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ORIGINAL PAPER

Optimality conditions for the continuous model of the final open pit problem

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Abstract
In this work we address the *Final Open Pit* problem in a continuous framework, that is, the problem of finding the optimal profile for an open pit that satisfies an additional slope and maximum capacity conditions on extraction. Using optimal control theory and calculus of variations tools, we provide optimality conditions for that problem. In particular, we prove that the distribution of gain along the lower border of the optimal pit must be zero, when the slope and capacity constraints are not active.

Keywords Final open pit · Optimal control · Calculus of variations · Optimality conditions

1 Introduction

The long term planning of a mine operation consists of defining a sequence for the extraction of material from the mine in order to maximize profit. As a first step in this process, decision-makers usually must decide the final pit limit, which corresponds to the identification of a maximum value on the total mass to be extracted from the site, which enables an upper bound on the discounted value of the profit over several periods to be defined. This first step is called the Final Open Pit or Ultimate Open Pit problem. A very early contribution to the practical resolution of this problem was proposed by Lerchs and Grossman [18] and, since then, a great variety of models and

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algorithms have been proposed. See Hustrulid et al [14] and Newman et al [19] for a more thorough introduction to open pit mine planning. The first effort to formally describe a practical mathematical model to solve this problem in an integrated way seems to be the work by Johnson [15].

Three different problems are usually considered for the economic valuation, design and planning of open pit mines. The first is the Final Open Pit (FOP) problem, which aims at finding the region of maximal economic value under geotechnical stability constraints. Another more realistic problem is what we call here the Capacity Final Open Pit (CFOP), which adds an additional constraint on the total capacity for extraction. The third problem is a multi-period version of the latter, which we call the Capacity Dynamic Open Pit (CDOP) problem, with the goal of finding an optimal sequence of volumes to be extracted with bounded capacities during each period.

The usual formulation of these problems consists of describing an ore reserve as a three-dimensional block model. Each block corresponds to a unitary volume of extraction, characterized by several physical and economic attributes, most of which are estimated from experimental sampling. Block models can be represented as directed graphs where nodes represent the blocks and arcs determineblock precedence (order of extraction). Block precedence is essentially induced by operational constraints, such as those derived from slope stability. This discrete approach usually gives rise to huge, combinatorially large-scale instances of Integer Programming, such as that presented by Cacetta [4]. A great number of publications dealing with discrete block modeling for open pit mines have been published over the last 60 years. The seminal methodology for obtaining the ultimate pit limit, introduced by Lerchs and Grossman [18], has been extensively applied in real mines for many years. The capacity dynamic problem is more difficult to solve and many methods using discrete optimization techniques have been proposed by Boland et al [2], Cacetta and Hill [5] and Hochbaum and Chen [13]. This problem is beyond the scope of this paper, but we can mention some dynamic programming formulations, for instance, Johnson and Sharp [16] and Wright [21]. Metaheuristic and evolutionary algorithms have also been extensively tested by Denby and Schofield [6] and Ferland et al [9].

In this paper we use an alternative approach to the above mentioned (CFOP) problem based on a continuous framework, proposed by Alvarez et al [1]. The basic idea is to describe pit contours by a continuous real-valued function, which maps each pair of horizontal coordinates to the corresponding vertical depth. Slope stability is ensured by means of a spatially distributed constraint on the local Lipschitz constant of the profile function. The maximal feasible local slope may vary throughout the site, depending on the geotechnical properties of the mineral deposit. The extraction capacity and operational costs are described by a possibly discontinuous effort density, a scalar function defined on the three-dimensional mining site. Concerning the continuous approach, we mention here the contribution by Ekeland and Queyranne [8], who proposed an alternative approach based on determining an optimum pit from an optimum dual solution of a particular transportation problem. Additionally, in [11], the authors derive duality results for the stationary open pit problem in the continuous framework, employing an additional condition called convex-likeness. The same authors, in [12], propose a partial differential equation model and show that, under

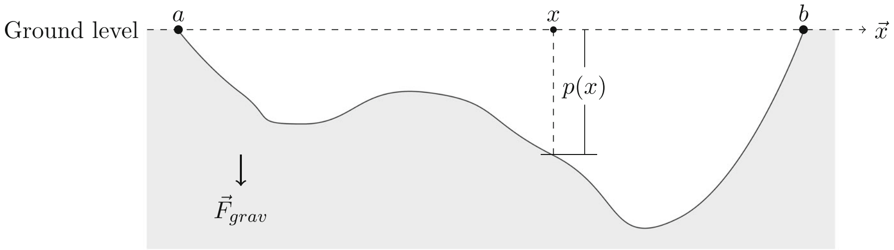

Fig. 1 Profile of an open pit on the plane

suitable assumptions, the physically stable excavation path is the solution of a certain Hamilton-Jacobi equation.

The economic value of the blocks is given by a gain density defined on the deposit, which can also be a discontinuous function. Our goal here is to extend the existence results develop by Alvarez et al [1] to the qualitative properties of the optimal solutions. This qualitative characterization is derived from the optimality conditions in the calculus of variations and control theory.

The paper is organized as follows. In Sect. 2 we describe the stationary problem in terms of continuous profile functions and we establish the basis of our approach, in the context of a "2D-mine", which permits to give a simple motivation of the real 3D problem and to derive relevant results that can be generalized to the real case. Section 3 is devoted to the study of the realistic 3D instance, extending the main results of the previous section. By using tools from the calculus of variations, we derive an operational characterization of the optimal profile, particulary to show that the gain function must take the value zero along the border of the optimal profile, unless the capacity or slope constraints are active. In Sect. 4 we briefly summarize the main contributions of this paper and indicate some avenues for future research.

