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ORIGINAL PAPER

Fractional optimal control problem for ordinary differential equation in weighted Lebesgue spaces

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Abstract
In this paper, a necessary and sufficient condition, such as the Pontryagin's maximum principle for a fractional optimal control problem with concentrated parameters, is given by the ordinary fractional differential equation with a coefficient in weighted Lebesgue spaces. We discuss a formulation of fractional optimal control problems by a fractional differential equation in the sense of Caputo fractional derivative. The statement of the fractional optimal control problem is studied by using a new version of the increment method that essentially uses the concept of an adjoint equation of the integral form.

Keywords Fractional optimal control problem · Initial value problem · Caputo fractional derivative · Weighed Lebesgue spaces · Pontryagin's maximum principle

1 Introduction

It is known that fractional optimal control problems described by ordinary fractional differential equations can be regarded as a generalization of classic optimal control problems. In the last time, fractional calculus plays an essential role in the various

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field of mathematics, physics, electronics, fluid filtration, control processing, signal processing, stochastic systems, engineering and many others (see, [4,5,11,16–19,22–24,27–29]). Development of fractional optimal control theory led to its application to practical problems such as a fractional order controlled objects, fractional optimization of dynamical systems and others. Many of these optimal control problems the solution of which the subject of numerous works, described fractional ordinary differential equations. The problem of optimal control of systems with concentrated parameters has numerous applications. For details, see [6].

The Pontryagin maximum principle is a fundamental result of the theory of necessary optimality conditions of the first order, which initially was proved in [25] for optimal control problems described by ordinary differential equations. The later works were dedicated to obtaining the necessary conditions for optimality in more complex control problems with concentrated and distributed parameters. The necessity of controlling the systems described by non-integer order models has led to developing fractional order control techniques. The Pontryagin maximum principle for fractional optimal control problems was proved in [3,12]. In the papers [1,2,13] a general formulation and a new solution scheme was given for a class of fractional optimal control problems for those systems. Recently, the optimal control problem in the processes described by the Goursat problem for a hyperbolic equation in variable exponent Sobolev spaces with dominating mixed derivatives was studied in [7] (see, also [8,20]). In [26] was considered a fractional order optimal control problems in which the dynamic control system involves integer and fractional order derivatives and the terminal time is free.

The present work is devoted to obtaining of necessary and sufficient condition such as the maximum principle of Pontryagin for a fractional optimal control problem with concentrated parameters described by an ordinary differential equation with coefficient in weighted Lebesgue spaces.

In this paper, the optimal control problem is investigated for an ordinary fractional differential equation with a coefficient in Lebesgue spaces and with initial value problem. The statement of optimal control problem is studied by using a new version of the increment method that essentially uses the concept of the adjoint equation of the integral form. The method also includes the case where the coefficients of the equation are non-smooth functions from weighted Lebesgue spaces. In this paper, it is shown that such a fractional optimal control problem can be investigated with the help of a new concept of the adjoint equation, which can be regarded as an auxiliary equation for determination of Lagrange multipliers. These fractional optimal control problems actually describe more complex control processes, which are very important in the theory of optimal processes.

The paper is organized as follows. Section 2 contains some preliminaries along with the standard ingredients used in the proofs. In Sect. 3, we give the problem statement, and in Sect. 4, we show the construction of an adjoint equation of the considered optimal control problem. In Sect. 5, we give the proof of the main result.

2 Preliminaries

Let $\mathbb{R}$ denote the set of real line and let $1 \leq p \leq \infty$. Suppose that $T > 0$ is a fixed number. We say that $v : (0, T) \mapsto \mathbb{R}$ is a weight function, if it is Lebesgue measurable, a.e. a positive and locally integrable function on $(0, T)$. We denote by $L_{p,v}(0, T)$ the space of Lebesgue measurable functions u on $(0, T)$ such that

$$\|u\|_{L_{p,v}(0,T)} = \|u\|_{p,v} = \left(\int_0^T |u(t)|^p v(t)\, dt\right)^{\frac{1}{p}} < \infty.$$

For $p = \infty$ we will use convention $\|u\|_{L_{\infty,v}(0,T)} = \|u\|_\infty = ess \sup\limits_{0<t<T} |u(t)|$.

