, 2, 3, and 5 there appear some differences with the finite dimensional case (see Examples 8 and 9 below).

Proposition 7 (The interior of Δ_c) *If $\theta \in \Delta_c$, then:*

(i) *In Scenarios 1 and 3, it is valid that*

$$\theta \in \operatorname{int}\Delta_c \Rightarrow c \in \operatorname{int}\operatorname{cone}(a).$$

(ii) *The following characterization holds in Scenarios 4 and 7:*

$$\theta \in \operatorname{int} \Delta_c \Leftrightarrow c \in \operatorname{int} \operatorname{cone}(a).$$

(iii) *In Scenarios 2 and 5, if* $c = \mathbf{0}$, *then* $\theta \in \operatorname{int} \Delta_c = \Delta_c = \Theta$ *while, for* $c \neq \mathbf{0}$,

$$[\theta \in \operatorname{int} \Delta_c \text{ and } \operatorname{int} \operatorname{cone}(a) \neq \emptyset] \Rightarrow c \in \operatorname{int} \operatorname{cone}(a).$$

(iv) $\operatorname{int} \Delta_c = \Delta_c$ *in Scenario 6, so* $\theta \in \operatorname{int} \Delta_c$ *always.*

Proof First observe that $\theta \in \operatorname{int} \Delta_c$ is a sufficient condition for $c \in \operatorname{int} \operatorname{cone}(a)$ in Scenarios 1, 3, 4 and 7 by using only perturbations of c . So (i) is true and also the implication ($\Rightarrow$) in (ii).

(ii) ($\Leftarrow$) If $c \in \operatorname{int} \operatorname{cone}(a)$, then $c + c' \in \operatorname{cone}(a)$ for all c' such that $\|c'\| \leq \varepsilon$, for some $\varepsilon > 0$. From (7) we have that $(a, b + b', c + c') \in \Delta_c$ for all $\|c'\| \leq \varepsilon$ and any b' in $\mathbb{R}^T$, so $\theta \in \operatorname{int} \Delta_c$ in Scenarios 4 and 7, where a remains fixed.

(iii) If $c = \mathbf{0}$, then $\Delta_c = \Theta$ in Scenarios 2 and 5. Let $c \neq \mathbf{0}$ and consider Scenario 5 (the case of Scenario 2 follows in a similar way). Assume that $\theta = a \in \operatorname{int} \Delta_c$ and $\operatorname{int} \operatorname{cone}(a) \neq \emptyset$. Since $\theta \in \Delta_c$ there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that $c = \sum_{t \in T} \lambda_t a_t$. Obviously, $\alpha := \sum_{t \in T} \lambda_t > 0$ (because $c \neq \mathbf{0}$). Let $\varepsilon > 0$ be such that $c \in \operatorname{cone}(a + \tilde{a})$ for all $\tilde{a} \in (X^*)^T$ with $\|\tilde{a}\|_\infty \leq \varepsilon$. Let $p \in X^*$ be such that $\|p\| = 1$ and set

$$a'_t := \varepsilon p, \forall t \in T.$$

Since $\|a'_t\| \leq \varepsilon$, $c \in \operatorname{cone}\left(a + a'\right)$. Therefore there exists $\beta \in \mathbb{R}_+^{(T)}$ such that $c = \sum_{t \in T} \beta_t \left(a_t + a'_t\right) = \sum_{t \in T} \beta_t a_t + \varepsilon (\sum_{t \in T} \beta_t) p$. Define $\mu := \sum_{t \in T} \beta_t > 0$. Then

$$c - \varepsilon \mu p \in \operatorname{cone}(a).$$

In the same way, for

$$a''_t := -\varepsilon p, \forall t \in T,$$

there exists $\gamma \in \mathbb{R}_+^{(T)}$ such that $c = \sum_{t \in T} \gamma_t \left(a_t + a''_t\right)$, and then

$$c + \varepsilon \nu p \in \operatorname{cone}(a),$$

where $\nu := \sum_{t \in T} \gamma_t > 0$. Let $\delta := \frac{1}{2} \min\{\varepsilon\mu, \varepsilon\nu\} > 0$. Then $\tau_1 := 1 - \frac{\delta}{\varepsilon\mu} \in (0, 1)$ and

$$c + \delta p = \tau_1 c + (1 - \tau_1)(c + \varepsilon \mu p) \in \operatorname{cone}(a).$$