2 The 2D open pit problem

To fix ideas, we begin by considering the idealized case of an open pit on the plane, that is, the framework where the profiles are modeled using a continuous function that depends only on a single space variable (denoted x for simplicity). Generically, we denote a profile of an open pit by $p : [a, b] \to \mathbb{R}_+$ where a and b are the extreme points of the open pit (there is no loss of generality in taking $a < b$) and where $p(x)$ represents the depth of the profile at the point $x \in [a, b]$; see Fig. 1.

2.1 Statement of the problem

For the sake of notation, we assume that the depth of a profile is always positive. In this framework, an admissible profile is a function $p : [a, b] \to \mathbb{R}$ that must satisfy some conditions, the first one being as follows: given an initial profile $p_0 : [a, b] \to \mathbb{R}$ an admissible profile has to satisfy

$$p_0(x) \le p(x), \qquad \forall x \in [a, b].$$

which means that a feasible profile must be deeper than the initial profile p_0.

Given a profile $p : [a, b] \to \mathbb{R}$, we define its *slope* at the point $x \in [a, b]$ as the *Lipschitz modulus* of p at x (see for example Dontchev and Rockafellar [7, Section 1D]), that is,

$$L_p(x) := \limsup_{\bar{x} \to x \leftarrow \hat{x}} \frac{|p(\bar{x}) - p(\hat{x})|}{|\bar{x} - \hat{x}|}.$$

Due to the risk of landslides, the slope of a profile cannot be too steep. Note that the maximal slope allowed may change depending on the position and depth in the pit. This constraint is then represented via the condition

$$L_p(x) \le \kappa(x, p(x)), \quad \forall x \in [a, b],$$

where $\kappa(x, z)$ represents the maximal slope at the point (x, z) allowed for a profile $p : [a, b] \to \mathbb{R}$ to be admissible. Note that if the profile is continuously differentiable on (a, b), then the slope agrees with the absolute value of the profile's derivative (see [7, Section 1D]), that is,

$$L_p(x) = |p'(x)|, \quad \forall x \in (a, b).$$

However, in our setting, working with smooth functions is too restrictive. For this reason we choose to work with a broader class of functions, namely, the collection of continuous functions whose derivatives exist almost everywhere on $[a, b]$ and which satisfy

$$p(x) = p(a) + \int_a^x p'(s)ds, \quad \forall x \in [a, b].$$

This class of functions is the so-called set of absolutely continuous functions, which we denote by $\mathcal{AC}[a, b]$. It turns out that absolutely continuous functions are well behaved with respect to the slope, in the sense that the slope agrees almost everywhere with the derivative of an absolutely continuous profile.

Lemma 1 *Let $p \in \mathcal{AC}[a, b]$, then $L_p(x) = |p'(x)|$ almost everywhere on $[a, b]$.*

On the other hand, due to physical or economic constraints, the capacity of extraction is indeed limited. Given a position x, the effort associated with extracting a block at depth $z \ge p_0(x)$ can be represented by a nonnegative quantity $e(x, z)$. Thus, given a maximal budget $c_{\max} > 0$, the capacity constraints associated with a profile $p : [a, b] \to \mathbb{R}$ can be expressed via the condition

$$\int_a^b \int_{p_0(x)}^{p(x)} e(x, z)dzdx \le c_{\max}.$$

Concerning optimality, the marginal profit at each $x \in [a, b]$ of an admissible profile $p : [a, b] \to \mathbb{R}$ is given by

$$\int_{p_0(x)}^{p(x)} g(x,z)dz$$

where $g(x, z)$ represents the profit earned (or gain) for carrying out extraction at the block (x, z) for any $z \in [p_0(x), p(x)]$. Therefore, the total profit associated with an admissible profile $p : [a, b] \to \mathbb{R}$ is given by

$$\int_a^b \int_{p_0(x)}^{p(x)} g(x,z)dzdx.$$

We are now in a position to formally state the 2D *Final Open Pit* problem:

$$\begin{cases} \text{Maximize } \int_a^b \int_{p_0(x)}^{p(x)} g(x,z)dzdx \\ \text{over all } p \in \mathcal{AC}[a,b] \text{ subject to } p(a) = p_0(a), \quad p(b) = p_0(b) \\ \quad p_0(x) \le p(x), \quad \text{for all } x \in [a,b], \\ \quad |p'(x)| \le \kappa(x, p(x)) \quad \text{for a.e. } x \in [a,b] \\ \quad \int_a^b \int_{p_0(x)}^{p(x)} e(x,z)dzdx \le c_{\max}. \end{cases} \tag{P$_{2D}$}$$

2.2 Standing assumptions

Throughout the remainder of this section, unless otherwise stated, we will assume $-\infty < a < b < +\infty$ and $c_{\max} > 0$ are fixed parameters of the problem. The initial profile $p_0 : [a, b] \to \mathbb{R}$ is a given continuously differentiable function.

The profit objective function $g : [a, b]\times\mathbb{R}$ is assumed to be a nonnegative, bounded and piecewise continuous function. The marginal cost of extraction $e : [a, b]\times\mathbb{R} \to \mathbb{R}$ is assumed to be a nonnegative, bounded and continuous function. Also, the maximal slope allowed $\kappa : [a, b] \times \mathbb{R} \to \mathbb{R}$ is assumed to be continuous, nonnegative and bounded with $p \mapsto \kappa(x, p)$ being continuously differentiable for any $x \in [a, b]$ fixed and such that $(x, q) \mapsto \nabla_q\kappa(x, q)$ is bounded on $[a, b] \times \mathbb{R}^n$.