Theorem 1 [9,21] *Let $1 \leq p \leq \infty$. Then the space $L_{p,v}(0, T)$ is a Banach space.*

Theorem 2 [9,21] *Let $1 \leq p < \infty$ and let $\frac{1}{p} + \frac{1}{q} = 1$. Then for every bounded linear functional ℓ on $L_{p,v}(0, T)$ there is a unique $g \in L_{q,v}(0, T)$ such that*

$$\ell(f) = \int_0^T f(x)g(x)v(x)dx, \quad \textit{for all} \quad f \in L_{p,v}(0, T).$$

Moreover $\|\ell\|_{L_{q,v}(0,T)} = \|g\|_{L_{q,v}(0,T)}$.

We need the following definition.

Definition 1 [10] Let v be a weight function on $(0, T)$. We say that a weight function v satisfies doubling condition, is there exists a constant $C \geq 1$ such that

$$\int_{x-2t}^{x+2t} v(y)dy \leq C \int_{x-t}^{x+t} v(y)dy$$

for all $x, t \in (0, T)$.

Let $\mathbb{N}$ be the set of natural numbers and $n \in \mathbb{N}$. Suppose $AC(0, T)$ is the space of absolutely continuous functions on $(0, T)$. By $AC^n(0, T)$ we denote the space of real-valued functions u which have continuous derivatives up to order $n - 1$ on $(0, T)$ such that $u^{(n-1)} \in AC(0, T)$. It is obvious that $AC^1(0, T) = AC(0, T)$ (see, [16]).

There are several definitions of a fractional derivative. In this section, we present a review of some definitions and preliminary facts which are particularly relevant for the results of this paper [16,24,27].

Definition 2 Let $f \in L_1(0, T)$. For almost all $t \in (0, T)$ and $\alpha > 0$, the left and right Riemann–Liouville fractional integrals of order α are defined by

$$I_{0+}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t - \tau)^{\alpha-1} f(\tau)d\tau,$$

and

$$I_{0-}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_{t}^{T} (\tau - t)^{\alpha-1} f(\tau) d\tau,$$

respectively, where Γ is the Euler gamma function.

Definition 3 Let $f \in AC^n(0, T)$. For almost all $t \in (0, T)$ and $\alpha > 0$, the left and right Riemann–Liouville fractional derivatives of order α are defined by

$${}^{RL}D_{0+}^{\alpha} f(t) := \frac{d^n}{dt^n} \left(I_{0+}^{n-\alpha} f(t) \right) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_{0}^{t} (t - \tau)^{n-\alpha-1} f(\tau) d\tau,$$

and

$${}^{RL}D_{0-}^{\alpha} f(t) := \frac{d^n}{dt^n} \left(I_{0-}^{n-\alpha} f(t) \right) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt} \right)^n \int_{t}^{T} (\tau - t)^{n-\alpha-1} f(\tau) d\tau,$$

respectively, where $n \in \mathbb{N}$ is such that $n - 1 < \alpha \leq n$.

Definition 4 Let $f \in AC^n(0, T)$. For almost all $t \in (0, T)$ and $\alpha > 0$, the left and right Caputo fractional derivatives are defined by

$${}^{C}D_{0+}^{\alpha} f(t) := I_{0+}^{n-\alpha} \frac{d^n}{dt^n} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t - \tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau,$$

and

$${}^{C}D_{0-}^{\alpha} f(t) := I_{0-}^{n-\alpha} \left(-\frac{d}{dt} \right)^n f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_{t}^{T} (\tau - t)^{n-\alpha-1} f^{(n)}(\tau) d\tau,$$

respectively, where $n \in \mathbb{N}$ is such that $n - 1 < \alpha \leq n$.

Remark 1 Let α be a whole number. Then the Riemann–Liouville and Caputo fractional derivatives coincides with the classical derivative $\frac{d^n f(t)}{dt^n}$.

It is obvious that the Caputo fractional derivative of a constant is equal to zero. This is not the case with the Riemann–Liouville fractional derivative. Indeed, if $c \neq 0$, then by the definition of Riemann–Liouville fractional derivative ${}^{RL}D_{0+}^{\alpha} c = \frac{c}{\Gamma(n-\alpha+1)} t^{n-\alpha}$.

