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Two-phase simplex method for linear semidefinite optimization

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Abstract

The linear semidefinite programming problem is considered. To solve it, the variant of the primal simplex method, that generalizes the corresponding method for linear programming problems, is proposed. The passage from one extreme point of the feasible set to another one is described. The main attention is given to pivoting in the case, when the extreme point is irregular, i.e. the "triangular" number of rank of the matrix in the basic point is less than number of equality type constraints in the problem. The approach for finding a starting extreme point is proposed too. The local convergence of the method is proven.

Keywords Linear semidefinite programming · Extreme points · Primal simplex method · Two-phase method · Local convergence

1 Introduction

Linear semidefinite programming problems are very important generalizations of linear programming problems [1–3]. The semidefinite optimization problem, represented in standard form, can be expressed as a problem of minimizing a linear objective function on the cone of positively semidefinite symmetric matrices subject to linear equality constraints [4,5]. Many convex nonlinear problems of mathematical programming, as well as problems of discrete and combinatorial optimization are reduced to such type statements [4,6]. Moreover, semidefinite programming includes second order cone programming as a special case [7].

To date, effective primal-dual methods proposed in [4,5] for solving linear semidefinite programming problems mainly generalize interior point techniques earlier developed for linear programming problems. The simplex-type methods had been

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studied lesser, and there are a number of reasons for this. Among them, one of the main ones is the lack of polyhedral property for the cone of positively semidefinite matrices. As a consequence, the feasible region has infinitely many extreme points. Nevertheless, there are some methods which are extensions of primal simplex method for linear programming.

In [8] a fairly universal generalization of the simplex method for problems with constraints defined in the form of linear matrix inequalities is proposed. Another version of the simplex method for semi-definite programming, using finite approximations of the cone of positively semidefinite matrices, is given in [9]. For problems of conic programming with arbitrary closed convex cones, the generalization of the simplex method was considered in [10]. Pivoting procedure for a class of second-order cone programming problems having one second-order cone, but possibly additional non-negative variables, have been proposed in [11]. Simplex-type algorithms for second-order cone programming via semi-infinite programming reformulation were considered also in [12].

The purpose of this paper is to develop a standard simplex pivoting procedure similar to one used in linear programming. The main attention is given to pivoting in the case, when the number of equality type constraints does not coincide with the number of variables in the problem, equal to the so-called "triangular" number (the number of elements of symmetric matrices on and below the main diagonal). The preliminary variant of the method was proposed earlier in [13]. Here we present also the special procedure for finding an extreme point of the feasible set in the case when only an arbitrary point of this set is known. The approach used at this first stage of the method is borrowed from the affine-scaling method proposed in [14]. In this method it is allowed to move along the faces of the feasible set and in addition to jump from one face to another.

The work consists of five sections. In Sect. 1, we give the formulation of the problem and present the optimality conditions. In Sect. 2, main assertions concerning the characterization of extreme points and their nondegeneracy are formulated. In Sects. 3 and 4, the passage from one extreme point of the feasible set to another one is described. In the end of the Sect. 4, we prove also the local convergence of the method. Finally, in Sect. 5 we consider the approach for finding an extreme point of the feasible set when only an arbitrary admissible point of this set is known.

Here is the notation used throughout this paper. We denote by $D(a)$ the diagonal matrix with a vector a on its diagonal. We use also "," for adjoining vectors or components of a vector in a row and ";" for adjoining them in a column. Thus for instance $a = [a^1; \ldots; a^n]$. The same rule is applied for matrices. The identity matrix of order n is denoted by I_n. The zero n-dimensional vector and the zero $(m \times n)$ matrix are denoted by 0_n and 0_{mn}, respectively.

2 The semidefinite optimization problem and optimality conditions

Let $\mathbb{S}^n$ denote the space of real symmetric matrices of order n, and let $\mathbb{S}^n_+$ denote the cone of positive semidefinite matrices from $\mathbb{S}^n$. To specify that a matrix $M \in \mathbb{S}^n$ is positively semidefinite, we will also use the inequality $M \succeq 0$. The dimension of $\mathbb{S}^n$ is equal to so-called nth "triangular" number $n_\triangle = n(n+1)/2$.

