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ORIGINAL PAPER

The subgradient extragradient method extended to pseudomonotone equilibrium problems and fixed point problems in Hilbert space

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Abstract
In this paper, we first introduce and analyze a new algorithm for solving equilibrium problems involving Lipschitz-type and pseudomonotone bifunctions in real Hilbert space. The algorithm uses a new step size, we prove the iterative sequence generated by the algorithm converge strongly to a common solution of equilibrium problem and a fixed point problem without the knowledge of the Lipschitz-type constants of bifunction. Finally, another similar algorithm is proposed and numerical experiments are reported to illustrate the efficiency of the proposed algorithms.

Keywords Equilibrium problems · Pseudomonotone bifunction · Subgradient extragradient method · Convex set

1 Introduction

In this paper, we consider the equilibrium problems (EP) of find $x^* \in C$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in C, \tag{1}$$

where C is a nonempty closed convex subset in a real Hilbert space H, $f : H \times H \longrightarrow \mathbb{R}$ is a bifunction. The set of solutions of (1) is denoted by $EP(f)$. This problem is also known as the Ky Fan's inequality due to his contribution to this field [1]. It unifies many important mathematical problems, such as optimization problems, com-

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plementary problems, variational inequality problems, Nash equilibrium problems, fixed point problems [2–4]. In recent decades, many methods have been proposed and analyzed for approximating solution of equilibrium problems [5–13]. One of the most common methods is proximal point method [5,6], but the method cannot be adapted to peudomonotone equilibrium problems [7].

Another fundamental method for equilibrium problem is the extragradient-like methods [6,7,9–12]. Some known methods use step sizes which depend on the Lipschitz-type constants of the bifunctions [6,12]. That fact can make some restrictions in applications because the Lipschitz-type constants are often unknown or difficult to estimate.

Recently, iterative methods for finding a common element of the set of solutions of equilibrium problems and the set of fixed points of operators in a real Hilbert space have further developed by many authors [14–16]. Very recently, Dang [8,10,11] proposed algorithms which use a step size sequence for solving strong pseudomonotone equilibrium problems in real Hilbert space.

The main purpose of this paper is to propose a new step size for finding a common element of the set of fixed points of a quasinonexpansive mapping and the set of solutions of equilibrium problems involving pseudomonotone and Lipschitz-type bifunctions.

The paper is organized as follows. In Sect. 2, we present some definitions and preliminaries that will be needed in the paper. In Sect. 3, we propose the new algorithms and analyze their convergence. Finally, some numerical experiments are provided.

2 Preliminaries

In this section, we recall some definitions and preliminaries for further use.

Definition 2.1 A bifunction $f : C \times C \longrightarrow \mathbb{R}$ is said to be as follows:

(i) *monotone* on C if $f(x, y) + f(y, x) \leq 0,\ \forall x, y \in C$.
(ii) *pseudomonotone* on C if $f(x, y) \geq 0 \Longrightarrow f(y, x) \leq 0,\ \forall x, y \in C$.
(iii) *strongly pseudomonotone* on C if there exists a constant $\gamma > 0$ such that $f(x, y) \geq 0 \Longrightarrow f(y, x) \leq -\gamma\|x - y\|^2,\ \forall x, y \in C$.
(iv) Lipschitz-type condition on C, if there exist two positive constants c_1, c_2 such that $f(x, y) + f(y, z) \geq f(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2,\ \forall x, y, z \in C$.

From the above definitions, we see that (i)⇒(ii) and (iii)⇒(ii).

Definition 2.2 A mapping $h : C \longrightarrow \mathbb{R}$ is called *subdifferentiable* at $x \in C$ if there exists a vector $w \in H$ such that $h(y) - h(x) \geq \langle w, y - x\rangle, \forall y \in C$.

Definition 2.3 Let $S : H \longrightarrow H$ is a mapping with $F(S) \neq \emptyset$. Then

(i) S is called *quasi-nonexpansive* if $\|S(x) - y\| \leq \|x - y\|,\ \forall x \in H, y \in F(S)$, where $F(S)$ is denoted the fixed point of S.
(ii) $I - S$ is called *demiclosed at zero* if $\{x_n\} \subset H$, $x_n \rightharpoonup x$ and $\|S(x_n) - x_n\| \to 0$, it follows that $x \in F(S)$.

Here, we assume that the bifunction f satisfies the following conditions:

$(A1)$ f is pseudomonotone on C and $f(x, x) = 0$ for all $x \in C$.
$(A2)$ f satisfies the Lipschitz-type condition on H.
$(A3)$ $f(x, .)$ is convex and subdifferentiable on H for every fixed $x \in H$.
$(A4)$ f is jointly weakly continuous on $H \times C$ in the sense that, if $x \in H$, $y \in C$ and $\{x_n\}$, $\{y_n\}$ converge weakly to x, y, respectively, then $f(x_n, y_n) \to f(x, y)$ as $n \to \infty$.

Remark 2.1 If f satisfies $(A1) - (A4)$, then the set of solutions $EP(f)$ of $EP(1)$ is closed and convex (see, [9,12]). If S is quasi-nonexpansive, then $F(S)$ is a closed convex subset of H [17, Proposition 1].

For a proper, convex and lower semicontinuous function $g : C \to (-\infty, +\infty]$ and $\lambda > 0$, the proximal mapping of g with λ is defined by

$$prox_{\lambda g}(x) = argmin\{\lambda g(y) + \frac{1}{2}\|x - y\|^2 : y \in C\}, \ x \in H. \tag{2}$$

The following lemma is a property of the proximal mapping [10,11,18].