Theorem 3 *Let $\alpha > 0$ and let $f \in C^{(n)}(0,T)$, $n = [\alpha]+1$. Then,*

$$ {}^{C}D^{\alpha}_{0+} I^{\alpha}_{0+} f(t) = f(t), \ \ {}^{C}D^{\alpha}_{0-} I^{\alpha}_{0-} f(t) = f(t), $$

and

$$ I^{\alpha}_{0+}\, {}^{C}D^{\alpha}_{0+} f(t) = f(t) - f(0), \ \ I^{\alpha}_{0-}\, {}^{C}D^{\alpha}_{0-} f(t) = f(T) - f(t), $$
$$ f'(0) = \cdots = f^{(n-1)}(0) = f'(T) = \cdots = f^{(n-1)}(T) = 0. $$

The Laplace transform $F(s)$ of a function $f(t)$ for $t > 0$ is defined as

$$ F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)dt. $$

We need the following lemma.

Lemma 1 [14] *Let $\alpha > 0$ and let $n-1 < \alpha \le n$, $n \in \mathbb{N}$. Suppose $f \in AC^n(0,T)$, $f(t) = 0$ for $t \ge T$ and let $\lim\limits_{t\to T^-} f^{(\ell)}(t) = 0$, $\ell = 0,1,\ldots,n-1$. Then the Laplace transform of the Caputo fractional derivatives of order α of a function f has the form*

$$ L\left[{}^{C}D^{\alpha}_{0+} f(x)\right] = \frac{s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)}{s^{n-\alpha}}. $$

Definition 5 Let $1 \le p < \infty$ and let $n = [\alpha]+1$. By $\widetilde{W}^{(\alpha)}_{p,v}(0,T)$ we define the following space of functions as

$$ \widetilde{W}^{(\alpha)}_{p,v}(0,T) := \left\{ u: \ u \in L_{p,v}(0,T) \cap AC^n(0,T), \ {}^{C}D^{\alpha}_{0+} u \in L_{p,v}(0,T) \right\}. $$

It is obvious that the expression

$$ \|u\|_{\widetilde{W}^{(\alpha)}_{p,v}(0,T)} = \|u\|_{p,v} + \left\| {}^{C}D^{\alpha}_{0+} u \right\|_{p,v} < \infty $$

defines a norm in $\widetilde{W}^{(\alpha)}_{p,v}(0,T)$.

Lemma 2 *Let $1 \le p < \infty$. Then, the space $\widetilde{W}^{(\alpha)}_{p,v}(0,T)$ is a Banach space.*

The proof of Lemma 2 immediately implies from the definition of this space.

3 Problem statement

Throughout this paper, we assume that $0 < \alpha < 1$. Let the controlled object be described by the equation

$$ (V_\alpha u)(t) \equiv {}^{C}D^{\alpha}_{0+} u(t) + a(t)u(t) = \varphi(t, \nu(t)), \tag{3.1} $$

the following initial value condition

$$V_0 u \equiv u(0) = \varphi_0 \tag{3.2}$$

where $a(t) \in L_{p,v}(0,T)$ and $\varphi_0 \in \mathbb{R}$. Let $\nu(t) = (\nu_1(t), \ldots, \nu_m(t))$ be m-dimensional control vector function and $\varphi(t, \nu(t))$ be a given function defined on $(0,T) \times \mathbb{R}^m$ and satisfying Caratheodory condition on $(0,T) \times \mathbb{R}^m$:

(1) $\varphi(t, \nu(t))$ is measurable by t in $(0,T)$ for all $\nu \in \mathbb{R}^m$;
(2) $\varphi(t, \nu(t))$ is continuous by ν in $\mathbb{R}^m$ for almost all $t \in (0,T)$;
(3) for any $\delta > 0$ there exists $\varphi_\delta^0(t) \in L_{p,v}(0,T)$ such that $|\varphi(t, \nu(t))| \le \varphi_\delta^0(t)$ for almost all $t \in (0,T)$ and $\|\nu\| = \sum_{i=1}^m |\nu_i| \le \delta$.

Since the coefficient of the Eq. (3.1) is non-smooth, we mean the solution of problem (3.1), (3.2) in the weak sense. Let a vector function $\nu(t)$ be measurable and bounded on $(0,T)$ and for almost every $t \in (0,T)$ it takes its value from the given set $\Omega \subset \mathbb{R}^m$. Then a vector function $\nu(t)$ is called admissible controls. The set of all admissible controls is denoted by Ω_∂.