Under these assumptions, the existence of an optimal profile is ensured, as proved by Alvarez et al [1]. Moreover, the fact that an optimal profile is absolutely continuous (Lipschitz continuous actually) is enforced by the boundedness and continuity of the maximal slope κ. This existence result concerns as well the 3D case studied in Sect. 3.

Remark 1 In the light of Alvarez et al. [1, Lemma 1], the feasible set of (P$_{2D}$) without the capacity constraint, is convex provided $z \mapsto \kappa(x, z)$ is concave for any $x \in [a, b]$ fixed. Moreover, by [1, Proposition 5], if $z \mapsto e(x, z)$ is monotonically increasing and $z \mapsto g(x, z)$ is monotonically decreasing (for $x \in [a, b]$ fixed), then the problem (P$_{2D}$) turns out to be a convex one. The previous paragraph can also be applied to the 3D case. However, under the assumptions we have done so far, these hypotheses cannot be assured. As a matter of fact, the problem (P$_{2D}$) may have several local minima, which are not necessarily global. It worths to mention then that in general setting of this manuscript we deal with non-convex problems. The previous comment also applies to the 3D case.

2.3 Basics on state constrained optimal control

Let us point out that the formulation of (P_{2D}0 is slightly more restrictive than what has been treated in Alvarez et al [1]. Essentially, we restrict our analysis to a small class of functions, those that are absolutely continuous. The main advantage of doing so is that now the *Final Open Pit* problem can be treated as an optimal control problem with state constraints, and optimality conditions can be derived by fairly standard methods.

For the sake of completeness, we state the main tool from optimal control theory we are going to use in the analysis provided in this section. Let us consider a general Mayer optimal control problem on $\mathbb{R}^n$:

$$\begin{cases} \text{Minimize} & \varphi(q(b)) \\ \text{over all} & q \in \mathcal{AC}^n[a,b] \text{ and measurable functions } u \\ \text{satisfying} & q'(x) = f(x, q(x), u(x)), \text{ for a.e. } x \in [a,b], \\ & u(x) \in U, \text{ for a.e. } x \in [a,b], \\ & h(x, q(x)) \leq 0, \text{ for any } x \in [a,b], \\ & (q(a), q(b)) \in E. \end{cases} \tag{P_M}$$

Here, $q \in \mathcal{AC}^n[a,b]$ means that $q : [a,b] \to \mathbb{R}^n$ and if $q = (q_1, \ldots, q_n)$, then each component $q_i \in \mathcal{AC}[a,b]$. Furthermore, for the purposes of our analysis, we only need to consider the case in which:

- $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function,
- $f : [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is such that, $x \mapsto f(x, \hat{q}, \hat{u})$ is measurable, $q \mapsto f(\hat{x}, q, \hat{u})$ is Lipschitz continuous (uniformly with respect to $(\hat{x}, \hat{u})$) and $u \mapsto f(\hat{x}, \hat{q}, u)$ is continuous for any $(\hat{x}, \hat{q}, \hat{u}) \in [a,b] \times \mathbb{R}^n \times U$ fixed,
- $h : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ is continuous, with $q \mapsto h(\hat{x}, q)$ being differentiable for any $\hat{x} \in [a,b]$ fixed and such that $(x,q) \mapsto \nabla_q h(x,q)$ is continuous on $[a,b] \times \mathbb{R}^n$,
- $U \subseteq \mathbb{R}^m$ is a given nonempty compact set,
- $E \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is a nonempty closed convex set.

It is worth recalling that the (convex) normal cone to a set $S \subseteq \mathbb{R}^k$ is defined by

$$N_S(s) := \{\eta \in \mathbb{R}^k \mid \langle \eta, \tilde{s} - s \rangle \leq 0, \ \forall \tilde{s} \in S\}, \quad \forall s \in S.$$

In particular, given $s_0 \in \mathbb{R}$, we have

$$N_{(-\infty, s_0]}(s) = \begin{cases} \{0\} & \text{if } s < s_0 \\ [0, +\infty) & \text{if } s = s_0, \end{cases} \quad \text{and} \quad N_{\{s_0\}}(s_0) = \mathbb{R}.$$

Definition 1 An arc $\bar{q} \in \mathcal{AC}^n[a,b]$ admissible for (P_M) is said to be a weak local minimizer of the problem (related to an optimal control $\bar{u}$) if there is $\varepsilon > 0$ such that

$$q \in \mathcal{AC}^n[a,b] \text{ is admissible for } (\mathrm{P_M}) \text{ and } \|q-\bar{q}\|_{W^{1,1}} \\ \leq \varepsilon \quad \Longrightarrow \quad \varphi(\bar{q}(b)) \leq \varphi(q(b)).$$

Here $\|\cdot\|_{W^{1,1}}$ stands for the usual norm of the Sobolev space $W^{1,1}([a,b];\mathbb{R}^n)$.

It turns out that, in this setting, weak local minimizers of $(\mathrm{P_M})$ satisfy Maximum Principle for State Constrained problems (Vinter [20, Theorem 9.3.1]).

