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ORIGINAL PAPER

Quasi ϵ-solutions in a semi-infinite programming problem with locally Lipschitz data

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Abstract

Under the fulfilment of the limiting constraint qualification, a necessary condition for a quasi ϵ-solution to a semi-infinite programming problem (SIP) by means of employing some advanced tools of variational analysis and generalized differential is established. Sufficient conditions for such a quasi ϵ-solution to problem (SIP) are also investigated in light of generalized convex functions defined in terms of the limiting subdifferential of locally Lipschitz functions. Finally, a Wolfe type dual model in approximate form is formulated, and weak, strong and converse-like duality theorems are proposed. Besides, we give some simple examples to illustrate the obtained results.

Keywords Semi-infinite programming · Quasi ϵ-solutions · Limiting constraint qualification · Optimality conditions · Duality

Mathematics Subject Classification 90C34 · 90C46 · 49N15

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1 Introduction

In this paper, we are interested in the study of a semi-infinite programming problem admiring the following form:

$$\min_{x \in C} \; f(x) \;\; \text{subject to} \;\; g_t(x) \leqq 0, t \in T, \tag{SIP}$$

where C that we call the *constraint set* of problem (SIP) is a nonempty closed (not necessarily convex) subset of $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function, and $g_t : \mathbb{R}^n \to \mathbb{R}$, $t \in T$, are locally Lipschitz with respect to x uniformly in t, and T is an index set (possibly infinite). We denote F as the feasible set of problem (SIP), given by

$$F := \{x \in C \colon g_t(x) \leqq 0, t \in T\}. \tag{1}$$

Recently, optimization problems with an infinite number of constraints have been studied in many research papers; see [1,4,10–12,15,16,23,25–27] and the references therein. In particular, some recent contributions to semi-infinite optimization problems are investigated by Goberna and López [11,12].

Besides, it is worth noting that semi-infinite programs with linear and convex inequality constraints have been widely studied and applied, problems with Lipschitzian data are pretty new in the literature. Especially, for the case of *exact* solutions, necessary optimality conditions were derived quite recently; see [20, Chapter 8] and the papers [21,22]. However, the results on optimality conditions for *approximate* solutions to problem (SIP) with Lipschitzian data seem to be developed, since sometimes the exact solutions do not exist while the approximate ones do even in the convex case, for example minimizing $f(x) = \frac{1}{x}$ over $x > 0$. Motivated by this, we will focus on a class of approximate solutions, i.e., quasi ϵ-solutions, to problem (SIP) in the paper. Note that some characterizations of such an approximate solution to robust *convex* optimization problems have been studied by Lee and Jiao [17] (see also [13,14]).

Below, let us recall the concept of a quasi ϵ-solution to problem (SIP), the geometric meaning of such an approximate solution is referred to [8,14,17].

Definition 1.1 Let $\epsilon \geqq 0$ be given. A point $\bar{x} \in F$ is said to be a *quasi ϵ-solution* to problem (SIP) if

$$f(\bar{x}) \leqq f(x) + \sqrt{\epsilon}\|x - \bar{x}\|, \; \forall x \in F.$$

Remark 1.1 (i) We say that $\bar{x} \in F$ is an *ϵ-solution* to problem (SIP) if $f(\bar{x}) \leqq f(x) + \epsilon$, for all $x \in F$. In addition, the concepts of the ϵ-solution and the quasi ϵ-solution are essentially different; see, for example [17].

(ii) The notion of a quasi ϵ-solution was motivated by the well-known Ekeland Variational Principle [9].

(iii) For nonconvex functions, it is crucial to use local concepts as the following one: a point $\bar{x}$ is a quasi ϵ-solution of f if $\bar{x}$ is a local minimum of the function $x \mapsto f(x) + \sqrt{\epsilon}\|x - \bar{x}\|$.

(iv) If $\bar{x}$ is a quasi ϵ-solution to problem (SIP), then there exists a ball $\mathbb{B}(\bar{x}) \subset \mathbb{R}^n$ around $\bar{x}$ with radius equal to $\sqrt{\epsilon}$ such that $f(\bar{x}) \leqq f(x)+\epsilon$ for all $x \in \mathbb{B}(\bar{x}) \cap F$. In this case, we can say that $\bar{x}$ is a *locally* ϵ-solution to problem (SIP).