Now consider the following optimal control problem: Find an admissible control $\nu(t)$ from Ω_∂, for which the solution of the problem (3.1), (3.2) $u \in \widetilde{W}_{p,v}^{(\alpha)}(0,T)$ that minimizes the multi-point functional

$$F(\nu) = \sum_{k=1}^{N} \alpha_k u(t_k) \to \min, \tag{3.3}$$

where $t_k \in (0,T]$ are the given fixed points, $\alpha_k \in \mathbb{R}$ are the given real numbers and N is a positive integer.

4 The construction of adjoint equation

To obtain the necessary and sufficient conditions for optimality, first we find the increment of the functional (3.3). Let $\nu(t)$ and $\nu(t) + \Delta\nu(t)$ be different admissible controls, $u(t)$ and $u(t) + \Delta u(t)$ solution of the problem (3.1), (3.2) in the space $\widetilde{W}_{p,v}^{(\alpha)}(0,T)$, respectively. Then the increment of the functional (3.3) is of the form

$$\Delta F(\nu) = \sum_{k=1}^{N} \alpha_k \Delta u(t_k). \tag{4.1}$$

Obviously, in this case the function $\Delta u \in \widetilde{W}_{p,v}^{(\alpha)}(0,T)$ is the solution of the equation

$$V_\alpha \Delta u(t) = \Delta\varphi(t), \tag{4.2}$$

satisfying trivial conditions

$$V_0 \Delta u = 0, \tag{4.3}$$

where $\Delta\varphi(t) = \varphi(t, \nu(t) + \Delta\nu(t)) - \varphi(t, \nu(t))$. The operator $V = (V_\alpha, V_0)$: $\widetilde{W}^{(\alpha)}_{p,v}(0,T) \mapsto E_{p,v} = L_{p,v}(0,T) \times \mathbb{R}$ generated by the problem (3.1), (3.2) is bounded by the above mentioned assumptions.

The integral representation of functions in $\widetilde{W}^{(\alpha)}_{p,v}(0,T)$ has the form

$$u(t) = u(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\ {}^C D^\alpha_{0+} u(\tau) d\tau. \tag{4.4}$$

Next, we show that the operator V has an adjoint operator $V^\star = (\omega_\alpha, \omega_0)$, which acts in the spaces $E_{q,v} = L_{q,v}(0,T) \times \mathbb{R}$ and satisfy the condition (4.2). Using the general form of a continuous linear functional on $E_{q,v}$, (see, [9,21]) we have

$$\begin{aligned} f(Vu) &= \int_0^T f_\alpha(t) \left(V_\alpha u\right)(t) v(t) dt + f_0 \left(V_0 u\right) \\ &= \int_0^T f_\alpha(t) v(t) \left[{}^C D^\alpha_{0+} u(t) + a(t) u(t)\right] dt + f_0 u(0). \end{aligned}$$

By (4.4), we get

$$\begin{aligned} f(Vu) &= \int_0^T f_\alpha(t) v(t) \left[{}^C D^\alpha_{0+} u(t) + a(t) \left(u(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\ {}^C D^\alpha_{0+} u(\tau) d\tau\right)\right] dt \\ &+ f_0 u(0) = \int_0^T (\omega_\alpha f)(t)\ {}^C D^\alpha_{0+} u(t) v(t) dt + \omega_0 f u(0) = \left(V^* f\right)(u), \end{aligned} \tag{4.5}$$

where $f = (f_\alpha(t), f_0) \in E_{q,v}$ is an arbitrary linear bounded functional on $E_{p,v}$, $u(t) \in \widetilde{W}^{(\alpha)}_{p,v}(0,T)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Expressions for the $\omega_\alpha f$ and $\omega_0 f$ is given as follows:

$$\begin{aligned} (\omega_\alpha f)(t) &\equiv f_\alpha(t) + \frac{1}{\Gamma(\alpha)} \int_t^T (\tau - t)^{\alpha-1} a(\tau) f_\alpha(\tau) d\tau = f_\alpha(t) + I^\alpha_{0-}(a f_\alpha)(t), \\ \omega_0 f &\equiv \int_0^T f_\alpha(t) a(t) v(t)\, dt + f_0. \end{aligned}$$

We need a following theorem.