Lemma 2 *Under the conditions stated above, if $\bar{q} \in \mathcal{AC}^n[a,b]$ is a weak local minimizer of $(\mathrm{P_M})$ related to the optimal control $\bar{u}$, then there exist $\lambda \in \mathcal{AC}^n[a,b]$, $\eta \in \{0,1\}$, a (positive) Radon measure μ on $[a,b]$, and a Borel measurable function $\gamma : [a,b] \to \mathbb{R}^n$ satisfying*

$$\gamma(x) = \nabla_q h(x,\bar{q}(x)), \quad \text{for } \mu-\text{a.e. } x \in [a,b],$$

such that

1. $(\lambda,\mu,\eta) \neq (0,0,0)$;
2. $-\lambda'(x) \in \partial_q^C H(x,\bar{q}(x),\xi(x),\bar{u}(x))$ *for a.e.* $x \in [a,b]$;
3. $(\lambda(a), -\xi(b)) \in \{0\} \times \{\eta\nabla\varphi(\bar{q}(b))\} + N_E(\bar{q}(a),\bar{q}(b))$;
4. $H(x,\bar{q}(x),\xi(x),\bar{u}(x)) = \max_{u\in U} H(x,\bar{q}(x),\xi(x),u)$ *for a.e.* $x \in [a,b]$;
5. $\mathrm{supp}(\mu) \subseteq \{x \in [a,b] \mid h(x,\bar{q}(x)) = 0\}$.

Here $H(x,q,\xi,u) = \langle \xi, f(x,q,u)\rangle$ *and*

$$\xi(x) = \lambda(x) + \int_{[a,x[} \gamma(s)\mu(ds) \quad \forall x \in [a,b[\quad \text{and} \quad \xi(b) = \lambda(b) + \int_{[a,b]} \gamma(s)\mu(ds)$$

2.4 Optimality conditions for the 2D open pit problem

In this part of the paper, we analyze the behavior of an optimal profile by using the tools from optimal control theory described earlier. The following result can in principle be stated for local optima as well. However, to keep the presentation of the paper simple, we prefer to present it only for a global optimum.

Theorem 1 *Let $\bar{p} \in \mathcal{AC}[a,b]$ be an optimal profile of the problem (P_{2D}). Then there are $\zeta \in \mathcal{AC}[a,b]$, $\eta \in \{0,1\}$, $\bar{\lambda} \leq 0$ and a (positive) Radon measure μ on $[a,b]$, with at least one of them not equal to zero, such that*

$$-\zeta'(x) \in \eta G(x,\bar{p}(x)) + \bar{\lambda} e(x,\bar{p}(x)) + |\mu([a,x[) - \zeta(x)|\partial_p\kappa(x,\bar{p}(x)), \quad \text{a.e. on } [a,b],$$

with $\mathrm{supp}(\mu) \subseteq \{x \in [a,b] \mid p_0(x) = \bar{p}(x)\}$, *and where*

$$G(x,p) = \mathrm{co}\left\{g(x,p^-), g(x,p^+)\right\}, \quad \forall x \in [a,b], \forall p \in \mathbb{R}.$$

Furthermore, we also have

$$\bar{\lambda}\left(\int_a^b \int_{p_0(x)}^{\bar{p}(x)} e(x,z)dzdx - c_{\max}\right) = 0$$

and $(\zeta(x) - \mu([a, x[))(|\bar{p}'(x)| - \kappa(x, \bar{p}(x))) = 0$ *for a.e.* $x \in [a, b]$.

Proof 1 The proof of the result is based on a transformation of the *Final Open pit* problem (P_{2D}) into a Mayer problem such as (P_M), for which $\bar{p}$ provides a weak local minimizer related to the optimal control

$$\bar{u}(x) := \begin{cases} \frac{\bar{p}'(x)}{\kappa(x,\bar{p}(x))} & \text{if } \kappa(x, \bar{p}(x)) \neq 0 \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

We divide the proof into several parts for the sake of exposition.

1. First we show that (P_{2D}) is an instance of the Mayer problem (P_M). The key points here are to interpret the slope condition as a controlled ordinary differential equation and to be able to handle the capacity constraints

$$\int_a^b \int_{p_0(x)}^{p(x)} e(x, z)dzdx \leq c_{\max} \quad (2)$$

as an end-point constraint of an additional state. Let $p \in \mathcal{AC}[a, b]$ be a given profile. On the one hand, note that for any $x \in [a, b]$ such that $\kappa(x, p(x)) \neq 0$, the condition $|p'(x)| \leq \kappa(x, p(x))$ is equivalent to $-1 \leq u(x) := \frac{p'(x)}{\kappa(x,p(x))} \leq 1$. This implies then that the condition $|p'(x)| \leq \kappa(x, p(x))$ is actually equivalent to

$$p'(x) = u(x)\kappa(x, p(x)), \quad \text{with } -1 \leq u(x) \leq 1, \quad \text{for a.e. } x \in [a, b].$$

On the other hand, note that (2) is actually an isoperimetric inequality constraint. To deal with it, we introduce a new auxiliary state. Let $q_1 : [a, b] \to \mathbb{R}$ be given by

$$q_1(t) = \int_a^t \int_{p_0(x)}^{p(x)} e(x, z)dzdx, \quad \forall t \in [a, b].$$

Thus, it is clear that (2) can be written as $q_1(b) \leq c_{\max}$. Furthermore, $q_1(a) = 0$ and the velocity of q_1 is given by the expression

$$q_1'(x) = \int_{p_0(x)}^{p(x)} e(x, z)dz, \quad \text{for a.e. } x \in [a, b].$$

Also, by defining $q_2 : [a, b] \to \mathbb{R}$ via the formula

$$q_2(t) = \int_a^t \int_{p_0(x)}^{p(x)} g(x, z)dzdx, \quad \forall t \in [a, b],$$

it is clear that the total profit is given by $q_2(b)$, and that this new state satisfies

$$q_2'(x) = \int_{p_0(x)}^{p(x)} g(x, z)dz, \quad \text{for a.e. } x \in [a, b], \qquad \text{with } q_2(a) = 0.$$