The inner product of two matrices M_1 and M_2 from $\mathbb{S}^n$ is defined as $M_1 \bullet M_2 = \text{tr}\,(M_1 M_2)$. In the case where M_1 and M_2 are positively semidefinite matrices from $\mathbb{S}^n$, the inequality $M_1 \bullet M_2 \geq 0$ takes place. Moreover, $M_1 \bullet M_2 = 0$ if and only if $M_1 M_2 = M_2 M_1 = 0_{nn}$. The cone $\mathbb{S}^n_+$ is self-dual.

Consider the linear semidefinite programming problem in standard form

$$\min C \bullet X, \\ A_i \bullet X = b^i, \quad i = 1, \ldots, m, \quad X \succeq 0. \tag{1}$$

Here the matrices C, X, and A_i belong to the space $\mathbb{S}^n$. We assume that the matrices A_i, $1 \leq i \leq m$, are linearly independent. The dual problem to (1) is

$$\max \langle b, u \rangle, \\ \sum_{i=1}^m u^i A_i + V = C, \quad V \succeq 0, \tag{2}$$

where $b = [b^1; \ldots; b^m]$, $V \in \mathbb{S}^n$, and angular brackets stand for usual Euclidean scalar product in $\mathbb{R}^m$. Below the matrix $V = C - \sum_{i=1}^m u^i A_i$, depending from $u \in \mathbb{R}^m$, is denoted by $V(u)$. We suppose that both problems (1) and (2) have solutions.

Let X_* be an optimal solution of problem (1), and let $[u_*, V_*]$ be an optimal solution of problem (2). There is the relation $V_* = V(u_*)$ between $u_* \in \mathbb{R}^m$ and V_*, and both symmetric positive semidefinite matrices $X_* \, V_*$ commute among themselves. Therefore, there exists the orthogonal matrix Q such that $X_* = QD(\eta_*)Q^T \, V_* = QD(\theta_*)Q^T$, where $D(\eta_*)$ and $D(\theta_*)$ are diagonal matrices with vectors η_* and θ_* at diagonals. These vectors are composed from eigenvalues of matrices X_* and V_* respectively, i.e.

$$\eta_* = \left[\eta_*^1; \ldots; \eta_*^n \right], \quad \theta_* = \left[\theta_*^1; \ldots; \theta_*^n \right].$$

For any pair of eigenvalues η_*^i and θ_*^i, $1 \leq i \leq m$, the following *complementary condition* holds, which means that $\eta_*^i \geq 0$, $\theta_*^i \geq 0$ and $\eta_*^i \theta_*^i = 0$. *The strict complementary condition* means in addition that $\eta_*^i + \theta_*^i > 0$ for all $1 \leq i \leq m$.

Consider the optimality conditions for pair of problems (1) and (2). Since we suppose that solutions of these problems exist, the following system of equalities and inequalities

$$X \bullet V = 0, \\ A_i \bullet X = b^i, \quad 1 \leq i \leq m, \\ V = C - \sum_{i=1}^m u^i A_i, \\ X \succeq 0, \quad V \succeq 0 \tag{3}$$

has the solution.

Let us write the equalities from (3) in different form using the operation of vectorization of matrices. For a square matrix M of order n, denote by $\text{vec}\,M$ the column vector of length n^2, in which columns of M are stacked one after another. If the matrix M is symmetric, then, instead of $\text{vec}\,M$, it is reasonable to use the column vector $\text{svec}M$ of smaller dimension $n_\triangle$. This vector contains the lower parts of columns of