Lemma 2.1 *For all $x \in H$, $y \in C$ and $\lambda > 0$, the following inequality holds:*

$$\lambda\{g(y) - g(prox_{\lambda g}(x))\} \geq \langle x - prox_{\lambda g}(x), y - prox_{\lambda g}(x)\rangle. \tag{3}$$

Remark 2.2 From Lemma 2.1, we note that if $x = prox_{\lambda g}(x)$, then

$$x \in Argmin\{g(y) : y \in C\} := \{x \in C : g(x) = \min_{y \in C} g(y)\}. \tag{4}$$

For a closed and convex $C \subseteq H$, the (metric) projection $P_C : H \longrightarrow C$ is defined, for all $x \in H$ such that $P_C(x) = argmin\{\| y - x \|: y \in C\}$. It is known that P_C has the following property.

Lemma 2.2 *Let C be a nonempty, closed and convex set in H and $x \in H$. Then*

$$\langle P_C(x) - x, y - P_C(x)\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.3 *Let $u,\ v \in H$. Then $\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v\rangle$.*

Lemma 2.4 [19] *Let $\{a_n\}$ be a nonnegative real sequence and $\exists N > 0$, $\forall n \geq N$, such that $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n$, where $\{\alpha_n\} \subset (0, 1)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\{b_n\}$ is a sequence such that $\limsup_{n\to\infty} b_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.5 [20] *Let $\{a_n\}$ be a nonnegative real sequence such that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$, and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, m_k *is the largest number* n *in the set* $\{1, 2, \ldots, k\}$ *such that* $a_n < a_{n+1}$.

The normal cone N_C to C at a point $x \in C$ is defined by

$$N_C(x) = \{w \in H : \langle w, y - x\rangle \le 0,\ \forall y \in C\}.$$

Lemma 2.6 [21, Chapter 7] *Let* C *be a nonempty convex subset of a real Hilbert space* H *and* $g : C \to \mathbb{R}$ *be a convex subdifferentiable and lower semicontinuous function on* C*. Then,* x^* *is a solution to the following convex problem*

$$min\{g(x) : x \in C\}$$

if and only if $0 \in \partial g(x^*) + N_C(x^*)$*, where* $\partial g(.)$ *denotes the subdifferential of* g *and* $N_C(x^*)$ *is the normal cone of* C *at* x^*.

3 Algorithm and its convergence

Inspired by the algorithms in [12–16,22–26] and viscosity scheme [27], we propose the following method.

Algorithm 3.1

(Step 0) Take $\lambda_0 > 0$, $x_0 \in H$, $\mu \in (0, 1)$.
(Step 1) Given the current iterate x_n, compute

$$y_n = argmin\{\lambda_n f(x_n, y) + \frac{1}{2}\|x_n - y\|^2,\ y \in C\} = prox_{\lambda_n f(x_n,.)}(x_n). \tag{5}$$

(Step 2) Choose $w_n \in \partial_2 f(x_n, y_n)$ such that $x_n - \lambda_n w_n - y_n \in N_C(y_n)$, compute

$$z_n = argmin\{\lambda_n f(y_n, y) + \frac{1}{2}\|x_n - y\|^2,\ y \in T_n\} = prox_{\lambda_n f(y_n,.)}(x_n), \tag{6}$$

where $T_n = \{v \in H | \langle x_n - \lambda_n w_n - y_n, v - y_n\rangle \le 0\}$.
(Step 3) Compute $t_n = \alpha_n x_0 + (1 - \alpha_n)z_n$, $x_{n+1} = \beta_n z_n + (1 - \beta_n)St_n$ and

$$\lambda_{n+1} = \begin{cases} min\{\frac{\mu(\|x_n - y_n\|^2 + \|z_n - y_n\|^2)}{2(f(x_n,z_n) - f(x_n,y_n) - f(y_n,z_n))}, \lambda_n\}, & if\ f(x_n, z_n) - f(x_n, y_n) - f(y_n, z_n) > 0, \\ \lambda_n, & otherwise. \end{cases} \tag{7}$$

Set $n := n + 1$ and return to step 1.

Remark 3.1 The domain of the first proximal mapping $y_n = prox_{\lambda_n f(x_n,.)}(x_n)$ is C. The domain of the second proximal mapping $z_n = prox_{\lambda_n f(y_n,.)}(x_n)$ is T_n.

Remark 3.2 We note that w_n exists and $C \subseteq T_n$. Since $y_n = argmin\{\lambda_n f(x_n, y) + \frac{1}{2}\|x_n - y\|^2,\ y \in C\}$, by Lemma 2.6, we obtain $0 \in \partial_2\{\lambda_n f(x_n, y) + \frac{1}{2}\|x_n - y\|^2\}(y_n) + N_C(y_n)$. That is , there exists $w_n \in \partial_2 f(x_n, y_n)$ such that $x_n - \lambda_n w_n - y_n \in N_C(y_n)$. Hence, there exists $w \in N_C(y_n)$ such that $\lambda_n w_n + y_n - x_n + w = 0$. Thus,

$$\langle x_n - y_n, y - y_n\rangle = \langle \lambda_n w_n, y - y_n\rangle + \langle w, y - y_n\rangle \leq \lambda_n \langle w_n, y - y_n\rangle.\ \forall y \in C.$$

That is $\langle x_n - \lambda_n w_n - y_n, y - y_n\rangle \leq 0,\ \forall y \in C$. Hence, $C \subseteq T_n$.

Lemma 3.1 *The sequence $\{\lambda_n\}$ generated by Algorithm 3.1 is a monotonically decreasing sequence with lower bound* $\min\{\frac{\mu}{2\max\{c_1, c_2\}}, \lambda_0\}$.