Therefore, setting $q(x) = (q_1(x), q_2(x), p(x))$ for any $x \in [a, b]$, we see that (P_{2D}) is an instance of the Mayer problem (P_M) with $\varphi(q) = -q_2$, $h(x, q) = p_0(x) - q_3$, $U = [-1, 1]$,

$$f(x, q, u) = \left(\int_{p_0(x)}^{q_3} e(x, z) dz, \int_{p_0(x)}^{q_3} g(x, z) dz, u\kappa(x, q_3) \right)$$

and

$$E = \left\{ (\alpha, \beta) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \alpha_1 = \alpha_2 = 0, \ \alpha_3 = p_0(a), \beta_1 \le c_{\max} \text{ and } \beta_3 = p_0(b) \right\}.$$

2. Now, since $\bar{p}$ is assumed to be an optimal solution of (P_{2D}), it follows that $\bar{p}$ provides a weak local minimizer of (P_M), related to the optimal control defined in (1).
Moreover, the condition under which Lemma 2 has been stated are satisfied by the data provided in the preceding part; the Lipschitz continuity of $q \mapsto f(\hat{x}, q, \hat{u})$ (uniformly with respect to $(\hat{x}, \hat{u}) \in [a, b] \times U$) comes from the fact that e and g are measurable bounded functions and κ is Lipschitz continuous in the second variable, uniformly with respect to the first one. Therefore, we can apply Lemma 2, and so, there exist $\lambda \in \mathcal{AC}^3[a, b]$, $\eta \in \{0, 1\}$, a (positive) Radon measure μ on $[a, b]$, and a Borel measurable function $\gamma : [a, b] \to \mathbb{R}^3$ fulfilling the conditions in Lemma 2. Note first that the Hamiltonian does not depend on q_1 nor on q_2, and also that

$$\nabla_q h(x, q) = (0, 0, -1).$$

Because of point 2 in Lemma 2 and the definition of $x \mapsto \xi(x)$, we can deduce that there are $\bar{\lambda}_1, \bar{\lambda}_2 \in \mathbb{R}$ such that

$$\xi_1(x) = \lambda_1(x) = \bar{\lambda}_1 \quad \text{and} \quad \xi_2(x) = \lambda_2(x) = \bar{\lambda}_2, \qquad \forall x \in [a, b],$$

and

$$\xi_3(x) = \lambda_3(x) - \mu([a, x[), \quad \forall x \in [a, b[, \qquad \text{and} \quad \xi_3(b) = \lambda_3(b) - \mu([a, b]).$$

Note also that $\nabla\varphi(q) = (0, -1, 0)$ and

$$N_E\left(\bar{q}(a), \bar{q}(b)\right) = \mathbb{R}^3 \times \left\{ \beta \in \mathbb{R}^3 \mid \beta_1 \ge 0, \ \beta_1(\bar{q}_1(b) - c_{\max}) = 0 \text{ and } \beta_2 = 0 \right\}.$$

By point 3 in Lemma 2, we have that $\bar{\lambda}_2 = \eta \in \{0, 1\}$ and $\bar{\lambda}_1 \le 0$ with

$$\bar{\lambda}_1 \left(\int_a^b \int_{p_0(x)}^{\bar{p}(x)} e(x, z) dz dx - c_{\max} \right) = 0.$$

By point 4 in Lemma 2 we have, since κ is nonnegative, that

$$\xi_3(x)\bar{u}(x)\kappa(x, \bar{p}(x)) = |\xi_3(x)|\kappa(x, \bar{p}(x)), \quad \text{for a.e. } x \in [a, b].$$

Note that whenever $\kappa(x, \bar{p}(x)) \neq 0$ (a.e. on $[a, b]$) we have that

$$\xi_3(x)\bar{p}'(x) = |\xi_3(x)|\kappa(x, \bar{p}(x)).$$

This implies that whenever $\bar{p}'(x) < \kappa(x, \bar{p}(x))$, then necessarily $\xi_3(x) = 0$, and so

$$\xi_3(x)\left(\bar{p}'(x) - \kappa(x, \bar{p}(x))\right), \quad \text{for a.e. } x \in [a, b].$$

Finally, note that since e is continuous and κ is continuously differentiable in the second variable, for any $x \in [a, b]$ $\xi \in \mathbb{R}^3$ and $u \in [-1, 1]$ fixed, such that $p \mapsto q(x, p)$ is continuous at $p = q_3$, we have

$$\nabla_q H(x, q, \xi, u) = \left(0, 0, \xi_1 e(x, q_3) + \xi_2 g(x, q_3) + \xi_3 u \partial_p \kappa(x, q_3)\right)$$

In particular, by point 2 in Lemma 2, for a.e. $x \in [a, b]$ such that $p \mapsto q(x, p)$ is continuous at $p = \bar{p}(x)$ we have

$$-\lambda_3'(x) = \bar{\lambda}_1 e(x, \bar{p}(x)) + \eta g(x, \bar{p}(x)) + \xi_3(x)\bar{u}(x)\partial_p \kappa(x, \bar{p}(x))$$

because in this case $q \mapsto H(x, q, \xi(x), \bar{u}(x))$ is continuously differentiable at

$$q = \left(\int_a^x \int_{p_0(\tilde{x})}^{p(\tilde{x})} e(\tilde{x}, z)dzd\tilde{x}, \int_a^x \int_{p_0(\tilde{x})}^{p(\tilde{x})} g(\tilde{x}, z)dzd\tilde{x}, \bar{p}(x)\right).$$