M starting from the diagonal element, and each off-diagonal element is multiplied by $\sqrt{2}$. Then the scalar product $M_1 \bullet M_2$ of two matrices $M_1 \in \mathbb{S}^n$ and $M_2 \in \mathbb{S}^n$ can be written as the usual scalar product in the space $\mathbb{R}^{n_\triangle}$, i.e. $M_1 \bullet M_2 = \langle \operatorname{svec} M_1, \operatorname{svec} M_2 \rangle$. Using the vector representation of matrices, we can rewrite the equalities from (3) in the following form

$$\langle \operatorname{svec} X, \operatorname{svec} V \rangle = 0, \quad \mathcal{A}_{\operatorname{svec}} \operatorname{svec} X = b, \qquad \operatorname{svec} V = \operatorname{svec} C - \mathcal{A}_{\operatorname{svec}}^T u. \quad (4)$$

Here $\mathcal{A}_{\operatorname{svec}}$ is the matrix of size $m \times n_\triangle$ with the rows $\operatorname{svec} A_i$, $1 \leq i \leq m$.

For passing from $\operatorname{vec} M$ to $\operatorname{svec} M$ and vice versa, for passing from $\operatorname{svec} M$ to $\operatorname{vec} M$, we will need special *elimination* and *duplication* matrices (see [15,16]), denoted by $\tilde{\mathcal{L}}_n$ and $\tilde{\mathcal{D}}_n$. They are full rank matrices of size $n_\triangle \times n^2$ and $n^2 \times n_\triangle$, respectively. If M is a symmetric matrix of order n, then $\operatorname{svec} M = \tilde{\mathcal{L}}_n \operatorname{vec} M$ and $\operatorname{vec} M = \tilde{\mathcal{D}}_n \operatorname{svec} M$. Observe that $\tilde{\mathcal{L}}_n$ and $\tilde{\mathcal{D}}_n$ are somewhat different from the elimination and duplication matrices $\mathcal{L}_n$ and $\mathcal{D}_n$ from [15]; specifically, $\tilde{\mathcal{L}}_n = D(\operatorname{svec} E_n)\mathcal{L}_n$ and $\tilde{\mathcal{D}}_n = \mathcal{D}_n D^{-1}(\operatorname{svec} E_n)$. Here E_n is the square matrix of order n with all elements equal to one.

Various ways of solving the system (4), supplemented by requirements that matrices X and V must be positive semidefinite, lead to various numerical methods for solving the problems (1) and (2). Below we will consider one of these ways which can be treated as the generalization of the simplex method for linear programming problems.

3 Extreme points of the feasible set

Let

$$\mathcal{F}_P = \left\{ X \in \mathbb{S}_+^n : A_i \bullet X = b^i, \ 1 \leq i \leq m \right\},$$

be the feasible set of problem (1). In what follows, we will concern with faces and extreme points of $\mathcal{F}_P$. The set $\mathcal{F}_P$ is an intersection of the cone $\mathbb{S}_+^n$ with the affine set $\mathcal{F}_A = \left\{ X \in \mathbb{S}^n : A_i \bullet X = b^i, \ 1 \leq i \leq m \right\}$. Since both these sets are convex, faces of the set $\mathcal{F}_P$ are intersections of faces of $\mathbb{S}_+^n$ and $\mathcal{F}_A$. Faces of the cone $\mathbb{S}_+^n$ are defined by subspaces $\mathcal{L}$ of the space $\mathbb{R}^n$, namely $\mathcal{G}$ is the face of $\mathbb{S}_+^n$ if and only if $\mathcal{G} = \mathcal{G}(\mathcal{L}) = \left\{ M \in \mathbb{S}_+^n : \mathcal{R}(M) \subseteq \mathcal{L} \right\}$. Here $\mathcal{R}(M)$ denote the space of columns of the matrix M. If the dimension of $\mathcal{L}$ is equal to r, then rank $M \leq r$ for all matrices M from $\mathcal{G}$. Furthermore, the dimension of $\mathcal{G}(\mathcal{L})$ is equal to $r_\triangle$. The matrix $M \in \mathcal{G}(\mathcal{L})$ can be represented as $M = Q \Lambda Q^T$, where Q is a full rank matrix of the size $n \times r$, and $\Lambda \in \mathbb{S}_+^r$. For all matrices $M \in \mathcal{G}(\mathcal{L})$ the matrix Q is the same. The dual face $\mathcal{G}^\star(\mathcal{L})$ to the $\mathcal{G}(\mathcal{L})$ is the face which is defined by the orthogonal subspace $\mathcal{L}^\perp$, i.e. $\mathcal{G}^\star(\mathcal{L}) = \mathcal{G}(\mathcal{L}^\perp)$.