Proof It is easy to see that $\{\lambda_n\}$ is a monotonically decreasing sequence. Since f satisfies the Lipschitz-type condition with constants c_1 and c_2, in the case of $f(x_n, z_n) - f(x_n, y_n) - f(y_n, z_n) > 0$, we have

$$\frac{\mu(\|x_n - y_n\|^2 + \|z_n - y_n\|^2)}{2(f(x_n, z_n) - f(x_n, y_n) - f(y_n, z_n))} \geq \frac{\mu(\|x_n - y_n\|^2 + \|z_n - y_n\|^2)}{2(c_1\|x_n - y_n\|^2 + c_2\|y_n - z_n\|^2)} \geq \frac{\mu}{2\max\{c_1, c_2\}}. \tag{8}$$

Thus, the sequence $\{\lambda_n\}$ has the lower bound $\min\{\frac{\mu}{2\max\{c_1, c_2\}}, \lambda_0\}$. □

Remark 3.3 It is obvious the limit of $\{\lambda_n\}$ exists and we denote $\lambda = \lim_{n\to\infty} \lambda_n$. Clearly $\lambda > 0$. If $\lambda_0 \leq \frac{\mu}{2\max\{c_1, c_2\}}$, Then $\{\lambda_n\}$ is a constant sequence.

The following lemma plays a crucial role in the proof of the Theorem 3.1.

Lemma 3.2 *Suppose that $S : H \longrightarrow H$ is a quasi-nonexpansive mapping. Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$ be sequences generated by Algorithm 3.1 and $F(S) \cap EP(f) \neq \emptyset$. Then the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$ are bounded.*

Proof Since $z_n = argmin\{\lambda_n f(y_n, y) + \frac{1}{2}\|x_n - y\|^2,\ y \in T_n\}$. By Lemma 2.1, we get

$$\lambda_n(f(y_n, y) - f(y_n, z_n)) \geq \langle x_n - z_n, y - z_n\rangle,\ \forall y \in T_n. \tag{9}$$

Note that $F(S) \cap EP(f) \subseteq EP(f) \subseteq C \subseteq T_n$. Let $u \in F(S) \cap EP(f)$, substituting $y = u$ into the last inequality, we have

$$\lambda_n(f(y_n, u) - f(y_n, z_n)) \geq \langle x_n - z_n, u - z_n\rangle. \tag{10}$$

As $u \in EP(f)$, we obtain $f(u, y_n) \geq 0$. Thus $f(y_n, u) \leq 0$ because of the pseudomonotonicity of f. Hence, from (10) and $\lambda_n > 0$, we obtain

$$-\lambda_n f(y_n, z_n) \geq \langle x_n - z_n, u - z_n\rangle. \tag{11}$$

Note that $w_n \in \partial_2 f(x_n, y_n)$, we get $f(x_n, y) - f(x_n, y_n) \geq \langle w_n, y - y_n\rangle, \forall y \in H$. In particular, substituting $y = z_n$ into the last inequality, we have $f(x_n, z_n) - f(x_n, y_n) \geq \langle w_n, z_n - y_n\rangle$. That is,

$$\lambda_n(f(x_n, z_n) - f(x_n, y_n)) \geq \lambda_n\langle w_n, z_n - y_n\rangle. \tag{12}$$

By the definition of T_n, we have $\langle x_n - \lambda_n w_n - y_n, z_n - y_n \rangle \leq 0$. Then

$$\lambda_n \langle w_n, z_n - y_n \rangle \geq \langle x_n - y_n, z_n - y_n \rangle. \tag{13}$$

Combining (11), (12) and (13), we have

$$\begin{aligned} &2\lambda_n (f(x_n, z_n) - f(x_n, y_n) - f(y_n, z_n)) \\ &\quad \geq 2\langle x_n - y_n, z_n - y_n \rangle + 2\langle x_n - z_n, u - z_n \rangle \\ &\quad = \|x_n - y_n\|^2 + \|y_n - z_n\|^2 - \|x_n - z_n\|^2 \\ &\qquad - \|x_n - u\|^2 + \|x_n - z_n\|^2 + \|z_n - u\|^2. \end{aligned} \tag{14}$$

Thus

$$\begin{aligned} \|z_n - u\|^2 \leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 \\ + 2\lambda_n (f(x_n, z_n) - f(x_n, y_n) - f(y_n, z_n)). \end{aligned} \tag{15}$$

By the definition of λ_n and (15), we obtain

$$\begin{aligned} \|z_n - u\|^2 &\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 \\ &\quad + 2\lambda_n (f(x_n, z_n) - f(x_n, y_n) - f(y_n, z_n)) \\ &= \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 \\ &\quad + 2\frac{\lambda_n}{\lambda_{n+1}} \lambda_{n+1} (f(x_n, z_n) - f(x_n, y_n) - f(y_n, z_n)) \\ &\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 \\ &\quad + \frac{\lambda_n}{\lambda_{n+1}} \mu (\|x_n - y_n\|^2 + \|z_n - y_n\|^2). \end{aligned} \tag{16}$$

Note that the limit

$$\lim_{n\to\infty} \lambda_n \frac{\mu}{\lambda_{n+1}} = \mu, \quad 0 < \mu < 1. \tag{17}$$

That is, $\exists N \geq 0$, such that $\forall n \geq N$, $0 < \lambda_n \frac{\mu}{\lambda_{n+1}} < 1$.

We obtain $\forall n \geq N$, $\|z_n - u\| \leq \|x_n - u\|$.