In the same way, $\tau_2 := 1 - \frac{\delta}{\varepsilon\nu} \in (0, 1)$ and

$$c - \delta p = \tau_2 c + (1 - \tau_2)(c - \varepsilon \nu p) \in \operatorname{cone}(a).$$

Then $[c - \delta p, c + \delta p] \subset \operatorname{cone}(a)$. Observe that it also holds for any $p \neq \mathbf{0}$, by considering $\frac{p}{\|p\|}$. So $c \in \operatorname{core} \operatorname{cone}(a)$. Therefore, from (4), $c \in \operatorname{int} \operatorname{cone}(a)$.

(iv) In Scenario 6, since (a, c) remains fixed, $\Delta_c \neq \emptyset$ implies that $\Delta_c = \Theta$. $\square$

The next examples show that $c \in \operatorname{int} \operatorname{cone}(a)$ is not a sufficient condition for θ to be an interior point of Δ_c in Scenarios 1 and 3, and also in Scenarios 2 and 5 for $c \neq \mathbf{0}$. This is an important difference with the finite dimensional setting.

Example 8 Suppose that we are in Scenario 3. Consider the construction from Lemma 1, with $Y = X^*$ and let $c := \mathbf{0} \in \operatorname{int} \operatorname{cone}(a) = X^*$. Given $\varepsilon > 0$, there exists k_0 such that $\frac{1}{k_0} < \varepsilon$. Take $a^\varepsilon \in (X^*)^T$ defined as in the proof of Lemma 1, and put $c^\varepsilon = \frac{1}{k_0} x_{k_0} \neq \mathbf{0}$. Then $c^\varepsilon \notin \operatorname{cone}(a^\varepsilon)$ because it is linearly independent with a_x^ε for all x with $a_x^\varepsilon \neq 0$. Then $\theta^\varepsilon := (a^\varepsilon, c^\varepsilon) \notin \Delta_c$ with $\|\theta^\varepsilon - \theta\|_\infty < \varepsilon$. Therefore $\theta \notin \operatorname{int} \Delta_c$ in Scenario 3 (and also in Scenario 1).

Example 9 Consider $X = \ell_1, X^* = \ell_\infty$. Let $B = \{x_i, i \in I\}$ be a normalized Hammel basis of ℓ_∞ and suppose w.l.o.g. that the canonical vectors $e^k \in B$ for all $k \in \mathbb{N}$, and put $c := \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$. Consider the construction from Lemma 1 with $Y = X^*$, $M := \left\{e^k,\ k \in \mathbb{N}\right\} \subset B$, $T := B \cup (-B)$, and

$$a_x := \begin{cases} \frac{1}{k}x, & \text{if} \quad x = \pm e^k,\ k \in \mathbb{N}, \\ x, & \text{if} \quad x \in T \backslash (M \cup (-M)). \end{cases}$$

So $c \in \operatorname{int} \operatorname{cone}(a) = \ell_\infty$ but $c \notin \operatorname{cone}(a^\varepsilon)$ for a^ε given as in Lemma 1, by replacing the a_x with $x = \pm e^k$ by $\mathbf{0}$ for k large enough. Therefore $\theta \notin \operatorname{int} \Delta_c$ in Scenarios 2 and 5.

Proposition 10 (The interior of Δ_i) *If $\theta \in \Delta_i$, then the following characterizations hold:*

(i) *In Scenarios 1 and 3,*

$$\theta \in \operatorname{int} \Delta_i \Leftrightarrow \mathbf{0} \notin \operatorname{cl} \operatorname{conv}(a) \text{ and } c \notin \operatorname{cl} \operatorname{cone}(a).$$

Moreover this equivalence is also valid in Scenarios 2 and 5 when the set $\{a_t\}_{t \in T}$ is bounded.

(ii)
$$\theta \in \operatorname{int} \Delta_i \Leftrightarrow c \notin \operatorname{cl} \operatorname{cone}(a),$$

in Scenarios 4 and 7.