The *minimal face* of the cone $\mathbb{S}_+^n$ containing the point X has the following form

$$\mathcal{G}_{\min}(X; \mathbb{S}_+^n) = \{ Y \in \mathbb{S}_+^n : \mathcal{R}(Y) \subseteq \mathcal{R}(X) \}.$$

Thus, if the matrix $X \in \mathbb{S}^n_+$ has rank r, then the face $\mathcal{G}_{\min}(X; \mathbb{S}^n_+)$ is isomorphic to the cone $\mathbb{S}^r_+$, and, therefore, has the dimension $r_\triangle$. The dual face $\mathcal{G}^*_{\min}(X; \mathbb{S}^n_+)$ is isomorphic to the cone $\mathbb{S}^{n-r}_+$ and has the dimension $(n-r)_\triangle$.

Getting over from the cone $\mathbb{S}^n_+$ to the feasible set $\mathcal{F}_P$, we obtain that the minimal face for the point $X \in \mathcal{F}_P$ with respect to the set $\mathcal{F}_P$ is defined now as

$$\mathcal{G}_{\min}(X; \mathcal{F}_P) = \mathcal{G}_{\min}(X; \mathbb{S}^n_+) \bigcap \mathcal{F}_A = \{Y \in \mathcal{F}_P : \mathcal{R}(Y) \subseteq \mathcal{R}(X)\}.$$

Let r be rank of the matrix $X \in \mathcal{F}_P$, and let

$$X = QD(\eta)Q^T, \qquad \eta = [\eta_B; \eta_N], \qquad \eta_B > 0_r, \qquad \eta_N = 0_{n-r}, \qquad (5)$$

be the Cholesky factorization of X. Here Q is an orthogonal matrix of order n. We denote by Q_B the $n \times r$ submatrix of Q formed from the first r columns of Q. We denote also by $A_i^{Q_B} = Q_B^T A_i Q_B, 1 \le i \le m$. The dimension of the face $\mathcal{G}_{\min}(X; \mathcal{F}_P)$ is equal to value

$$\dim \mathcal{G}_{\min}(X; \mathcal{F}_P) = r_\triangle - \text{rank}\left[A_1^{Q_B}, \ldots, A_m^{Q_B}\right]. \qquad (6)$$

The point X is an extreme point of the feasible set $\mathcal{F}_P$ if and only if the dimension of the face $\mathcal{G}_{\min}(X; \mathcal{F}_P)$ is equal to zero. According to (6) the matrix $X \in \mathcal{F}_P$ of the rank r is an extreme point of the set $\mathcal{F}_P$ if and only if

$$\text{rank}\left[A_1^{Q_B}, \ldots, A_m^{Q_B}\right] = r_\triangle. \qquad (7)$$

For linearly independent matrices $A_1, A_2, \ldots, A_m$ this equality can take place only in the case, where $r_\triangle \le m$ [10].

Therefore, the point $X \in \mathcal{F}_P$ may be extreme only when the rank r of X is such that $r_\triangle \le m$. We say that the extreme point $X \in \mathcal{F}_P$ is *regular*, if $r_\triangle = m$. Otherwise, in the case, where $r_\triangle < m$, we call the extreme point X *irregular*. Note that all extreme points of $\mathcal{F}_P$ are irregular, if m is not a "triangular" number.