Example 1.1 This example aims to illustrate Remark 1.1 (iv). Let $f : \mathbb{R} \to \mathbb{R}$, and $f(x) = x^2$, $F = \mathbb{R}$. Moreover, let $\epsilon = \frac{1}{4}$ be given. Then, by definition, the quasi ϵ-solution set is $[-\frac{1}{4}, \frac{1}{4}]$, and we pick $\bar{x} = \frac{1}{4}$, observe that Remark 1.1 (iv) holds, since there exists a ball $\mathbb{B}(\bar{x}) \subset \mathbb{R}$ around $\bar{x} = \frac{1}{4}$ with radius equal to $\sqrt{\epsilon} = \frac{1}{2}$, i.e., $\mathbb{B}(\bar{x}) = \{x \in \mathbb{R}\colon -\frac{1}{4} \leqq x \leqq \frac{3}{4}\}$ such that $f(\bar{x}) \leqq f(x) + \epsilon$ for all $x \in \mathbb{B}(\bar{x}) \cap F = \mathbb{B}(\bar{x})$.

We make the following contributions to the semi-infinite programming (SIP) in the paper.

- We establish a *necessary condition* for a quasi ϵ-solution to problem (SIP) by means of employing some advanced tools of variational analysis and generalized differentiation (due to Mordukhovich [19,20]), under the fulfilment of the limiting constraint qualification.
- We also investigate *sufficient conditions* for such a quasi ϵ-solution to problem (SIP) in light of generalized convex functions defined in terms of the limiting subdifferential of locally Lipschitzian data.
- After the *dual model* in the sense of Wolfe (stated in approximate form) being formulated, we propose the weak, strong and converse-like duality theorems.

The rest of the paper is organized as follows. Section 2 presents some notations and preliminaries. Section 3 establishes necessary and sufficient conditions for a quasi ϵ-solution to problem (SIP). Section 4 studies duality results between the primal problem and its dual one in the sense of Wolfe. Finally, conclusions are given in Sect. 5.

2 Preliminaries

In this section, we recall briefly some standard notation of variational analysis and generalized differentiation widely used in the present paper; see [19,20] for more details. Let $\mathbb{R}^n$ denote the Euclidean space equipped with the usual Euclidean norm $\|\cdot\|$. The notation $\langle\cdot,\cdot\rangle$ signifies the inner product in $\mathbb{R}^n$. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$. Denote by $\mathbb{B}(\bar{x}) := \{x \in \mathbb{R}^n\colon \|x - \bar{x}\| \leqq 1\}$ as a ball in $\mathbb{R}^n$ around $\bar{x}$ with radius equal to 1. As usual, the *polar cone* of a set $\Omega \subset \mathbb{R}^n$ is defined by

$$\Omega^\circ := \{y \in \mathbb{R}^n\colon \langle y, x\rangle \leqq 0, \quad \forall x \in \Omega\}. \tag{2}$$

Let φ be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := [-\infty, +\infty]$. We say $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ is *lower semicontinuous* (l.s.c.) at $\bar{x} \in \mathbb{R}^n$ if $\liminf_{x\to\bar{x}} \varphi(x) \geqq \varphi(\bar{x})$.

Along with single-valued mappings usually denoted by $f\colon \mathbb{R}^n \to \mathbb{R}^m$, we consider set-valued mappings (or multifunctions) $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, with values $F(x) \subset \mathbb{R}^m$ in the collection of all the subsets of $\mathbb{R}^m$. The limiting construction

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{ y \in \mathbb{R}^m : \exists x_k \to \bar{x},\ y_k \to y \text{ with } y_k \in F(x_k) \quad \text{for all } k \in \mathbb{N} \Big\} \tag{3}$$

is known as the *Painlevé–Kuratowski upper/outer limit* of F at $\bar{x}$, where $\mathbb{N} := \{1, 2, \ldots\}$.

Given a set $\Omega \subset \mathbb{R}^n$, associate with it the *distance function*

$$\operatorname{dist}(x; \Omega) := \inf_{z\in\Omega} \|x - z\|, \quad x \in \mathbb{R}^n,$$

and define the *Euclidean projector* of $x \in \mathbb{R}^n$ to Ω by

$$\Pi(x; \Omega) := \{w \in \Omega : \|x - w\| = \operatorname{dist}(x; \Omega)\}.$$

Under the imposed local closedness of Ω around $\bar{x} \in \Omega$, we have $\Pi(x; \Omega) \neq \emptyset$ for all $x \in \mathbb{R}^n$ sufficiently close to this point.