Thus, $\forall n \geq N$,

$$\begin{aligned} \|x_{n+1} - u\| &= \|\beta_n z_n + (1 - \beta_n) S t_n - u\| \\ &\leq \beta_n \|z_n - u\| + (1 - \beta_n) \|S t_n - u\| \\ &\leq \beta_n \|z_n - u\| + (1 - \beta_n) \|t_n - u\| \\ &\leq \beta_n \|z_n - u\| + (1 - \beta_n) \|\alpha_n x_0 + (1 - \alpha_n) z_n - u\| \\ &\leq \beta_n \|z_n - u\| + (1 - \beta_n)(\alpha_n \|x_0 - u\| + (1 - \alpha_n) \|z_n - u\|) \\ &\leq (\beta_n + (1 - \beta_n)(1 - \alpha_n)) \|z_n - u\| + (1 - \beta_n) \alpha_n \|x_0 - u\| \end{aligned}$$

$$\leq max\{\|x_0 - u\|, \|x_n - u\|\}$$
$$\leq \cdots \leq max\{\|x_0 - u\|, \|x_N - u\|\}. \tag{18}$$

Hence the sequence $\{x_n\}$ is bounded. Clearly, $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$ are bounded. That is the desired result. □

Theorem 3.1 *Let S be a quasi-nonexpansive mapping such that $I - S$ is demi-closed at zero. Assume that $\{\alpha_n\} \subset (0, 1)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ and $\beta_n \in [a, b] \subset (0, 1)$. Moreover, the assumptions $(A1)-(A4)$ and $F(S) \cap EP(f) \neq \emptyset$ hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $x^* = P_{F(S)\cap EP(f)}(x_0)$.*

Proof Let $x^* = P_{F(S)\cap EP(f)}(x_0)$, by Lemma 2.2, we have

$$\langle x_0 - x^*, z - x^*\rangle \leq 0, \ \forall z \in F(S) \cap EP(f). \tag{19}$$

By the proof of Lemma 3.2, we get $\exists N_1 \geq 0$, $\forall n \geq N_1$, $\|z_n - x^*\| \leq \|x_n - x^*\|$. From Lemma 3.2, we have the sequence $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$ are bounded. By lemma 2.3, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\beta_n z_n + (1-\beta_n)St_n - x^*\|^2 \\
&= \beta_n\|z_n - x^*\|^2 + (1-\beta_n)\|St_n - x^*\|^2 - \beta_n(1-\beta_n)\|St_n - z_n\|^2 \\
&\leq \beta_n\|z_n - x^*\|^2 + (1-\beta_n)\|t_n - x^*\|^2 - \beta_n(1-\beta_n)\|St_n - z_n\|^2 \\
&= \beta_n\|z_n - x^*\|^2 + (1-\beta_n)\|\alpha_n x_0 + (1-\alpha_n)z_n - x^*\|^2 \\
&\quad - \beta_n(1-\beta_n)\|St_n - z_n\|^2 \\
&\leq \beta_n\|z_n - x^*\|^2 - \beta_n(1-\beta_n)\|St_n - z_n\|^2 \\
&\quad + (1-\beta_n)(2\alpha_n\langle x_0 - x^*, t_n - x^*\rangle + (1-\alpha_n)\|z_n - x^*\|^2) \\
&= (\beta_n + (1-\beta_n)(1-\alpha_n))\|z_n - x^*\|^2 + 2(1-\beta_n)\alpha_n\langle x_0 - x^*, t_n - x^*\rangle \\
&\quad - \beta_n(1-\beta_n)\|St_n - z_n\|^2.
\end{aligned} \tag{20}$$

Hence, for $\forall n \geq N_1$, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq (\beta_n + (1-\beta_n)(1-\alpha_n))\|x_n - x^*\|^2 + 2(1-\beta_n)\alpha_n\langle x_0 - x^*, t_n - x^*\rangle \\
&\quad - \beta_n(1-\beta_n)\|St_n - z_n\|^2.
\end{aligned} \tag{21}$$

Moreover, by (20) and (16), we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq (\beta_n + (1-\beta_n)(1-\alpha_n))\|z_n - x^*\|^2 + 2(1-\beta_n)\alpha_n\langle x_0 - x^*, t_n - x^*\rangle \\
&\leq \|x_n - x^*\|^2 - (1 - \lambda_n\frac{\mu}{\lambda_{n+1}})(\|y_n - x_n\|^2 \\
&\quad + \|y_n - z_n\|^2) + 2(1-\beta_n)\alpha_n\langle x_0 - x^*, t_n - x^*\rangle.
\end{aligned} \tag{22}$$

Case 1 Suppose that there exists $N_2 \in \mathbb{N}(N_2 \geq N_1)$, such that $\|x_{n+1} - x^*\| \leq \|x_n - x^*\|$, $\forall n \geq N_2$. Then $\lim_{n\to\infty} \|x_n - x^*\|$ exists and we denote $\lim_{n\to\infty} \|x_n - x^*\| = l$. From (22), we get

$$(1 - \lambda_n \frac{\mu}{\lambda_{n+1}})(\|y_n - x_n\|^2 + \|y_n - z_n\|^2) \\ \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2(1 - \beta_n)\alpha_n \langle x_0 - x^*, t_n - x^*\rangle. \quad (23)$$

Combining (23) and $\lim_{n\to\infty}(1-\lambda_n \frac{\mu}{\lambda_{n+1}}) = 1-\mu > 0$, we obtain $\lim_{n\to\infty} \|x_n - y_n\| = 0$, $\lim_{n\to\infty} \|z_n - y_n\| = 0$ and $\lim_{n\to\infty} \|z_n - x_n\| = 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ that converges weakly to some $z_0 \in H$, such that

$$\limsup_{n\to\infty} \langle x_0 - x^*, x_n - x^*\rangle = \lim_{k\to\infty} \langle x_0 - x^*, x_{n_k} - x^*\rangle = \langle x_0 - x^*, z_0 - x^*\rangle. \quad (24)$$

Then $y_{n_k} \rightharpoonup z_0$ and $z_0 \in C$. Since $y_{n_k} = prox_{\lambda_{n_k} f(x_{n_k},.)}(x_{n_k})$, by Lemma 2.1, we have

$$\lambda_{n_k}(f(x_{n_k}, y) - f(x_{n_k}, y_{n_k})) \geq \langle x_{n_k} - y_{n_k}, y - y_{n_k}\rangle, \ \forall y \in C. \quad (25)$$