Theorem 4 *Let* $1 < p < \infty$ *and let* $a(t) \in L_{p,v}(0,T)$ *and* $f_\alpha(t) \in L_{q,v}(0,T)$. *Suppose* $v : (0,T) \mapsto (0,\infty)$ *is a weight function satisfying the condition*

$$B(v, \alpha, q) = \sup_{0<\tau<T} \frac{1}{v(\tau)} \left(\int_0^\tau (\tau - t)^{q(\alpha-1)} v(t)\, dt\right)^{1/q} < \infty.$$

Then

$$\|\omega_\alpha f\|_{q,v} \leq \left(1 + \frac{B(v,\alpha,q)}{\Gamma(\alpha)} \|a\|_{p,v}\right) \|f_\alpha\|_{q,v}$$

and

$$|\omega_0 f| \leq \|a\|_{p,v} \|f_\alpha\|_{q,v} + |f_0|\,.$$

Proof Obviously,

$$\|\omega_\alpha f\|_{q,v} \leq \|f_\alpha\|_{q,v} + \left\|I^\alpha_{0-}(af_\alpha)\right\|_{q,v}\,.$$

Let (a,b) be a subset of $(0,T)$. We denote by $\chi_{(a,b)}(\tau)$ the characteristic function of (a,b). Applying the generalized Minkowski inequality, we have

$$\begin{aligned}
\left\|I^\alpha_{0-}(af_\alpha)\right\|_{q,v} &= \frac{1}{\Gamma(\alpha)} \left(\int_0^T \left|\int_t^T (\tau-t)^{\alpha-1} a(\tau) f_\alpha(\tau) d\tau\right|^q v(t)dt\right)^{1/q} \\
&= \frac{1}{\Gamma(\alpha)} \left(\int_0^T \left|\int_0^T (\tau-t)^{\alpha-1} a(\tau) f_\alpha(\tau) v(t)^{\frac{1}{q}} \chi_{(t,T)}(\tau) d\tau\right|^q dt\right)^{1/q} \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^T \left(\int_0^T (\tau-t)^{q(\alpha-1)} \left|a(\tau) f_\alpha(\tau) \chi_{(t,T)}(\tau)\right|^q v(t)\,dt\right)^{1/q} d\tau \\
&= \frac{1}{\Gamma(\alpha)} \int_0^T |a(\tau)|\,|f_\alpha(\tau)| \left(\int_0^\tau (\tau-t)^{q(\alpha-1)} v(t)\,dt\right)^{1/q} d\tau \\
&= \frac{1}{\Gamma(\alpha)} \int_0^T |a(\tau)|\,|f_\alpha(\tau)|\,v(\tau)\,\frac{1}{v(\tau)} \left(\int_0^\tau (\tau-t)^{q(\alpha-1)} v(t)\,dt\right)^{1/q} d\tau \\
&\leq \frac{B(v,\alpha,q)}{\Gamma(\alpha)} \int_0^T |a(\tau)|\,|f_\alpha(\tau)|\,v(\tau) d\tau.
\end{aligned}$$

Applying the Hölder inequality, we get

$$\begin{aligned}
\int_0^T |a(\tau)|\,|f_\alpha(\tau)|\,v(\tau)d\tau &= \int_0^T |a(\tau)|\,|f_\alpha(\tau)|\,v(\tau)^{\frac{1}{p}+\frac{1}{q}} d\tau \\
&= \int_0^T \left(|a(\tau)| v(\tau)^{\frac{1}{p}}\right)\left(|f_\alpha(\tau)|\,v(\tau)^{\frac{1}{q}}\right) d\tau \\
&\leq \left\|a v^{\frac{1}{p}}\right\|_p \left\|f_\alpha v^{\frac{1}{q}}\right\|_q = \|a\|_{p,v} \|f_\alpha\|_{q,v}\,.
\end{aligned}$$

Therefore, we get

$$\left\|I^\alpha_{0-}(af_\alpha)\right\|_{q,v} \leq \frac{B(v,\alpha,q)}{\Gamma(\alpha)} \|a\|_{p,v} \|f_\alpha\|_{q,v}\,.$$

Thus,

$$\|\omega_\alpha f\|_{q,v} \le \left(1 + \frac{B(v,\alpha,q)}{\Gamma(\alpha)}\|a\|_{p,v}\right)\|f_\alpha\|_{q,v}\,.$$

Again using Hölder inequality, it is easy to show that

$$|\omega_0 f| \le \|a\|_{p,v}\,\|f_\alpha\|_{q,v} + |f_0|\,.$$

This completes the proof of Theorem 4. □

By Theorem 4 we conclude that $V^\star = (\omega_\alpha, \omega_0)$ is a bounded operator from $L_{q,v}(0,T)$ to $E_{q,v}$ and satisfies the condition (4.2). Indeed,

$$\left\|V^\star f\right\|_{E_{q,v}} = \|\omega_\alpha f\|_{q,v} + |\omega_0 f|\,.$$