Definition 2.1 [20, Definition 1.1] Let $\Omega \subset \mathbb{R}^n$ with $\bar{x} \in \Omega$. The (*basic*) *normal come* to Ω at $\bar{x}$ is defined by

$$N_\Omega(\bar{x}) := \operatorname*{Lim\,sup}_{x\to\bar{x}} [\operatorname{cone}(x - \Pi(x; \Omega)]$$

via the outer limit (3). Each $v \in N_\Omega(\bar{x})$ is called a *basic* or *limiting normal* to Ω at $\bar{x}$ and is represented as follows: there are sequences $x_k \to \bar{x}$, $w_k \in \Pi(x_k; \Omega)$, and $\alpha_k \geqq 0$ such that $\alpha_k(x_k - w_k) \to v$ as $k \to \infty$.

For an extended real-valued function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ we set

$$\operatorname{epi}\varphi := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : \varphi \leqq r\}.$$

The *limiting/Mordukhovich subdifferential* of φ at $\bar{x} \in \mathbb{R}^n$ with $|\varphi(\bar{x})| < \infty$ is defined by

$$\partial\varphi(\bar{x}) := \{y \in \mathbb{R}^n : (y, -1) \in N_{\operatorname{epi}\varphi}(\bar{x}, \varphi(\bar{x}))\}.$$

If $|\varphi(\bar{x})| = \infty$, one puts $\partial\varphi(\bar{x}) := \emptyset$. It is known [19,20] that when φ is a convex function, the above-defined subdifferential coincides with the subdifferential in the sense of convex analysis [24].

Let δ_Ω be the *indicator function* defined by $\delta_\Omega(x) := 0$ if $x \in \Omega$, and $\delta_\Omega(x) := \infty$ if $x \notin \Omega$. We have a relation between the basic normal cone and the limiting/Mordukhovich subdifferential of the indicator function as follows (see e.g., [20, Proposition 1.19]):

$$\partial\delta_\Omega(\bar{x}) = N_\Omega(\bar{x}), \quad \forall \bar{x} \in \Omega. \tag{4}$$

The nonsmooth version of Fermat's rule (see e.g., [19, Proposition 1.114]), which is an important fact for many applications, can be formulated as follows: If $\bar{x} \in \mathbb{R}^n$ is a local minimizer for $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$, then

$$0 \in \partial\varphi(\bar{x}). \tag{5}$$

The following limiting subdifferential sum rule is needed for our study.

Lemma 2.1 [20, Corollary 2.21] *Let $\varphi_i : \mathbb{R}^n \to \overline{\mathbb{R}},\ i = 1, 2, \ldots, k,\ k \geqq 2$, be lower semicontinuous around $\bar{x} \in \mathbb{R}^n$, and let all these functions except, possibly, one be Lipschitz continuous around $\bar{x}$. Then one has*

$$\partial(\varphi_1 + \varphi_2 + \cdots + \varphi_k)(\bar{x}) \subset \partial\varphi_1(\bar{x}) + \partial\varphi_2(\bar{x}) + \cdots + \partial\varphi_k(\bar{x}). \tag{6}$$

Now, we recall the following linear space that is used for semi-infinite programming; see [10] for details. We denote by $\mathbb{R}_+^{|T|}$ the collection of all the functions $\lambda \colon T \to \mathbb{R}$, which are positive at finitely many points of T and equal to zero at infinitely other points, mathematically say,

$$\mathbb{R}^{|T|} := \{\lambda = (\lambda_t)_{t\in T} : \lambda_t = 0 \text{ for all } t \in T \text{ but only finitely many } \lambda_t \neq 0\}.$$

With $\lambda \in \mathbb{R}^{|T|}$, its supporting set, $T(\lambda) = \{t \in T \colon \lambda_t \neq 0\}$, is a finite subset of T. The nonnegative cone of $\mathbb{R}^{|T|}$ is denoted by:

$$\mathbb{R}_+^{|T|} = \{\lambda = (\lambda_t)_{t\in T} \in \mathbb{R}^{|T|} \colon \lambda_t \geqq 0, t \in T\}.$$

For $g_t, t \in T$,

$$\sum_{t\in T} \lambda_t g_t = \begin{cases} \sum_{t\in T(\lambda)} \lambda_t g_t & \text{if } T(\lambda) \neq \emptyset, \\ 0 & \text{if } T(\lambda) = \emptyset. \end{cases}$$

3 ϵ-Optimality conditions

In this section, we establish the necessary and sufficient optimality conditions for a quasi ϵ-solution to problem (SIP). In connection with the constraint set C of problem (SIP), we use the set of *active constraint multipliers* at $\bar{x} \in C$ defined by

$$A(\bar{x}) := \{\lambda \in \mathbb{R}_+^{|T|} \colon \lambda_t g_t(\bar{x}) = 0 \quad \text{for all } t \in T\}. \tag{7}$$

Below, we recall the concept of the so-called limiting constraint qualification, which can be seen in [2–4,6].