Passing to the limit in the last inequality as $k \longrightarrow \infty$ and using assumptions (A1), (A4) and $\lim_{k\to\infty} \lambda_{n_k} = \lambda > 0$, we get $f(z_0, y) \geq 0$, $\forall y \in C$. That is $z_0 \in EP(f)$. Next we prove $z_0 \in F(S)$. Indeed, it follow from (21) that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 + \alpha_n(1 - \beta_n)\|x_n - x^*\|^2 \\ &\quad - 2(1 - \beta_n)\alpha_n\langle x_0 - x^*, t_n - x^*\rangle \\ &\leq \liminf_{n\to\infty}(-\beta_n(1 - \beta_n)\|St_n - z_n\|^2) \\ &= -\limsup_{n\to\infty} \beta_n(1 - \beta_n)\|St_n - z_n\|^2 \\ &\leq -a(1 - b)\limsup_{n\to\infty} \|St_n - z_n\|^2. \end{aligned} \quad (26)$$

Hence $\lim_{n\to\infty} \|St_n - z_n\| = 0$. Since $t_n - z_n = \alpha_n(x_0 - z_n)$, we get $\lim_{n\to\infty} \|t_n - z_n\| = 0$. Consequently, $t_{n_k} \rightharpoonup z_0$ and $\lim_{n\to\infty} \|t_n - St_n\| = 0$. Using the demiclosedness of the mapping $I - S$, we have $z_0 \in F(S)$. That is $z_0 \in F(S) \cap EP(f)$. Using (19) and (24), we obtain $\limsup_{n\to\infty} \langle x_0 - x^*, x_n - x^*\rangle = \langle x_0 - x^*, z_0 - x^*\rangle \leq 0$. Hence, we get

$$\limsup_{n\to\infty}\langle x_0-x^*, t_n-x^*\rangle \leq \limsup_{n\to\infty}\langle x_0-x^*, t_n-x_n\rangle + \limsup_{n\to\infty}\langle x_0-x^*, x_n-x^*\rangle \leq 0. \quad (27)$$

By (21), for $\forall n \geq N_2$, we have

$$\|x_{n+1}-x^*\|^2 \leq (\beta_n + (1-\beta_n)(1-\alpha_n))\|x_n - x^*\|^2 + 2(1-\beta_n)\alpha_n\langle x_0 - x^*, t_n - x^*\rangle$$

$$= (1-\alpha_n(1-\beta_n))\|x_n - x^*\|^2 + (1-\beta_n)\alpha_n 2\langle x_0 - x^*, t_n - x^*\rangle. \tag{28}$$

It follows from (27), (28) and Lemma 2.4, we obtain $\lim_{n\to\infty} \|x_n - x^*\|^2 = 0$.

Case 2 There exists a subsequence $\{\|x_{n_j} - x^*\|\}$ of $\{\|x_n - x^*\|\}$ such that $\|x_{n_j} - x^*\| < \|x_{n_j+1} - x^*\|$ for all $j \in \mathbb{N}$. From Lemma 2.5, there exists a nondecreasing sequence m_k of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$0 \le \|x_{m_k} - x^*\| \le \|x_{m_k+1} - x^*\| \;\; and \;\; \|x_k - x^*\| \le \|x_{m_k+1} - x^*\|. \tag{29}$$

By (22), we have

$$\begin{aligned}&(1-\lambda_{m_k}\frac{\mu}{\lambda_{m_k+1}})(\|y_{m_k} - x_{m_k}\|^2 + \|y_{m_k} - z_{m_k}\|^2)\\ &\quad \le \|x_{m_k} - x^*\|^2 - \|x_{m_k+1} - x^*\|^2 + 2(1-\beta_{m_k})\alpha_{m_k}\langle x_0 - x^*, t_{m_k} - x^*\rangle. \end{aligned}\tag{30}$$

Combining (30) and $\lim_{k\to\infty}(1-\lambda_{m_k}\frac{\mu}{\lambda_{m_k+1}}) = 1-\mu > 0$, we obtain $\lim_{k\to\infty}\|x_{m_k} - y_{m_k}\| = 0$ and $\lim_{k\to\infty}\|z_{m_k} - y_{m_k}\| = 0$. Using the same argument as in the proof of Case 1, we have $\limsup_{k\to\infty}\langle x_0 - x^*, t_{m_k} - x^*\rangle \le 0$. Using (29) and the same argument as in the proof of (28), for all $m_k \ge N_1$, we have

$$\begin{aligned}\|x_{m_k+1}-x^*\|^2 &\le (1-\alpha_{m_k}(1-\beta_{m_k}))\|x_{m_k}-x^*\|^2 + (1-\beta_{m_k})\alpha_{m_k}2\langle x_0 - x^*, t_{m_k} - x^*\rangle\\ &\le (1-\alpha_{m_k}(1-\beta_{m_k}))\|x_{m_k+1}-x^*\|^2 + (1-\beta_{m_k})\alpha_{m_k}2\langle x_0 - x^*, t_{m_k} - x^*\rangle.\end{aligned}\tag{31}$$

This implies that

$$\|x_{m_k+1} - x^*\|^2 \le 2\langle x_0 - x^*, t_{m_k} - x^*\rangle, \quad \forall\, m_k \ge N_1. \tag{32}$$

Since $\limsup_{k\to\infty} 2\langle x_0 - x^*, t_{m_k} - x^*\rangle \le 0$, we obtain $\lim_{k\to\infty}\|x_{m_k+1} - x^*\|^2 = 0$ and $\lim_{k\to\infty}\|x_{m_k} - x^*\|^2 = 0$. Since $\|x_k - x^*\| \le \|x_{m_k+1} - x^*\|$, we have $\lim_{k\to\infty}\|x_k - x^*\| = 0$. Therefore $x_k \to x^*$. That is the desired result. □