Since the functions g is piecewise continuous, for any $x \in [a, b]$, if $p \mapsto q(x, p)$ is not continuous at $p = \bar{p}(x)$ we have that

$$\partial_q^C H(x, q, \xi, u) = \{(0, 0)\} \times \left(\xi_1 e(x, q_3) + \xi_2 G(x, q_3) + \xi_3 u \partial_p \kappa(x, q_3)\right)$$

where

$$G(x, p) = \text{co}\{g(x, p^-), g(x, p^+)\}, \qquad \forall x \in [a, b], \ p \in \mathbb{R}.$$

Also, on the one hand, by Maximum Principle (point 4 in Lemma 2), for a.e. $x \in [a, b]$ such that $\kappa(x, \bar{p}(x)) > 0$ we must have that $\xi_3(x)\bar{u}(x) = |\xi_3(x)|$. On the other hand, if $\kappa(x, \bar{p}(x)) = 0$, we must have that $\partial_p \kappa(x, \bar{p}(x)) = 0$ because $p = \bar{p}(x)$ is a local minimum of $p \mapsto \kappa(x, p)$. Combining these two issues we get

$$\xi_3(x)\bar{u}(x)\partial_p \kappa(x, \bar{p}(x)) = |\xi_3(x)|\partial_p \kappa(x, \bar{p}(x)), \quad \text{for a.e. } x \in [a, b].$$

Therefore, setting $\bar{\lambda} = \bar{\lambda}_1$ and $\zeta = \lambda_3$ the conclusion follows.

We now state a direct consequence of the preceding theorem in the case when the slope condition is not active, and the state constraint is only active at the end-points.

Corollary 1 *Let $\bar{p} \in \mathcal{AC}[a, b]$ be an optimal profile of the problem (P_{2D}). Suppose that*

$$p_0(x) < \bar{p}(x), \quad \forall x \in]a, b[\quad \textit{and} \quad |\bar{p}'(x)| < \kappa(x, \bar{p}(x)), \quad \textit{for a.e. } x \in [a, b]. \quad (3)$$

Then there are $\eta \in \{0, 1\}$ and $\bar{\lambda} \leq 0$, such that

$$0 \in \eta G(x, \bar{p}(x)) + \bar{\lambda} e(x, \bar{p}(x)), \quad \textit{a.e. on } [a, b].$$

with

$$\bar{\lambda} \left(\int_a^b \int_{p_0(x)}^{\bar{p}(x)} e(x, z) dz dx - c_{\max} \right) = 0.$$

In particular, if $p \mapsto g(x, p)$ is continuous for any $x \in [a, b]$ fixed, then the condition reduces to

$$\eta g(x, \bar{p}(x)) + \bar{\lambda} e(x, \bar{p}(x)) = 0, \quad \forall x \in [a, b].$$

Moreover,

1. *if the marginal cost associated with extracting a block at any depth is zero (there is no capacity constraint), that is, $e(x, z) = 0$ for any $x \in [a, b]$ and $z \geq p_0(x)$, then the marginal gain of extracting a block at any depth must be zero on the subsection $[a, b]$, that is,*

$$g(x, \bar{p}(x)) = 0, \quad \forall x \in [a, b].$$

2. *if the marginal cost associated with extracting a block at any depth is positive (there is an effective capacity constraint), that is, $e(x, z) > 0$ for any $x \in [a, b]$ and $z \geq p_0(x)$, then $\eta = 1$ and*

$$g(x, \bar{p}(x)) + \bar{\lambda} e(x, \bar{p}(x)) = 0, \quad \forall x \in [a, b].$$

Proof 2 It is enough to apply directly Theorem 1, and check that $\zeta(x) = \mu([a, x[)$ for a.e. $x \in [a, b]$ and note that $\mu([a, x[) = 0$ for a.e. $x \in [a, b]$ because $\text{supp}(\mu) \subseteq \{a, b\}$. □

3 The 3D open pit problem

We now turn into the more realistic case of an open pit in the 3D space. The profiles in this framework are modeled using a continuous function that depends on the two horizontal space variable (denoted x and y for simplicity). Generically, we denote a profile of an open pit by $p : \Omega \to \mathbb{R}$ where $\Omega \subseteq \mathbb{R}^2$ is the bounded domain in $\mathbb{R}^2$ that represents the open pit and where $p(x, y)$ represents the depth of the profile at the point $(x, y) \in \Omega$.

3.1 Statement of the problem

As done for the 2D case, we assume that the depth of a profile is always positive. The final open pit problem in the 3D case has the same structure as in the 2D case. This means that for a given initial profile $p_0 : \overline{\Omega} \to \mathbb{R}$, the total profit and total extraction associated with an admissible profile $p : \overline{\Omega} \to \mathbb{R}$ are given respectively by

$$\int_\Omega \int_{p_0(x,y)}^{p(x,y)} g(x, y, z)dzdxdy \quad \text{and} \quad \int_\Omega \int_{p_0(x,y)}^{p(x,y)} e(x, y, z)dzdxdy.$$

The maximal slope allowed is also considered to be bounded, and thus profiles are Lipschitz continuous mappings. The associated constraint is then represented via the condition

$$L_p(x, y) := \limsup_{(\bar{x},\bar{y})\to(x,y)\leftarrow(\hat{x},\hat{y})} \frac{|p(\bar{x}, \bar{y}) - p(\hat{x}, \hat{y})|}{\sqrt{|\bar{x} - \hat{x}|^2 + |\bar{y} - \hat{y}|^2}} \leq \kappa(x, y, p(x, y)), \quad \forall (x, y) \in \Omega.$$