Definition 3.1 Let $\bar{x} \in F$. We say that the following *limiting constraint qualification* (LCQ) is satisfied at $\bar{x}$ iff

$$N_F(\bar{x}) \subseteq \bigcup_{\lambda \in A(\bar{x})} \left[\sum_{t \in T} \lambda_t \partial g_t(\bar{x}) \right] + N_C(\bar{x}). \tag{LCQ}$$

Remark 3.1 It is worth noting that, when considering F defined in (1) with $C := \mathbb{R}^n$, the condition given in (LCQ) is exactly the limiting constraint qualification introduced in [3] if we keep the parameter fixed. Indeed, the paper [3] used the parameterized constraint qualification to evaluate the limiting subdifferential of the optimal value/marginal function of a parametric optimization problem; moreover, the authors also pointed out that the (LCQ) covers almost the existing constraint qualifications of the Mangasarian–Fromovitz and the Farkas–Minkowski types. Furthermore, the reader is referred to [7,18] for some sufficient conditions ensuring the (LCQ) in the case when g_t are convex for all $t \in T$.

Now, we give a Karush–Kuhn–Tucker (KKT) necessary optimality condition for a quasi ϵ-solution to problem (SIP) under the fulfilment of the (LCQ).

Theorem 3.1 (Necessary Optimality Condition) *Let f and g_t, $t \in T$ be locally Lipschitz functions, where T is an arbitrary index set. Let the* (LCQ) *be satisfied at $\bar{x} \in F := \{x \in C \colon g_t(x) \leqq 0, t \in T\}$. If $\bar{x}$ is a quasi ϵ-solution to f over F, then there exist $\lambda \in A(\bar{x})$ defined in* (7) *such that*

$$0 \in \partial f(\bar{x}) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + N_C(\bar{x}) + \sqrt{\epsilon}\mathbb{B}(\bar{x}). \tag{8}$$

Proof Let $\bar{x}$ be a quasi ϵ-solution to f over F, then by definition

$$f(\bar{x}) + \sqrt{\epsilon}\|\bar{x} - \bar{x}\| \leqq f(x) + \sqrt{\epsilon}\|x - \bar{x}\|, \quad \forall x \in F;$$

in other words, $\bar{x}$ is a minimizer of the following problem

$$\min_{x \in F} \left\{ f(x) + \sqrt{\epsilon}\|x - \bar{x}\| \right\}.$$

Equivalently, $\bar{x}$ is an optimal solution of the following unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} \left\{ f(x) + \sqrt{\epsilon}\|x - \bar{x}\| + \delta_F(x) \right\}. \tag{9}$$

Using the nonsmooth version of Fermat's rule (5) to problem (9), we have

$$0 \in \partial\big(f + \sqrt{\epsilon}\| \cdot - \bar{x}\| + \delta_F(\cdot)\big)(\bar{x}). \tag{10}$$

Since functions f and $\|\cdot - \bar{x}\|$ are Lipschitz continuous around $\bar{x}$ and the function $\delta_F(\cdot)$ is l.s.c around this point, it follows from the sum rule (6) applied to (10), from the fact that $\partial\|\cdot - \bar{x}\| = \mathbb{B}(\bar{x})$ at $\bar{x}$, and from the relation in (4) that

$$0 \in \partial f(\bar{x}) + N_F(\bar{x}) + \sqrt{\epsilon}\mathbb{B}(\bar{x}). \tag{11}$$

On the other hand, the (LCQ) being satisfied at $\bar{x} \in F$ yields that

$$N_F(\bar{x}) \subseteq \bigcup_{\lambda \in A(\bar{x})} \left[\sum_{t \in T} \lambda_t \partial g_t(\bar{x})\right] + N_C(\bar{x}),$$

where the set $A(\bar{x})$ was defined in (7). This, along with (11), tells that

$$0 \in \partial f(\bar{x}) + \bigcup_{\lambda \in A(\bar{x})} \left[\sum_{t \in T} \lambda_t \partial g_t(\bar{x})\right] + N_C(\bar{x}) + \sqrt{\epsilon}\mathbb{B}(\bar{x}).$$

Thus, the desired result is obtained. □

The following simple example shows that the fulfilment of (LCQ) at the point in question is essential in Theorem 3.1.