The tangent space to the cone $\mathbb{S}^n_+$ at the point $X \in \mathcal{F}_P$, for which the factorization (5) holds, has the following form [17]:

$$\mathcal{T}_X = \left\{ Q \begin{bmatrix} G & F \\ F^T & 0 \end{bmatrix} Q^T : G \in \mathbb{S}^r, \ F \in \mathbb{R}^{r \times (n-r)} \right\}.$$

The dimension of $\mathcal{T}_X$ is defined by the rank of the matrix X and is equal to $n_\triangle - (n-r)_\triangle$. Let us give the definition of a nondegenerate point [18].

Definition 1 The point $X \in \mathcal{F}_P$ is called nondegenerate, if $\mathcal{T}_X + \mathcal{N}_A = \mathbb{S}^n$, where $\mathcal{N}_A$ is the subspace in $\mathbb{S}^n$, which is parallel to the affine set $\mathcal{F}_A$.

The dimension of the subspace $\mathcal{N}_A$ is equal to $n_\triangle - m$. Since $\dim \mathbb{S}^n = n_\triangle$, the equality $\mathcal{T}_X + \mathcal{N}_A = \mathbb{S}^n$ takes place if and only if $\dim \mathcal{T}_X + \dim \mathcal{N}_A \ge n_\triangle$. Thus, in

order the extreme point $X \in \mathcal{F}_P$ be nondegenerate, the following inequalities between the dimension of the space $\mathbb{S}^n$, the rank of the matrix X and the number of equalities m must be fulfilled: $r_\triangle \leq m \leq n_\triangle - (n - r)_\triangle$.

Let for a point $X \in \mathcal{F}_P$ the factorization (5) be valid, and let Q_B and Q_N be the sub-matrices of the orthogonal matrix Q, consisting from the first r and subsequent $n - r$ columns of Q, respectively. As it is shown in [18], in order that the point X be nondegenerate, it is necessary and sufficient that the matrices

$$\begin{bmatrix} Q_B^T A_i Q_B & Q_B^T A_i Q_N \\ Q_N^T A_i Q_B & 0_{(n-r)(n-r)} \end{bmatrix}, \quad 1 \leq i \leq m,$$

be linearly independent.

In what follows, we assume that all extreme points of the set $\mathcal{F}_P$ are nondegenerate. Moreover, if the point X is irregular, then the rank r of X satisfies to the condition $r_\triangle > m - r$. In this case we say, that the problem (1) is *quasi-regular*.

4 Pivoting at regular point

Let the starting extreme point $X_0 \in \mathcal{F}_P$ be given, and let the sequence of extreme points $\{X_k\}$ be generated in such a way, that the values of the objective function in problem (1) monotonically decrease from iteration to iteration.

Assume that $X \in \mathcal{F}_P$ is a current extreme point. Assume also that the rank of X is equal to $r < n$, and the Cholesky factorization (5) is valid for X. It is possible to rewrite (5) in the following form

$$X = Q_B D(\eta_B) Q_B^T. \tag{8}$$

Here $\eta_B = [\eta^1; \ldots; \eta^r] > 0_r$, and, as before, Q_B is the submatrix of the orthogonal matrix Q, consisting from the first r columns of Q. For simplicity, we assume in this section that X is a regular extreme point.

If X is not an optimal solution, it is desirable to pass to the new extreme point with the lesser value of the objective function. Let us describe this passage, using the optimality conditions (3). The idea of updating the point X is similar to one using in the simplex method for linear programming. First of all, we define from (3) the dual variable $u \in \mathbb{R}^m$ and calculate the weak dual variable $V = V(u)$.

It follows from properties of the matrix trace that

$$X \bullet V = \mathrm{tr}\left(Q_B D(\eta_B) Q_B^T V \right) = \mathrm{tr}\left(D(\eta_B) V^{Q_B} \right) = D(\eta_B) \bullet V^{Q_B},$$

where $V^{Q_B} = Q_B^T V Q_B$. Therefore, the equality $X \bullet V = 0$ is certainly satisfiable for the matrix V such that $V^{Q_B} = 0_{rr}$.