Therefore, the *Final Open Pit* problem in the 3D case is the following:

$$\begin{cases} \text{Maximize} & \int_\Omega \int_{p_0(x,y)}^{p(x,y)} g(x, y, z)dzdxdy \\ \text{over all} & p \in \mathrm{Lip}\left(\overline{\Omega}\right) \\ \text{subject to} & p(x, y) = p_0(x, y), \quad \text{for any } (x, y) \in \partial\Omega \\ & p_0(x, y) \leq p(x, y), \quad \text{for any } (x, y) \in \Omega, \\ & L_p(x, y) \leq \kappa(x, y, p(x, y)) \quad \text{for any } (x, y) \in \Omega \\ & \int_\Omega \int_{p_0(x,y)}^{p(x,y)} e(x, y, z)dzdxdy \leq c_{\max}. \end{cases} \tag{P$_{3D}$}$$

Remark 2 A more general model that considers profiles having a time dependance has been studied by Álvarez et al in [1]. The analysis of this problem, called Capacitated Dynamic Open Pit, becomes more difficult and we plan to study it in details elsewhere.

Remark 3 Similarly as for the 2D case, a control setting can be introduced to deal with the 3D case; see the proof of Theorem 1. This is certainly a suitable approach to handle Theorem 2, however, in this setting the control is distributed and also subject to constraints; see the discussion in Sect. 4. Moreover, optimality conditions for problems of this kind are known to be harder to handle and for this reason we take another path to prove Theorem 2 base on classical calculus of variations.

3.2 Standing assumptions

Throughout the remainder of this section, unless otherwise stated, we will assume $\Omega \subseteq \mathbb{R}^2$ is an open bounded domain and $c_{\max} > 0$ is fixed parameter of the problem. The initial profile $p_0 : \overline{\Omega} \to \mathbb{R}$ is a given continuously differentiable function.

The densities of gain and effort are now $g : \Omega \times \mathbb{R}$ and $e : \Omega \times \mathbb{R} \to \mathbb{R}$, respectively. They are assumed to be bounded, measurable and the second one (the densities of effort) nonnegative. Also, the maximal slope allowed $\kappa : \Omega \times \mathbb{R} \to \mathbb{R}$ is assumed to be continuous, nonnegative and bounded with $p \mapsto \kappa(x, p)$ being continuously differentiable for any $x \in \Omega$ fixed and such that $(x, q) \mapsto \partial_p \kappa(x, p)$ is bounded on $\Omega \times \mathbb{R}^n$.

3.3 Optimality conditions

We now present some necessary optimality conditions that extend the one given for the 2D case. The conditions obtained in this case do not require the continuity of the gain function g nor the continuity of the effort e. However, because of the nonholonomic character of the slope constraints, the result we present is only valid for the case when optimal profiles do not saturate this condition (see assumption (4) below). Nonholonomic constraints are hard to handle in calculus of variations of multiple integral and require technical assumptions which may be too strong for the scope of this paper. The main difficulty is that the construction of suitable *variations* is not always ensured; see for instance [10, Chapter 2]. It remains then as an open problem and future work to provide necessary optimality conditions for the general case where nonholonomic restriction may be active.

Theorem 2 *Let $\bar{p} \in \mathrm{Lip}\left(\overline{\Omega}\right)$ be an optimal profile of (P_{3D}). Assume that $\bar{p} \neq p_0$ and let $\Omega_0 \subseteq \Omega$ be the open domain of $\mathbb{R}^2$ given by*

$$\Omega_0 = \left\{(x, y) \in \mathbb{R}^2 \mid p_0(x, y) < \bar{p}(x, y)\right\}.$$

If the slope constraints is not active on Ω_0, that is,

$$\sup_{(x,y)\in\Omega_0} \left\{L_{\bar{p}}(x, y) - \kappa(x, y, \bar{p}(x, y))\right\} < 0, \tag{4}$$

then there is $\bar{\lambda} \leq 0$ such that

$$g(x, y, \bar{p}(x, y)) + \bar{\lambda} e(x, y, \bar{p}(x, y)) = 0, \ \ a.e.\, in\, \Omega_0.$$

Furthermore, $\bar{\lambda}$ satisfies the following properties:

1. *If $\int_\Omega \int_{p_0(x,y)}^{\bar{p}(x,y)} e(x, y, z)dzdxdy < c_{\max}$ then $\bar{\lambda} = 0$.*
2. *$\bar{\lambda}$ can be taken to be any value $\lambda = -\int_\Omega g(x, y, \bar{p}(x, y))\psi(x, y)dxdy$, provided that $\psi \in \mathcal{C}_0^\infty(\Omega)$ is such that $\int_\Omega e(x, y, \bar{p}(x, y))\psi(x, y)dxdy = 1$.*

Proof 3 The proof follows rather standard arguments in calculus of variations, adapted to be able to handle the integral inequality constraint of isoperimetric type. Assume first that $e(x, y, \bar{p}(x, y))$ is not identically zero in Ω_0. Take some $\psi \in \mathcal{C}_0^\infty(\Omega_0)$ such that

$$\int_{\Omega_0} e(x, y, \bar{p}(x, y))\psi(x, y)dxdy = 1.$$

Since $e(x, y, \bar{p}(x, y)) \geq 0$ in $(x, y) \in \Omega_0$ and it is not identically zero, the existence of such function ψ is guaranteed. Now take $\varphi \in \mathcal{C}_0^\infty(\Omega_0)$ arbitrary and define for $s, t \in \mathbb{R}$ the profile $p_{s,t} \in \mathrm{Lip}\left(\overline{\Omega}\right)$ given by $p_{s,t}(x, y) = \bar{p}(x, y) + s\varphi(x, y) + t\psi(x, y)$. Consider the functions $(s, t) \mapsto f(s, t)$ and $(s, t) \mapsto h(s, t)$ defined on $\mathbb{R}^2$ via the formulas

$$f(s, t) := \int_{\Omega_0} \int_{p_0(x,y)}^{p_{s,t}(x,y)} g(x, y, z)dzdxdy \quad \text{and} \quad h(s, t) := \int_{\Omega_0} \int_{p_0(x,y)}^{p_{s,t}(x,y)} e(x, y, z)dzdxdy.$$