Example 3.1 We consider problem (SIP) with $C = \mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := 2x$, and let $g_t : \mathbb{R} \to \mathbb{R}$ be given by $g_t(x) := tx^2$ for $x \in \mathbb{R}$ and for $t \in T := [1, 2]$. Then the feasible set $F = \{0\}$ and thus, $\bar{x} := 0$ is the optimal solution of f over F, and $\bar{x} = 0$ is also a quasi ϵ-solution of f over F; in addition, $\bar{x} = 0$ is the optimal solution of $f(\cdot) + \sqrt{\epsilon}\|\cdot - \bar{x}\|$ over F. Since $\partial g_t(\bar{x}) = 2t\bar{x} = 0$ at $\bar{x} = 0$ for all $t \in T$,

$$\bigcup_{\lambda \in A(\bar{x})} \left[\sum_{t \in T} \lambda_t \partial g_t(\bar{x})\right] + N_C(\bar{x}) = \{0\}.$$

On the other hand, $N_F(\bar{x}) = \mathbb{R}$. Therefore, the (LCQ) does not hold at $\bar{x}$, and also Theorem 3.1 goes awry.

Before we discuss the sufficient conditions for quasi ϵ-solutions to problem (SIP), we would introduce the concept of generalized convexity, which is motivated by [4].

Definition 3.2 Let $g_T := (g_t)_{t \in T}$. We say that the pair (f, g_T) is *generalized convex* on C at $\bar{x} \in C$ iff, for any $x \in C$, $\xi \in \partial f(\bar{x})$ and $\xi_t \in \partial g_t(\bar{x}), t \in T$, there exists $\omega \in N_C(\bar{x})^\circ$ satisfying

$$\begin{aligned}
&f(x) - f(\bar{x}) \geqq \langle \xi, \omega \rangle, \\
&g_t(x) - g_t(\bar{x}) \geqq \langle \xi_t, \omega \rangle, \ \forall t \in T, \\
&\langle b, \omega \rangle \leqq \|x - \bar{x}\|, \ \forall b \in \mathbb{B}(\bar{x}).
\end{aligned}$$

Remark 3.2 Observe that, if C is convex and f and g_t, $t \in T$ are convex, then (f, g_T) is generalized convex on C at any $\bar{x} \in C$ with $\omega := x - \bar{x}$ for each $x \in C$. Furthermore, [4, Example 3.2] showed that the class of generalized convex functions is properly larger than the one of convex functions.

The next theorem in this section provides a KKT type sufficient optimality condition for a quasi ϵ-solution to problem (SIP) under the satisfaction of the generalized convexity; and the proof is motivated by [5, Theorem 3.13] and [27, Theorem 3.3].

Theorem 3.2 (Sufficient Optimality Condition) *Let $\bar{x} \in F$ satisfy (8). If (f, g_T) is generalized convex on C at $\bar{x}$, then $\bar{x}$ is a quasi ϵ-solution to problem (SIP).*

Proof Since the point $\bar{x} \in F$ satisfies condition (8), there exist $\lambda \in A(\bar{x})$ defined in (7), and $\xi \in \partial f(\bar{x})$, $\xi_t \in \partial g_t(\bar{x})$, $t \in T$, $b \in \mathbb{B}(\bar{x})$ such that

$$-\left(\xi + \sum_{t \in T} \lambda_t \xi_t + \sqrt{\epsilon} b\right) \in N_C(\bar{x}). \tag{12}$$

Assume to the contrary that $\bar{x}$ is not a quasi ϵ-solution to problem (SIP), then there exists an $\hat{x} \in F$ such that

$$f(\hat{x}) + \sqrt{\epsilon}\|\hat{x} - \bar{x}\| < f(\bar{x}). \tag{13}$$

Since (f, g_T) is generalized convex on C at $\bar{x}$, for $\hat{x}$ above, there exists $\omega \in N_C(\bar{x})^\circ$ such that

$$\langle \xi, \omega \rangle \leqq [f(\hat{x}) - f(\bar{x})], \tag{14}$$

$$\sum_{t \in T} \lambda_t \langle \xi_t, \omega \rangle \leqq \sum_{t \in T} \lambda_t [g_t(\hat{x}) - g_t(\bar{x})], \tag{15}$$

$$\langle b, \omega \rangle \leqq \|\hat{x} - \bar{x}\|, \ \forall b \in \mathbb{B}(\bar{x}). \tag{16}$$

By definition of polar cone (2), it follows from (12) and the relation $\omega \in N_C(\bar{x})^\circ$ that

$$0 \leqq \langle \xi, \omega \rangle + \sum_{t \in T} \lambda_t \langle \xi_t, \omega \rangle + \langle \sqrt{\epsilon} b, \omega \rangle. \tag{17}$$