If $f(x, y) = \langle A(x), y - x\rangle, \forall x, y \in H$, where $A : H \to H$ is a mapping. Then the equilibrium problem become the variational inequality. That is, find $x^* \in C$ such that

$$\langle A(x^*), y - x^*\rangle \ge 0, \quad \forall y \in C.$$

Moreover, we have

$$\begin{aligned}&argmin\{\lambda_n f(x_n, y) + \frac{1}{2}\|x_n - y\|^2, \; y \in C\}\\ &\quad = argmin\{\lambda_n\langle A(x_n), y - x_n\rangle + \frac{1}{2}\|x_n - y\|^2 + \frac{\lambda_n^2}{2}\|Ax_n\|^2, \; y \in C\}\end{aligned}$$

$$= argmin\{\frac{1}{2}\|(x_n - \lambda_n A(x_n)) - y\|^2,\ y \in C\}$$
$$= P_C(x_n - \lambda_n A(x_n))$$

and

$$argmin\{\lambda_n f(y_n, y) + \frac{1}{2}\|x_n - y\|^2,\ y \in T_n\} = P_{T_n}(x_n - \lambda_n A(y_n)).$$

Remark 3.4 Let $f(x, y) = \langle A(x), y - x\rangle, \forall x, y \in H$. If A is Lipschitz-continuous, i.e., there exists $L > 0$ such that

$$\| A(x) - A(y) \| \leq L \| x - y \|,\ \ \forall x, y \in H.$$

Then the condition (A2) holds for f with $c_2 = c_1 = \frac{L}{2}$. If A is monotone and Lipschitz-continuous, the conditions (A1), (A3) and (A4) can be dropped in Theorem 3.1. It is obvious that $f(x, y)$ satisfies $(A1)$ and $(A3)$. Since $f(x, y) = \langle A(x), y - x\rangle$ and (25), we get

$$\lambda_{n_k}(\langle A(x_{n_k}), y - x_{n_k}\rangle - \langle A(x_{n_k}), y_{n_k} - x_{n_k}\rangle) \geq \langle x_{n_k} - y_{n_k}, y - y_{n_k}\rangle,\ \forall y \in C.$$

That is

$$\lambda_{n_k}\langle A(x_{n_k}), y - y_{n_k}\rangle \geq \langle x_{n_k} - y_{n_k}, y - y_{n_k}\rangle,\ \forall y \in C.$$

Using the monotonicity of A, for $\forall y \in C$ we have

$$\begin{aligned} 0 &\leq \langle y_{n_k} - x_{n_k}, y - y_{n_k}\rangle + \lambda_{n_k}\langle A(x_{n_k}), y - y_{n_k}\rangle \\ &= \langle y_{n_k} - x_{n_k}, y - y_{n_k}\rangle + \lambda_{n_k}\langle A(x_{n_k}), y - x_{n_k}\rangle + \lambda_{n_k}\langle A(x_{n_k}), x_{n_k} - y_{n_k}\rangle \\ &\leq \langle y_{n_k} - x_{n_k}, y - y_{n_k}\rangle + \lambda_{n_k}\langle A(y), y - x_{n_k}\rangle + \lambda_{n_k}\langle A(x_{n_k}), x_{n_k} - y_{n_k}\rangle. \end{aligned}$$

Let $k \to \infty$, using the facts $\lim_{k\to\infty} \|y_{n_k} - x_{n_k}\| = 0$, $\{y_{n_k}\}$ is bounded and $\lim_{k\to\infty} \lambda_{n_k} = \lambda > 0$, we obtain $\langle A(y), y - z_0\rangle \geq 0,\ \ \forall y \in C$. Using the similar proof in [22] (Using Minty Lemma), we have $z_0 \in EP(f)$.

Corollary 3.1 *Let S be a quasi-nonexpansive mapping such that $I - S$ is demiclosed at zero and $f(x, y) = \langle A(x), y - x\rangle, \forall x, y \in H$. Let $A : H \to H$ is a monotone and Lipschitz-continuous mapping and $F(S) \cap EP(f) \neq \emptyset$. Assume that $\{\alpha_n\} \subset (0, 1)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ and $\beta_n \in [a, b] \subset (0, 1)$. Then the sequence $\{x_n\}$ generated by*

$$\begin{cases} \lambda_0 > 0, x_0 \in H, \mu \in (0, 1), \\ y_n = P_C(x_n - \lambda_n A(x_n)), \\ T_n = \{x \in H | \langle x_n - \lambda_n A(x_n) - y_n, x - y_n\rangle \leq 0\}, \\ z_n = P_{T_n}(x_n - \lambda_n A(y_n)), \\ t_n = \alpha_n x_0 + (1 - \alpha_n) z_n,\ \ x_{n+1} = \beta_n z_n + (1 - \beta_n) S t_n \end{cases}$$

converges strongly to $x^* = P_{F(S)\cap EP(f)}(x_0)$.

Now we introduce another algorithm, which is Algorithm 3.1 does not consider the fixed point. The proof of convergence is similar to Algorithm 3.1. We omitted the proof. The algorithm is of the form

Algorithm 3.2

(Step 0) Take $\lambda_0 > 0$, $x_0 \in H$, $\mu \in (0, 1)$.
(Step 1) Given the current iterate x_n, compute

$$y_n = argmin\{\lambda_n f(x_n, y) + \frac{1}{2}\|x_n - y\|^2,\ y \in C\} = prox_{\lambda_n f(x_n,.)}(x_n).$$

If $x_n = y_n$, then stop: x_n is a solution. Otherwise, go to Step 2.
(Step 2) Choose $w_n \in \partial_2 f(x_n, y_n)$ such that $x_n - \lambda_n w_n - y_n \in N_C(y_n)$, compute

$$z_n = argmin\{\lambda_n f(y_n, y) + \frac{1}{2}\|x_n - y\|^2,\ y \in T_n\} = prox_{\lambda_n f(y_n,.)}(x_n),$$

where $T_n = \{v \in H | \langle x_n - \lambda_n w_n - y_n, v - y_n\rangle \le 0\}$.
(Step 3) Compute $x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) z_n$ and

$$\lambda_{n+1} = \begin{cases} min\{\frac{\mu(\|x_n - y_n\|^2 + \|z_n - y_n\|^2)}{2(f(x_n,z_n) - f(x_n,y_n) - f(y_n,z_n))}, \lambda_n\}, & if\ f(x_n, z_n) - f(x_n, y_n) - f(y_n, z_n) > 0, \\ \lambda_n, & otherwise. \end{cases}$$

Set $n := n + 1$ and return to step 1.