By the definition of Ω_0, the continuity of $p \mapsto \kappa(x, y, p)$ and (4), it follows then that there is $\delta > 0$ such that for any $(x, y) \in \Omega_0$ and any $s, t \in (-\delta, \delta)$ we have

$$p_{s,t}(x, y) > p_0(x, y) \quad \text{and} \quad L_{p_{s,t}}(x, y) < \kappa(x, y, p_{s,t}(x, y)).$$

Since $\bar{p}$ is an optimal profile for the final open pit problem (P_{3D}), it follows that $(0, 0)$ is a local maximum of the problem

$$\text{Maximize } f(s, t) \text{ over all} s, t \in \mathbb{R} \text{subject to} h(s, t) \leq c_{\max}.$$

This nonlinear optimization problem satisfies the so-called Mangasarian-Fromovitz condition because

$$\partial_t h(0, 0) = \int_{\Omega_0} e(x, y, \bar{p}(x, y))\psi(x, y)dxdy = 1.$$

Where the first equality is justified by the Dominated Convergence Theorem and the fact that $e \in L^\infty(\Omega \times \mathbb{R})$. Therefore, by the Karush-Kuhn-Tucker theorem, there is $\bar{\lambda} \leq 0$, which in principle depends on ψ and φ, such that

$$\nabla f(0, 0) + \bar{\lambda}\nabla h(0, 0) = 0 \quad \text{and} \quad \bar{\lambda}(h(0, 0) - c_{\max}) = 0.$$

Moreover, similarly as justified above, it is not difficult to see that

$$\partial_s h(0, 0) = \int_{\Omega_0} e(x, y, \bar{p}(x, y))\varphi(x, y)dxdy,$$
$$\partial_s f(0, 0) = \int_{\Omega_0} g(x, y, \bar{p}(x, y))\varphi(x, y)dxdy,$$

$$\partial_t f(0,0) = \int_{\Omega_0} g(x, y, \bar{p}(x, y))\psi(x, y)dxdy.$$

On the one hand, by the condition over the partial derivatives with respect to the t variable we get $\bar{\lambda}$ does not depend on φ because

$$\int_{\Omega_0} g(x, y, \bar{p}(x, y))\psi(x, y)dxdy = \partial_t f(0,0) = -\bar{\lambda}\partial_t h(0,0) = -\bar{\lambda}.$$

On the other hand, by the condition over the partial derivatives with respect to the s variable we get

$$\int_{\Omega_0} \left(g(x, y, \bar{p}(x, y)) + \bar{\lambda} e(x, y, \bar{p}(x, y))\right)\varphi(x, y)dxdy = 0.$$

But, since $\varphi \in \mathcal{C}_0^\infty(\Omega)$ is arbitrary and $(x, y) \mapsto g(x, y, \bar{p}(x, y)) + \bar{\lambda} e(x, y, \bar{p}(x, y))$ belongs in particular to $L^2(\Omega_0)$, by the fundamental lemma of the calculus of variations (cf. [17, Lemma 3.2.3]) the conclusion follows.

Finally, for the case that $e(x, y, \bar{p}(x, y))$ is identically zero in Ω_0 it is enough to define $s \mapsto f(s)$ on $\mathbb{R}$ via the formula

$$f(s) := \int_{\Omega_0} \int_{p_0(x,y)}^{\bar{p}(x,y)+s\varphi(x,y)} g(x, y, z)dzdxdy,$$

with $\varphi \in \mathcal{C}_0^\infty(\Omega_0)$ arbitrary and check that $s = 0$ is a local minimum of f. Then the conclusion follows by using the Fermat rule ($f'(0) = 0$) and fundamental lemma of the calculus of variations. □

Remark 4 Note in particular that Theorem 2 says that if the marginal cost associated with extracting a block at any depth is zero (there is no capacity constraint), that is, $e(x, y, z) = 0$ for any $(x, y) \in \Omega$ and $z \geq p_0(x, y)$, then the marginal gain of extracting a block at any depth must be zero on the subsection Ω_0, that is,

$$g(x, y, \bar{p}(x, y)) = 0, \qquad a.e. \, in \, \Omega_0.$$

4 Future work and final remarks

In this paper we have provided necessary optimality conditions for a profile of an open pit mine to be an optimum for the final open pit problem in the 2D as well as on the 3D. Both settings involve isoperimetric restriction, which are hard to handle in general. Nonholonomic restrictions, such as the slope condition has been treated only for the 2D case. The 3D case remains as an open question that deserves some attention, and which we plan to address in a future work.

Obtaining numerical solutions is still an open problem. For this purpose, a classical direct method or an indirect method using the results of this paper can be implemented.

Preliminary simulations have been done with the help of the INRIA solver for optimal control problems BOCOP [3]. This is an issue that need to be investigated in more details.

Finally, let us mention that, similarly as done for the 2D case, the maximal slope condition in the 3D can actually be subsumed by a control type condition

$$\nabla p(x, y) = \kappa(x, y, p(x, y))\,(\cos(\theta), \sin(\theta))\,, \quad \text{for a.e. } (x, y) \in \Omega,\ \theta \in [0, 2\pi).$$

Thus, a possible way to address the final open pit problem is to study the optimal control problem associated with this constraints. This issue needs to be investigated in details.

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