Thus, by Theorem 4, we get

$$\begin{aligned}\left\|V^\star f\right\|_{E_{q,v}} &\le \left(1 + \left(1 + \frac{B(v,\alpha,q)}{\Gamma(\alpha)}\right)\|a\|_{p,v}\right)\|f_\alpha\|_{q,v} + |f_0| \\ &\le 2\max\left\{\left(1 + \left(1 + \frac{B(v,\alpha,q)}{\Gamma(\alpha)}\right)\|a\|_{p,v}\right)\|f_\alpha\|_{q,v}\,,\ |f_0|\right\}.\end{aligned}$$

Now in (4.5) instead of $u(t)$ substitute the solution of the problem (4.2), (4.3). Then, the equality

$$\begin{aligned} f(V\Delta u) &= \int_0^T f_\alpha(t)\Delta\varphi(t)\,v(t)dt \\ &= \int_0^T (\omega_\alpha f)(t)\ {}^C D^\alpha_{0+}\Delta u(t)\,v(t)dt \equiv \left(V^\star f\right)(\Delta u) \end{aligned} \tag{4.6}$$

holds for all $f \in E_{q,v}$. In other words,

$$-\int_0^T f_\alpha(t)\Delta\varphi(t)\,v(t)dt + \int_0^T (\omega_\alpha f)(t)\ {}^C D^\alpha_{0+}\Delta u(t)\,v(t)dt = 0. \tag{4.7}$$

Therefore the function $\Delta u(t)$ as an element of $\widetilde{W}^{(\alpha)}_{p,v}(0,T)$ satisfies the condition (4.3). Using the integral representation (4.4), we have

$$\alpha_k \Delta u\,(t_k) = \int_0^T B_k(t)\ {}^C D^\alpha_{0+}\Delta u(t)dt, \quad k = 1,\ldots,N,$$

where

$$B_k(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\,\alpha_k\,(t_k - t)^{\alpha-1}\,, & t < t_k \\ 0, & t \ge t_k. \end{cases}$$

Therefore, the increment (4.1) of the functional (3.3) can be represented as

$$\Delta F(\nu) = \int_0^T \sum_{k=1}^N B_k(t)\ {}^C D_{0+}^\alpha \Delta u(t) dt,$$

or

$$\Delta F(\nu) = \int_0^T B(t)\ {}^C D_{0+}^\alpha \Delta u(t) dt, \tag{4.8}$$

and

$$B(t) = \sum_{k=1}^N B_k(t).$$

By (4.7) the increment (4.8) can be represented in the form

$$\Delta F(\nu) = \int_0^T [B(t) + (\omega_\alpha f)(t)\, v(t)]\ {}^C D_{0+}^\alpha \Delta u(t) dt - \int_0^T f_\alpha(t) \Delta\varphi(t)\, v(t) dt. \tag{4.9}$$

Since ω_α depends only on f_α, equality (4.9) holds for all $f_\alpha(t) \in L_{q,v}(0, T)$. The adjoint equation corresponding to the optimal control problem (3.1)–(3.3) has the form

$$(\omega_\alpha f_\alpha)(t)\, v(t) + B(t) = 0, \quad t \in (0, T). \tag{4.10}$$

As the function of $f_\alpha(t)$ we take the solution of the Eq. (4.10) in $L_{q,v}(0, T)$. Then equality (4.9) has the simple form

$$\Delta F(\nu) = -\int_0^T f_\alpha(t) \Delta\varphi(t)\, v(t)\, dt.$$

5 Main result

Now, for a fixed $\tau \in (0, T)$ we consider the following needle variation of the admissible control $\nu(t)$:

$$\Delta\nu_\varepsilon(t) = \begin{cases} \widehat{\nu} - \nu(t), & t \in G_\varepsilon \\ 0, & t \in (0, T) \backslash G_\varepsilon, \end{cases}$$