Remark 3.5 The domain of the first proximal mapping $y_n = prox_{\lambda_n f(x_n,.)}(x_n)$ is C. The domain of the second proximal mapping $z_n = prox_{\lambda_n f(y_n,.)}(x_n)$ is T_n.

Remark 3.6 Under assumptions $(A1) - (A4)$, from Lemma 2.1 and Remark 2.2, we obtain that if Algorithm 3.2 terminates at some iterate, i.e., $x_n = y_n$, then $x_n \in EP(f)$.

Theorem 3.2 *Assume that* $\{\alpha_n\} \subset (0, 1)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$. *Moreover, the assumptions* $(A1)-(A4)$ *and* $EP(f) \neq \emptyset$ *hold. Then the sequence* $\{x_n\}$ *generated by Algorithm 3.2 converges strongly to* $x^* = P_{EP(f)}(x_0)$.

4 Numerical experiments

In this section, we present some numerical experiments. We compare our algorithms with the Dang's algorithm (Algorithm in [12]) and Algorithm 1 in [15]. We take $\alpha_n = \frac{1}{10000(n+2)}$ for all the methods. To terminate the algorithms, we use the condition $\|y_n - x_n\| \le \varepsilon$ for all the algorithms.

Problem 1 We consider the equilibrium problem for the following bifunction $f : H \times H \to \mathbb{R}$ which comes from the Nash-Cournot equilibrium model in [10,12]

$$f(x, y) = \langle Px + Qy + q, y - x\rangle,$$

Table 1 Problem 1

m	ε	Dang's Alg.		Algorithm 3.2		Alg.1 in [15]	
		Iter.	Time	Iter.	Time	Iter.	Time
10	10^{-6}	425	3.73	360	2.81	396	3.33
	10^{-6}	350	2.45	405	3.14	349	3.42
100	10^{-6}	1192	79.72	1184	66.88	1155	151.99
	10^{-6}	1212	80.76	1149	77.41	1211	99.96
200	10^{-6}	1836	214.01	1728	310.04	1778	513.87
	10^{-6}	1816	339.85	1746	333.90	1670	604.50

Table 2 Problem 2

x_0	ε	Algorithm 3.2		Dang's Alg.	
		Iter.	Time	Iter.	Time
$\frac{1}{50}t^2$	10^{-3}	9	14.08	9	11.79
$\frac{1}{100}(1-t^2)$	10^{-3}	8	4.74	8	3.87
$\frac{1}{50}sin(t)$	10^{-3}	10	42.28	10	36.03

where $q \in \mathbb{R}^m$ is chosen randomly with its elements in $[-m, m]$, and the matrices P and Q are two square matrices of order m such that Q is symmetric positive semidefinite and $Q-P$ is negative semidefinite. In this case, the bifunction f satisfies $(A1)-(A4)$ with the Lipschitz-type constants $c_1 = c_2 = \frac{\|P-Q\|}{2}$, see [9, Lemma 6.2]. For Alg.3.2, we take $\lambda_0 = \frac{1}{4c_1}$ and $\mu = 0.9$. For Dang's algorithm and Algorithm 1 in [15], we take $\lambda = \frac{1}{4c_1}$. For Algorithm 1 in [15], we take $S = I$ and $F(x) = x - x_0$.

For numerical experiments: we suppose that the feasible set $C \subset \mathbb{R}^m$ has the form of

$$C = \{x \in \mathbb{R}^m : Ax \le b\},$$

where A is a matrix of the size $k \times m$ ($m = 10, 100, 200$ and $k = 100$) with its entries generated randomly in $[-2, 2]$ and $b \in \mathbb{R}^k$ is a vector with its elements generated randomly in $[1, 3]$. The numerical results are showed in Table 1.

Problem 2 Let $H = L^2([0, 1])$ with norm $\|x\| = (\int_0^1 |x(t)|^2 dt)^{\frac{1}{2}}$ and inner product $\langle x, y\rangle = \int_0^1 x(t)y(t)dt$, $x, y \in H$. The bifunction f is defined by $f(x, y) = \langle Ax, y - x\rangle$ and the operator $A : H \to H$ is defined by $Ax(t) = max(0, x(t))$, $t \in [0, 1]$ for all $x \in H$. It can be easily verified that A is Lipschitz-continuous and monotone. The feasible set is $C = \{x \in H : \int_0^1 (t^2 + 1)x(t)dt \le 1\}$. Observe that $0 \in EP(f)$ and so $EP(f) \ne \emptyset$. We take $\lambda_0 = 0.7$ and $\mu = 0.9$ for Alg.3.2 and $\lambda = 0.7$ for Dang's algorithm. The results are presented in Table 2.