Hence, along with (17), it follows from (14)–(16) that

$$\begin{aligned} 0 &\leqq [f(\hat{x}) - f(\bar{x})] + \sum_{t \in T} \lambda_t [g_t(\hat{x}) - g_t(\bar{x})] + \sqrt{\epsilon}\|\hat{x} - \bar{x}\|. \\ &\leqq [f(\hat{x}) - f(\bar{x})] + \sqrt{\epsilon}\|\hat{x} - \bar{x}\|, \end{aligned} \tag{18}$$

where (18) follows due to the fact $\lambda_t g_t(\bar{x}) = 0$, and $\lambda_t g_t(\hat{x}) \leqq 0$ for all $t \in T$.

Clearly, (18) contradicts to (13). Thus, $\bar{x}$ is a quasi ϵ-solution to problem (SIP). □

We close this section by designing an example, which aims to demonstrate the importance of the generalized convexity assumption imposed in Theorem 3.2. In other words, a feasible point $\bar{x}$ satisfying condition (8) in Theorem 3.1 may not be a quasi ϵ-solution to problem (SIP) if the generalized convexity of (f, g_T) on C at $\bar{x}$ was violated.

Example 3.2 Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$, let $g_t : \mathbb{R} \to \mathbb{R}$ be given by $g_t(x) := tx^2,\ x \in \mathbb{R},\ t \in T := [-2, -1]$, and let $C = \mathbb{R}$. Observe that the feasible set $F = \mathbb{R}$. Take $\bar{x} = 0 \in F$, clearly $\bar{x} = 0$ satisfies condition (8) in Theorem 3.1. However, $\bar{x} = 0$ is not a quasi ϵ-solution to problem (SIP). The reason is that the generalized convexity of (f, g_T) on C at $\bar{x}$ was violated.

4 ϵ-Wolfe type duality

In this section, we address a Wolfe type dual problem (stated in approximate form) to the primal one and establish duality relations between them. For $y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}_+^{|T|}$, put

$$\mathcal{L}(y, \lambda) := f(y) + \sum_{t \in T} \lambda_t g_t(y).$$

In connection with the primal problem (SIP), we consider a dual problem in the sense of Wolfe (in approximate form) as follows:

$$\max\ \mathcal{L}(y, \lambda)\ \text{ subject to }\ (y, \lambda) \in F_W, \tag{D_W}$$

where the feasible set F_W is given by

$$F_W := \left\{ (y, \lambda) \in C \times \mathbb{R}_+^{|T|} : 0 \in \partial f(y) + \sum_{t \in T} \lambda_t \partial g_t(y) + N_C(y) + \sqrt{\epsilon}\mathbb{B}(\bar{x}) \right\}.$$

Similar to the notion of a quasi ϵ-solution to the primal problem (SIP) stated in Definition 1.1, we define such an approximate solution to the dual problem (D_W).

Definition 4.1 Let $\epsilon \geqq 0$ be given. We say $(\bar{y}, \bar{\lambda}) \in F_W$ is a *quasi ϵ-solution* to problem (D_W) if

$$\mathcal{L}(y, \lambda) \leqq \mathcal{L}(\bar{y}, \bar{\lambda}) + \sqrt{\epsilon}\|\bar{y} - y\|,\ \forall (y, \lambda) \in F_D.$$

The following theorem tells a weak duality relation between the primal problem (SIP) and the dual problem (D_W).

Theorem 4.1 (Weak Duality) *For any feasible point x of the primal problem* (SIP) *and any feasible point (y, λ) of the dual problem* (D_W), *if (f, g_T) is generalized convex on C at y, then*

$$f(x) \geqq \mathcal{L}(y, \lambda) - \sqrt{\epsilon}\|x - y\|.$$

Proof Since $(y, \lambda) \in F_W$, there exist $\lambda \in \mathbb{R}_+^{|T|}$, $\xi \in \partial f(y)$, $\xi_t \in \partial g_t(y)$, $t \in T$, and $b \in \mathbb{B}(\bar{x})$ such that

$$-\left(\xi + \sum_{t \in T} \lambda_t \xi_t + \sqrt{\epsilon} b\right) \in N_C(y). \tag{19}$$

Assume to the contrary that

$$f(x) < \mathcal{L}(y, \lambda) - \sqrt{\epsilon}\|x - y\|,$$

i.e.,

$$f(x) - f(y) - \sum_{t \in T} \lambda_t g_t(y) + \sqrt{\epsilon}\|x - y\| < 0. \tag{20}$$