(iii) $\operatorname{int} \Delta_i = \Delta_i$ *in Scenario 6, so $\theta \in \operatorname{int} \Delta_i$ always.*

Proof It is almost the same proof that appears in [17, Lemma 7 (i)–(iii)], mostly based in separation arguments. We only need to consider the norm closure. □

Observe that $\mathbf{0} \notin \operatorname{cl}^* \operatorname{conv}(a)$ and $c \notin \operatorname{cl}^* \operatorname{cone}(a)$ implies that $\mathbf{0} \notin \operatorname{cl} \operatorname{conv}(a)$ and $c \notin \operatorname{cl} \operatorname{cone}(a)$. If X is reflexive we have the equality of both the weak* and the norm closure of convex sets. However, it is well known that in the dual of every non-reflexive Banach space there are closed convex sets that are not w^*−closed, even bounded closed convex sets (see, e.g., [5,10]). The next example shows that we can not replace cl by cl* in the equivalences in Proposition 10 when X is not reflexive.

Example 11 Let X be a non-reflexive Banach space, then the canonical injection $J : X \to X^{**}$ is not surjective. Let $\xi \in X^{**}$ with $\xi \notin J(X)$. Then the convex cone $\ker(\xi) = \{z \in X^* : \langle \xi, z\rangle = 0\}$ is closed but it is not w^*-closed, that is to say

$$\ker(\xi) = \operatorname{cl}\ker(\xi) \subsetneq \operatorname{cl}^* \ker(\xi).$$

Take an arbitrary $c \in \operatorname{cl}^* \ker(\xi) \setminus \operatorname{cl}\ker(\xi)$ and define $T := -c + \ker(\xi)$. Consider $\{a_t : t \in T\}$, where each a_t is defined by $a_t = t$. Then

$$\mathbf{0} \notin \operatorname{cl}\operatorname{conv}(a) = -c + \ker(\xi) \text{ and } c \notin \operatorname{cl}\operatorname{cone}(a) = \operatorname{cl}\{-\mathbb{R}_+ c + \ker(\xi)\},$$

but

$$\mathbf{0} \in \operatorname{cl}^* \operatorname{conv}(a) = -c + \operatorname{cl}^* \ker(\xi) \text{ and } c \in \operatorname{cl}^* \operatorname{cone}(a).$$

6 Primal-dual stability

Following very similar reasonings as in LSIP [17], we may obtain the next conditions, some of them being necessary and sufficient for the primal-dual stability w.r.t. consistency, others only necessary or only sufficient. Indeed, we characterize, in the LIP framework, the primal-dual stability w.r.t. consistency in the case of fixed data a and, furthermore, we apply this characterization to analyze the stability of the duality gap functions in scenarios 4, 6 and 7.

Theorem 12 (The interior of Θ_{cc}) *Let $\theta = (a, b, c) \in \Theta_{cc}$. Then the following statements hold:*

(i) *In Scenario 1,*

$$\theta \in \operatorname{int}\Theta_{cc} \Rightarrow (\mathbf{0}, 0) \notin \operatorname{cl}^* \operatorname{conv}(a, b) \text{ and } c \in \operatorname{int}\operatorname{cone}(a) \tag{11}$$

(ii) *The condition (11) is true in Scenario 2 whenever* $\operatorname{int}\operatorname{cone}(a) \neq \emptyset$ *holds and* $c \neq \mathbf{0}$; *in Scenario 3 provided that* $\sup_{t\in T} b_t > 0$; *and in Scenario 5 in case that* $\sup_{t\in T} b_t > 0$ *and* $c \neq \mathbf{0}$.

(iii) *Assume that* $\sup_{t\in T} b_t \leq 0$. *Then*

$$\theta \in \operatorname{int}\Theta_{cc} \Rightarrow c \in \operatorname{int}\operatorname{cone}(a)$$

in Scenario 3 and, for $c \neq \mathbf{0}$, *also in Scenario 5.*

(iv) *Assume that* $c = \mathbf{0}$. *Then, in Scenario 2*

$$\theta \in \operatorname{int}\Theta_{cc} \Leftrightarrow (\mathbf{0}, 0) \notin \operatorname{cl}^* \operatorname{conv}(a, b),$$

and also in Scenario 5 when $\sup_{t\in T} b_t > 0$. *Moreover, in Scenario 5 with* $\sup_{t\in T} b_t \leq 0$, *it is always true that* $\theta \in \operatorname{int}\Theta_{cc}$.