Table 3 Problem 3

x_0	ε	Algorithm 3.1		Alg.1 in [15]	
		Iter.	Time	Iter.	Time
10	10^{-6}	19	0.0007	690	0.0008
	10^{-10}	59	0.0037	6896552	4.01
-20	10^{-6}	17	0.0007	1380	0.0014
	10^{-10}	118	0.0031	13793104	8.82

Problem 3 The third problem was considered in [28], where $f(x, y) = \langle A(x), y - x \rangle$, $\forall x, y \in \mathbb{R}$. The mapping $A : \mathbb{R} \to \mathbb{R}$ is defined by $A(x) = x + sinx$ and $S : \mathbb{R} \to \mathbb{R}$ is defined by $S(x) = \frac{x}{2} sinx$. The feasible set is $C = [-2\pi, 2\pi]$. It is easy to show that A is monotone and Lipschitz-continuous with $L = 2$, S is a quasinonexpansive mapping such that $I - S$ is demiclosed at zero and $0 = F(S) \cap EP(f) \neq \emptyset$, see [28, Example 4.1]. We take $\lambda_0 = 0.4$, $\beta_n = \frac{1}{2}$ and $\mu = 0.9$ for Alg.3.1. For Algorithm 1 in [15], we take $\beta = 0$, $\beta_n = \frac{1}{2}$, $F(x) = x - x_0$ and $\lambda_n = \lambda = 0.4$. The numerical results are showed in Table 3.

From the aforementioned numerical results, we see that the proposed algorithms are effective.

5 Conclusions

In this paper, we consider the convergence results for equilibrium problem involving the Lipschitz-type condition and pseudomonotone bifunctions but the Lipschitz-type constants are unknown. We modify the Halpern subgradient extragradient methods with a new step size. We prove the sequence generated by Algorithm 3.1 converge strongly to a common solution of an equilibrium problem and a fixed point problem. Another algorithm is proposed and some numerical experiments confirm the effectiveness of the proposed algorithms.

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References

1. Fan, K.: A Minimax Inequality and Applications, Inequalities III, pp. 103–113. Academic Press, New York (1972)
2. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123–145 (1994)
3. Facchinei, F., Pang, J.S.: Finite-Dimensional Variational Inequalities and Complementarity Problem. Springer, New York (2003)
4. Konnov, I.V.: Equilibrium Models and Variational Inequalities. Elsevier, Amsterdam (2007)

5. Konnov, I.V.: Application of the proximal point method to nonmonotone equilibrium problems. J. Optim. Theory Appl. **119**, 317–333 (2003)
6. Moudafi, A.: Proximal point algorithm extended to equilibrium problem. J. Nat. Geom. **15**, 91–100 (1999)
7. Dang, V.H.: Convergence analysis of a new algorithm for strongly pseudomontone equilibrium problems. Numer. Algorithms **77**, 983–1001 (2018)
8. Dang, V.H., Duong, V.T.: New extragradient-like algorithms for strongly pseudomonotone variational inequalities. J. Glob. Optim. **70**, 385–399 (2018)
9. Quoc, T.D., Muu, L.D., Hien, N.V.: Extragradient algorithms extended to equilibrium problems. Optimization. **57**, 749–776 (2008)
10. Dang, V.H.: New inertial algorithm for a class of equilibrium problems. Numer. Algorithms **80**, 1413–1436 (2019)
11. Dang, V.H., Yeol, J.C., Yibin, X.: Modified extragradient algorithms for solving equilibrium problems. Optimization **67**, 2003–2029 (2018)
12. Dang, V.H.: Halpern subgradient extragradient method extended to equilibrium problems. RACSAM **111**, 823–840 (2017)
13. Hung, P.G., Muu, L.D.: The Tikhonov regularization extended to equilibrium problems involving pseudomonotone bifunctions. Nonlinear Anal. **74**, 6121–6129 (2011)
14. Anh, P.N.: A hybrid extragradient method extended to fixed point problems and equilibrium problems. Optimization **62**(2), 271–283 (2013)
15. Vuong, P.T., Strodiot, J.J., Nguyen, V.H.: On extragradient-viscosity methods for solving equilibrium and fixed point problems in a Hilbert space. Optimization **64**(2), 429–451 (2015)
16. Anh, P.N.: Strong convergence theorems for nonexpansive mappings and Ky Fan inequalities. J. Optim. Theory Appl. **154**(1), 303–320 (2012)
17. Yamada, I., Ogura, N.: Hybrid steepest descent method for variational inequality operators over the problem certain fixed point set of quasi-nonexpansive mappings. Numer. Funct. Anal. Optim. **25**, 619–655 (2004)
18. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces. Springer, New York (2011)
19. Xu, H.K.: Iterative algorithm for nonlinear operators. J. Lond. Math. Soc. **66**(2), 240–256 (2002)
20. Mainge, P.E.: The viscosity approximation process for quasi-nonexpansive mapping in Hilbert space. Comput. Math. Appl. **59**, 74–79 (2010)
21. Tiel, J.V.: Convex Analysis: An Introductory Text. Wiley, New York (1984)
22. Yang, J., Liu, H.W., Liu, Z.X.: Modified subgradient extragradient algorithms for solving monotone variational inequalities. Optimization **67**(12), 2247–2258 (2018)
23. Yang, J., Liu, H.W.: Strong convergence result for solving monotone variational inequalities in Hilbert space. Numer. Algorithms **80**, 741–756 (2019)
24. Yang, J., Liu, H.W.: A modified projected gradient method for monotone variational inequalities. J. Optim. Theory Appl. **179**(1), 197–211 (2018)
25. Rapeepan, K., Satit, S.: Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert space. J. Optim. Theory Appl. **163**, 399–412 (2014)
26. Censor, Y., Gibali, A., Reich, S.: The subgradient extragradient method for solving variational inequalities in Hilbert space. J. Optim. Theory Appl. **148**, 318–335 (2011)
27. Moudafi, A.: Viscosity methods for fixed points problems. J. Math. Anal. Appl. **241**, 46–55 (2000)
28. Thong, D.V., Hieu, D.V.: New extragradient methods for solving variational inequality problems and fixed point problems. J. Fixed Point Theory Appl. **20**, 129 (2018)