By definition of polar cone (2) and the generalized convexity of (f, g_T) on C at y, it follows from (19) that, for such x, there exists $\omega \in N_C(y)^\circ$ such that

$$\begin{aligned} 0 &\leqq \langle \xi, \omega \rangle + \sum_{t \in T} \lambda_t \langle \xi_t, \omega \rangle + \langle \sqrt{\epsilon} b, \omega \rangle \\ &\leqq [f(x) - f(y)] + \sum_{t \in T} \lambda_t [g_t(x) - g_t(y)] + \sqrt{\epsilon}\|x - y\| \\ &\leqq [f(x) - f(y)] - \sum_{t \in T} \lambda_t g_t(y) + \sqrt{\epsilon}\|x - y\|, \end{aligned} \tag{21}$$

where (21) holds for $x \in C$ implying $\lambda_t g_t(x) \leqq 0$.

Combining (21) and (20) arrives at a contradiction, hence the desired result holds. □

The forthcoming theorem shows a strong duality relation between the primal problem (SIP) and the dual problem (D_W).

Theorem 4.2 (Strong Duality) *Let $\bar{x}$ be a quasi ϵ-solution to the primal problem* (SIP) *such that the* (LCQ) *is satisfied at this point. Then there exists $\bar{\lambda} \in \mathbb{R}_+^{|T|}$ such that $(\bar{x}, \bar{\lambda}) \in F_W$ and $f(\bar{x}) = \mathcal{L}(\bar{x}, \bar{\lambda})$. If in addition, (f, g_T) is generalized convex on C at any $y \in C$, then $(\bar{x}, \bar{\lambda})$ is a quasi ϵ-solution to problem* (D_W).

Proof Thanks to Theorem 3.1, there exist $\bar{\lambda} \in A(\bar{x})$ defined in (7) such that

$$0 \in \partial f(\bar{x}) + \sum_{t \in T} \bar{\lambda}_t \partial g_t(\bar{x}) + N_C(\bar{x}) + \sqrt{\epsilon}\mathbb{B}(\bar{x}).$$

Then $(\bar{x}, \bar{\lambda}) \in F_W$. In addition, since $\bar{\lambda} \in A(\bar{x})$ defined in (7), then $\bar{\lambda}_t g_t(\bar{x}) = 0$ for all $t \in T$. This implies that $\sum_{t \in T} \bar{\lambda}_t g_t(\bar{x}) = 0$, and hence

$$f(\bar{x}) = f(\bar{x}) + \sum_{t \in T} \bar{\lambda}_t g_t(\bar{x}) = \mathcal{L}(\bar{x}, \bar{\lambda}).$$

Furthermore, as (f, g_T) is generalized convex on C at any $y \in C$, it follows from Theorem 4.1 that

$$\mathcal{L}(\bar{x}, \bar{\lambda}) = f(\bar{x}) \geqq \mathcal{L}(y, \lambda) - \sqrt{\epsilon}\|\bar{x} - y\|,$$

for any $(y, \lambda) \in F_W$; in other words, $(\bar{x}, \bar{\lambda})$ is a quasi ϵ-solution to the dual problem (D_W). □

Remark 4.1 Note that the (LCQ) imposed in Theorem 4.2 plays a key role. Namely, if $\bar{x}$ is a quasi ϵ-solution to the primal problem (SIP) at which the (LCQ) is not satisfied, then one may not find out a $\bar{\lambda} \in \mathbb{R}_+^{|T|}$ described in Theorem 4.2 such that $(\bar{x}, \bar{\lambda}) \in F_W$ and $f(\bar{x}) = \mathcal{L}(\bar{x}, \bar{\lambda})$, even in the convex case. One may refer to [5, Example 4.4] for more information. Besides, it is also worth noting that the generalized convexity of (f, g_T) on C stated in Theorem 4.2 cannot be omitted; see [5, Example 4.5] for instance.

We now present the converse-like duality relation for a quasi ϵ-solution between the primal problem (SIP) and the dual problem (D_W).

Theorem 4.3 (Converse-like Duality) *Let $(\bar{x}, \bar{\lambda}) \in F_W$ such that $f(\bar{x}) = \mathcal{L}(\bar{x}, \bar{\lambda})$. If $\bar{x} \in F$ and (f, g_T) is generalized convex on C at $\bar{x}$, then $\bar{x}$ is a quasi ϵ-solution to problem* (SIP).