(v) *Table 3 characterizes the membership of* θ *to the interior* $\operatorname{int}\Theta_{cc}$ *in Scenarios 4, 6, and 7.*

Table 3 Characterization of Θ_{cc} in Scenarios 4, 6 and 7

Scenario	Parameter	Characterization of $\theta \in \operatorname{int} \Theta_{cc}$
4	(b, c)	$(\mathbf{0}, 0) \notin \operatorname{cl}^* \operatorname{conv}(a, b)$ and $c \in \operatorname{int} \operatorname{cone}(a)$
6	b	$(\mathbf{0}, 0) \notin \operatorname{cl}^* \operatorname{conv}(a, b)$
7	c	$c \in \operatorname{int} \operatorname{cone}(a)$

Proof It follows immediately by gathering the results in Propositions 6 and 7. □

We finally combine the results in the last two sections to get characterizations of the primal-dual stability w.r.t. inconsistency.

Theorem 13 (The interior of Θ_{ii}) *If* $\theta \in \Theta_{ii}$, *the following statements hold:*

(i) *In Scenarios 1, 2, 3 and 5, if* $\sup_{t\in T} b_t = +\infty$ *and* $\{a_t\}_{t\in T}$ *is bounded, then*

$$\theta \in \operatorname{int} \Theta_{ii} \Leftrightarrow \mathbf{0} \notin \operatorname{cl} \operatorname{conv}(a) \ \textit{and}\ c \notin \operatorname{cl} \operatorname{cone}(a).$$

(ii) *In Scenario 4, if* $\inf_{t\in T} b_t > 0$, *then*

$$\theta \in \operatorname{int} \Theta_{ii} \Leftrightarrow c \notin \operatorname{cl} \operatorname{cone}(a).$$

(iii) *In Scenario 6, if* $\inf_{t\in T} b_t > 0$, *then*

$$\theta \in \operatorname{int} \Theta_{ii} \Leftrightarrow c \notin \operatorname{cone}(a).$$

(iv) *In Scenario 7,*

$$\theta \in \operatorname{int} \Theta_{ii} \Leftrightarrow c \notin \operatorname{cl} \operatorname{cone}(a).$$

Proof (i) follows from Propositions 6(i) and 10(i); (ii) from Propositions 6(ii) and 10 (ii); (iii) from Propositions 6(ii) and 10(iii), together with (7); and, finally, (iv) from Propositions 6(iii) and 10(ii). □

7 Stability of the duality gap

Inspired in [17], we are interested in the preservation of the zero duality gap under perturbations of some parameter θ (i.e., the duality gap g is identically 0 in an open neighborhood of θ), the so-called $0-$*stability* around θ. Observe that $\theta \in \operatorname{int} \Theta_{cc}$ is a necessary condition for the $0-$stability of g at θ. We will also refer to the $\infty-$*stability* when g is identically $+\infty$ in an open neighborhood of θ, which is equivalent to $\theta \in \operatorname{int}(\Theta \backslash \Theta_{cc})$. Finally, notice that g presents a high instability at any $\theta \in \operatorname{bd} \Theta_{cc}$ because g takes both, finite values and the $+\infty$ value, on any neighborhood of θ.

The last two results, which are immediate consequences of Theorem 12, (8) and (9), gather together sufficient conditions and necessary conditions for 0-stability around a given parameter, firstly for those scenarios where a remains fixed and, secondly those scenarios where a can be perturbed.

Proposition 14 (0-stability with deterministic a) *Let* $\bar{\theta} = \left(\bar{a}, \bar{b}, \bar{c}\right) \in \Theta_{cc}$. *Then:*

(i) *In Scenario 7, the following statements are true:*

  (i.1) *If* (KC) *holds at* $\theta = \left(\bar{a}, \bar{b}, c\right)$ *for all* c *in some neighborhood of* $\bar{c}$, *then* g *is* 0-*stable around* $\bar{\theta}$.
  (i.2) *If* g *is* 0-*stable around* $\bar{\theta}$, *then* $\bar{c} \in \operatorname{int}\operatorname{cone}(\bar{a})$.

(ii) *In Scenario 6, the following statements are true:*

  (ii.1) *If* (HC) *holds at* $\theta = (\bar{a}, b, \bar{c})$ *for all* b *in some neighborhood of* $\bar{b}$, *then* g *is* 0-*stable around* $\bar{\theta}$.
  (ii.2) *If* g *is* 0-*stable around* $\bar{\theta}$, *then* $(\mathbf{0}, 0) \notin \mathrm{cl}^* \operatorname{conv}\left(\bar{a}, \bar{b}\right)$.