where $\widehat{\nu} \in \Omega_\partial$, $\varepsilon > 0$ is a sufficiently small parameter and $G_\varepsilon = \left(\tau - \frac{\varepsilon}{2}, \tau + \frac{\varepsilon}{2}\right) \subset (0, T)$. The control $\nu_\varepsilon(t)$ defined by the equality $\nu_\varepsilon(t) = \nu(t) + \Delta\nu_\varepsilon(t)$ is an admissible control for all sufficiently small $\varepsilon > 0$ and all $\widehat{\nu} \in \Omega_\partial$ called a needle perturbation given by the control $\nu(t)$, where $\tau \in (0, T)$ is some fixed point. Obviously,

$$\begin{aligned} F(\nu_\varepsilon) - F(\nu) &= -\int_0^T f_\alpha(t)\varphi(t, \nu(t) + \Delta\nu_\varepsilon(t))\, \upsilon(t)\, dt + \int_0^T f_\alpha(t)\Delta\varphi(t)\, \upsilon(t)\, dt \\ &= -\int_0^T f_\alpha(t)\left[\varphi(t, \widehat{\nu}(t)) - \varphi(t, \nu(t))\right] \upsilon(t) dt \\ &= -\left(\int_{G_\varepsilon} f_\alpha(t)\left[\varphi(t, \widehat{\nu}(t)) - \varphi(t, \nu(t))\right] \upsilon(t) dt \right. \\ &\quad \left. + \int_{(0,T)\backslash G_\varepsilon} f_\alpha(t)\left[\varphi(t, \widehat{\nu}(t)) - \varphi(t, \nu(t))\right] \upsilon(t) dt \right). \end{aligned}$$

Let $t \in (0, T)\backslash G_\varepsilon$. Since $\widehat{\nu}(t) = \nu(t)$, it follows that

$$F(\nu_\varepsilon) - F(\nu) = -\int_{G_\varepsilon} f_\alpha(t)\left[\varphi(t, \widehat{\nu}(t)) - \varphi(t, \nu(t))\right] \upsilon(t) dt. \tag{5.1}$$

Since the optimal control problem is linear, the following theorem follows from (5.1).

Theorem 5 *Let $1 < p < \infty$ and let $f_\alpha(t) \in L_{q,\upsilon}(0, T)$ be a solution of the adjoint equation (4.10). Suppose $\upsilon : (0, T) \mapsto (0, \infty)$ is a weight function satisfies doubling condition. Then for the optimality of the admissible control $\nu(t)$, it is necessary and sufficient that for almost all $t \in (0, T)$ it satisfy the Pontryagin maximum condition*

$$\max_{\widehat{\nu} \in \Omega_\partial} H(t, f_\alpha(t), \widehat{\nu}) = H(t, f_\alpha(t), \nu),$$

where $H(t, f_\alpha(t), \nu) = f_\alpha(t) \cdot \varphi(t, \nu)$ is the Hamilton–Pontryagin function.

Proof Suppose that a control $\nu(t) \in \Omega_\partial$ gives the minimum value of the functional (3.3). Then by (5.1), we have

$$-\int_{G_\varepsilon} \left[H(t, f_\alpha(t), \widehat{\nu}) - H(t, f_\alpha(t), \nu(t))\right] \upsilon(t) dt \geq 0. \tag{5.2}$$

Dividing the both sides of (5.2) by $\int_{G_\varepsilon} \upsilon(t)\, dt$ and passing to the limit as $\varepsilon \to +0$, for almost all $\tau \in (0, T)$ and using the analog of the Lebesgue differentiation theorem in $L_{p,\upsilon}(0, T)$ (see, [10]) for all $\nu \in \Omega_\partial$, we get

$$H(\tau, f_\alpha(\tau), \nu(\tau)) - H(\tau, f_\alpha(\tau), \widehat{\nu}) \geq 0. \tag{5.3}$$

Thus, for optimal control $\nu(t) \in \Omega_\partial$, it is necessary to satisfy the condition (5.3). Besides, the equality

$$\Delta F(\nu) = -\int_0^T \Delta H(t, f_\alpha(t), \nu(t))\, \upsilon(t) dt$$

shows that this condition is also sufficient for optimal control $\nu(t)$, where $\Delta H\,(t,\, f_\alpha,\, \nu) = H\,(t,\, f_\alpha(t), \nu + \Delta\nu) - H\,(t,\, f_\alpha(t), \nu(t))$.