Proof Since $(\bar{x}, \bar{\lambda}) \in F_W$, there exist $\bar{\lambda} \in \mathbb{R}_+^{|T|}$, $\xi \in \partial f(\bar{x})$, $\xi_t \in \partial g_t(\bar{x})$, $t \in T$, and $b \in \mathbb{B}(\bar{x})$ such that

$$-\left(\xi + \sum_{t \in T} \bar{\lambda}_t \xi_t + \sqrt{\epsilon} b\right) \in N_C(\bar{x}). \tag{22}$$

Assume to the contrary that $\bar{x}$ is not a quasi ϵ-solution to problem (SIP), then there exists an $\hat{x} \in F$ such that

$$f(\hat{x}) + \sqrt{\epsilon}\|\hat{x} - \bar{x}\| < f(\bar{x}). \tag{23}$$

On the other hand, since (f, g_T) is generalized convex on C at $\bar{x}$, for $\hat{x}$ above, there exists $\omega \in N_C(\bar{x})^\circ$ such that

$$\langle \xi, \omega \rangle \leqq [f(\hat{x}) - f(\bar{x})], \tag{24}$$

$$\sum_{t \in T} \bar{\lambda}_t \langle \xi_t, \omega \rangle \leqq \sum_{t \in T} \bar{\lambda}_t [g_t(\hat{x}) - g_t(\bar{x})], \tag{25}$$

$$\langle b, \omega \rangle \leqq \|\hat{x} - \bar{x}\|, \ \forall b \in \mathbb{B}(\bar{x}). \tag{26}$$

By definition of polar cone (2), it follows from (22) and the relation $\omega \in N_C(\bar{x})^\circ$ that

$$0 \leqq \langle \xi, \omega \rangle + \sum_{t \in T} \bar{\lambda}_t \langle \xi_t, \omega \rangle + \sqrt{\epsilon}\langle b, \omega \rangle. \tag{27}$$

Now, along with (27), it follows from (24)–(26) that

$$0 \leqq [f(\hat{x}) - f(\bar{x})] + \sum_{t \in T} \bar{\lambda}_t [g_t(\hat{x}) - g_t(\bar{x})] + \sqrt{\epsilon}\|\hat{x} - \bar{x}\|. \tag{28}$$

In addition, since $f(\bar{x}) = \mathcal{L}(\bar{x}, \bar{\lambda})$, then $\sum_{t \in T} \bar{\lambda}_t g_t(\bar{x}) = 0$, and since $\hat{x} \in F$, then $\sum_{t \in T} \bar{\lambda}_t g_t(\hat{x}) \leqq 0$, it follows from (28) that

$$\begin{aligned} f(\bar{x}) &= f(\bar{x}) + \sum_{t \in T} \bar{\lambda}_t g_t(\bar{x}) \\ &\leqq f(\hat{x}) + \sum_{t \in T} \bar{\lambda}_t g_t(\hat{x}) + \sqrt{\epsilon}\|\hat{x} - \bar{x}\| \\ &\leqq f(\hat{x}) + \sqrt{\epsilon}\|\hat{x} - \bar{x}\|. \end{aligned}$$

This together with (23) gives a contradiction, and the proof is completed. □

Remark 4.2 For $y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}_+^{|T|}$, put $L(y, \lambda) := f(y)$. In connection with the primal problem (SIP), we consider a dual problem in the sense of Mond–Weir (in approximate form) as follows:

$$\max \; L(y, \lambda) \;\; \text{subject to} \;\; (y, \lambda) \in F_{MW}, \tag{D_{MW}}$$

where the feasible set F_{MW} is given by

$$F_{MW} := \Bigg\{ (y, \lambda) \in C \times \mathbb{R}_+^{|T|} : 0 \in \partial f(y) + \sum_{t \in T} \lambda_t \partial g_t(y) + N_C(y) + \sqrt{\epsilon}\mathbb{B}(\bar{x}), \\ \sum_{t \in T} \lambda_t g_t(y) \geqq 0. \Bigg\}$$

Then, similar results to Theorems 4.1, 4.2 and 4.3 in the sense of Mond–Weir type dual problem (D_{MW}) can be verified.

5 Conclusions

In this paper, we studied a necessary optimality condition for a quasi ϵ-solution to a semi-infinite programming problem (SIP), and we proposed it under the fulfilment of limiting constraint qualification, which was different to the ones in [26,27]. Then, a sufficient optimality condition for such a quasi ϵ-solution to problem (SIP) was proposed in light of generalized convex functions defined in terms of the limiting subdifferential of locally Lipschitz functions. Finally, we formulated a Wolfe type dual model in approximate form, and studied weak, strong and converse-like duality theorems. All in all, we investigated some characterizations of a quasi ϵ-solution to a semi-infinite programming problem (SIP).

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