(iii) *In Scenario 4, the following statements are true:*

  (iii.1) *If* (KC) *or* (HC) *hold at* $\theta = (\bar{a}, b, c)$ *for every* (b, c) *in some neighborhood of* $\left(\bar{b}, \bar{c}\right)$, *then* g *is* 0-*stable around* $\bar{\theta}$.
  (iii.2) *If* g *is* 0-*stable around* $\bar{\theta}$, *then* $\bar{c} \in \operatorname{int}\operatorname{cone}(\bar{a})$ *and* $(\mathbf{0}, 0) \notin \mathrm{cl}^* \operatorname{conv}\left(\bar{a}, \bar{b}\right)$.

Proposition 15 (0-stability with uncertain a) *Let* $\bar{\theta} = \left(\bar{a}, \bar{b}, \bar{c}\right) \in \Theta_{cc}$. *Then, the following statements hold:*

(i) $\bar{c} \in \operatorname{int}\operatorname{cone}(\bar{a})$ *and* $(\mathbf{0}, 0) \notin \mathrm{cl}^* \operatorname{conv}\left(\bar{a}, \bar{b}\right)$ *are necessary conditions for* g *to be* $0-$*stable around* $\bar{\theta}$ *in the following cases:*

  (i.1) *In Scenario 1.*
  (i.2) *In Scenario 2, with* $\operatorname{int}\operatorname{cone}(\bar{a}) \neq \emptyset$ *and* $\bar{c} \neq \mathbf{0}$.
  (i.3) *In Scenario 3, with* $\sup_{t\in T} \bar{b}_t > 0$.
  (i.4) *In Scenario 5, with* $\sup_{t\in T} \bar{b}_t > 0$ *and* $\bar{c} \neq \mathbf{0}$.

(ii) $\bar{c} \in \operatorname{int}\operatorname{cone}(\bar{a})$ *is a necessary condition for* g *to be* $0-$*stable around* $\bar{\theta}$ *in the following cases:*

  (ii.1) *In Scenario 3, with* $\sup_{t\in T} \bar{b}_t \leq 0$.
  (ii.2) *In Scenario 5, with* $\sup_{t\in T} \bar{b}_t \leq 0$ *and* $\bar{c} \neq \mathbf{0}$.

(iii) *Assume also that there exists a neighborhood* V *of* $\bar{\theta}$ *such that every* $\theta \in V$ *satisfies at least one of the conditions* (KC) *or* (HC). *Then,* $(\mathbf{0}, 0) \notin \mathrm{cl}^* \operatorname{conv}\left(\bar{a}, \bar{b}\right)$ *is a necessary and sufficient condition for* g *to be* $0-$ *stable around* $\bar{\theta}$ *in the following cases:*

  (iii.1) *In Scenario 2, with* $\bar{c} = \mathbf{0}$.
  (iii.2) *In Scenario 5, with* $\bar{c} = \mathbf{0}$ *and* $\sup_{t\in T} \bar{b}_t > 0$.

Remark In Scenario 5 with $\bar{c} = \mathbf{0}$ and $\sup_{t\in T} \bar{b}_t \leq 0$, it is always true that $\theta = \left(a, \bar{b}, \bar{c}\right) \in \operatorname{int}\Theta_{cc} = \Theta$. Moreover, there is no duality gap at θ because, in this case, $v^P(\theta) = v^D(\theta) = 0$.

Finally, regarding the ∞-stability of g, observe that in Scenarios 4, 6, and 7 it is immediate to obtain the equality $\operatorname{int}\left(\Theta \diagdown \Theta_{cc}\right) = \operatorname{int}(\Pi_i) \cup \operatorname{int}(\Delta_i)$, so in this case we can make use of the conditions in Propositions 6 and 10 to describe the ∞-stability of g around any parameter $\theta \notin \Theta_{cc}$. Getting a complete description of int $\left(\Theta \diagdown \Theta_{cc}\right)$ remains as an open, difficult, problem in the other possible scenarios. Nonetheless, Theorem 13 provides sufficient conditions for the $\infty-$stability of g at some parameter θ because int $(\Theta_{ii}) \subset$ int $\left(\Theta \diagdown \Theta_{cc}\right)$, for every scenario.

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