This completes the proof. □

Remark 2 Theorem 5 shows that for the solvability of the optimal control problem (3.1)–(3.3), it is sufficient to find a solution $f_\alpha(t) \in L_{q,v}(0, T)$ of the integral equation (4.10). Then the optimal control $\nu(t)$ can be found as an element of the Ω_∂, which gives the maximum value to the functional $H\,(t,\, f_\alpha(t),\, \nu(t))$ in Ω_∂ with respect to the function ν.

Remark 3 Note that similarly results can be proved for optimal control problem (3.1)–(3.3) without restriction $0 < \alpha < 1$.

For power weight function we have the following corollary.

Corollary 1 *Let* $1 < p < \infty$ *and let* $v(t) = t^\beta$, $\max\left\{\frac{1}{p}, \frac{\beta+1}{p}\right\} < \alpha < 1$ *and* $-1 < \beta \le p-1$. *Suppose that* $f_\alpha \in L_{q,\,t^\beta}(0, T)$ *is a solution of the adjoint equation* (4.10)*. Then for the optimality of the admissible control* $\nu(t)$*, it is necessary and sufficient that for almost all* $t \in (0, T)$ *satisfy the Pontryagin maximum condition*

$$\max_{\widehat{\nu}\in\Omega_\partial} H\,(t,\, f_\alpha(t), \widehat{\nu}) = H\,(t,\, f_\alpha(t), \nu)\,,$$

where $H\,(t,\, f_\alpha(t), \nu) = f_\alpha(t)\cdot\varphi(t, \nu)$ *is the Hamilton–Pontryagin function.*

Example 1 Let $a(t) = 0$ in the left hand side of Eq. (3.1). Then the adjoint equation (4.10) has the simple form

$$f_\alpha(t) + B(t) = 0, \quad \text{a.e. } t \in (0, T).$$

Thus, $f_\alpha(t) = -B(t)$. The case $B(t) = 0$ is trivial, so we can assume $B(t) \neq 0$. By definition of the Hamilton–Pontryagin function, we have

$$H\,(t,\, f_\alpha(t), \nu) = -B(t)\,\varphi(t, \nu).$$

By Theorem 5, we get

$$\max_{\widehat{\nu}\in\Omega_\partial} H\,(t,\, f_\alpha(t), \widehat{\nu}) = \max_{\widehat{\nu}\in\Omega_\partial} \left[-B(t)\varphi\,(t, \widehat{\nu})\right].$$

Thus, in this case, the Pontryagin maximum principle is expressed by $\varphi(t, \nu(t)) = {}^C D^\alpha_{0+} u(t)$.

Example 2 Let $a(t) = 1$ in the left hand side of Eq. (3.1). Then the adjoint equation (4.10) has the form

$$f_\alpha(t) + I^\alpha_{0-}\,(f_\alpha) + B(t) = 0, \quad \text{a.e. } t \in (0, T).$$

By definition of the Hamilton–Pontryagin function, we have

$$H\left(t, f_\alpha(t), \nu\right) = -\left(I_{0-}^{\alpha}\left(f_\alpha\right) + B(t)\right)\varphi(t, \nu).$$

By Theorem 5, we get

$$\max_{\widehat{\nu}\in\Omega_\partial} H\left(t, f_\alpha(t), \nu\right) = \max_{\widehat{\nu}\in\Omega_\partial}\left[-\left(I_{0-}^{\alpha}\left(f_\alpha\right) + B(t)\right)\varphi(t, \nu)\right].$$

Let $\nu(t)$ be a fixed control and let $\overline{u}(t) = \begin{cases} u(t), & 0 < t < T, \\ 0, & t \geq T. \end{cases}$

Let us consider the problem

$$\begin{cases} {}^{C}D_{0+}^{\alpha}\overline{u}(t) + \overline{u}(t) = \varphi(t, \nu(t)), & \text{a.e. } t \in (0, \infty), \\ \overline{u}(0) = \varphi_0. \end{cases}$$

It is well known that the Mittag–Leffler function with two parameters is defined as (see, [16])

$$E_{\beta,\gamma}(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(k\beta+\gamma)},$$

where $\beta, \gamma > 0$. Applying Lemma 1, similarly as in [15], we can prove that

$$\overline{u}(t) = \varphi_0 E_{\alpha,1}\left(-t^{\alpha}\right) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-(t-\tau)^{\alpha})\varphi(\tau, \nu(\tau))d\tau. \quad (5.4)$$

Thus, in this case, the Pontryagin maximum principle is expressed by (5.4).

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