Denote by $\mathcal{A}_{\mathrm{svec}}^{Q_B}$ the $m \times r_\triangle$ matrix, whose rows are the vectors $\mathrm{svec}(A_i^{Q_B})$, $1 \leq i \leq m$. Then the equality $V^{Q_B} = 0_{rr}$ is reduced to the following system of linear

equations with respect to the vector u:

$$\mathrm{svec}\, V^{Q_B} = \mathrm{svec}\, C^{Q_B} - (\mathcal{A}_{\mathrm{svec}}^{Q_B})^T u = 0_{r_{\triangle}}, \tag{9}$$

where $C^{Q_B} = Q_B^T C Q_B$.

Since by assumption the extreme point X is regular, (9) is a system of m equations with respect to m unknowns. Moreover, according to (7) the square matrix $\mathcal{A}_{\mathrm{svec}}^{Q_B}$ is nonsingular. Solving the system (9), we obtain

$$u = \left(\mathcal{A}_{\mathrm{svec}}^{Q_B}\right)^{-T} \mathrm{svec}\, C^{Q_B}, \tag{10}$$

where notation M^{-T} is used instead of $(M^T)^{-1}$. In the case, when the matrix $V = V(u)$ is positive semidefinite, X with $[u, V]$ are the solutions of problems (1) and (2) respectively.

Assume further that V is not a positive semidefinite matrix. Observe also that the matrix V is similar to the matrix $V^Q = Q^T V Q$, and thus, its eigenvalues coincide with eigenvalues of V^Q. But V^Q is a bordering matrix, since the upper left block of V^Q is zero. Therefore, if off-diagonal blocks of V^Q are non-zero, then among all eigenvalues of V^Q there exist at least one negative [16].

Consider the Cholesky factorization $V = H D(\theta) H^T$ of the matrix V, where H is an orthogonal matrix, and θ is the vector of eigenvalues of V. Denote by h_j the jth column of the matrix H (the eigenvector of V). Then V can be written also in the form

$$V = \sum_{j=1}^{n} \theta^j h_j h_j^T.$$

Let θ^k be a negative eigenvalue, and let h_k be the corresponding eigenvector. Then $V^{h_k} = h_k^T V h_k = \theta^k < 0$ or, in somewhat other representation,

$$V^{h_k} = C^{h_k} - \langle u, \mathcal{A}^{h_k} \rangle = \theta^k < 0. \tag{11}$$

Here we introduce the value $C^{h_k} = h_k^T C h_k$ and the m-dimensional vector $\mathcal{A}^{h_k}$ with components $h_k^T A_i h_k$, $1 \leq i \leq m$.

Proposition 1 *The vector h_k does not belong to the subspace $\mathcal{R}(Q_B)$ generated by the columns of the matrix Q_B.*

Proof Indeed, if we assume that $h_k = Q_B z$ for some nonzero vector $z \in \mathbb{R}^r$, then we would have $V h_k = V Q_B z = \theta^k Q_B z$. Multiplying this equality on the left by the matrix Q_B^T, we get $V^{Q_B} z = \theta^k z$, which is impossible, because V^{Q_B} is the zero matrix. The assertion is proven. □

Using the rank one matrix $h_k h_k^T$, we pass to the new point $\bar{X}$ by setting

$$\bar{X} = X + \alpha \Delta X, \quad \Delta X = Q_B \Delta Z Q_B^T + h_k h_k^T, \tag{12}$$

where $\Delta Z \in \mathbb{S}^r$, and $\alpha > 0$.

The matrix ΔZ we select in such a way that

$$A_i \bullet Q_B \Delta Z Q_B^T + A_i \bullet h_k h_k^T = 0, \quad 1 \le i \le m. \tag{13}$$

Then the updated point $\bar{X}$ satisfies all equality constraints in problem (1).

Since matrices $M_1 M_2$ and $M_2 M_1$ have the same trace, the equalities (13) can be rewritten as

$$\left(A_i^{Q_B} \right) \bullet \Delta Z + h_k^T A_i h_k = 0, \quad 1 \le i \le m. \tag{14}$$

If we replace matrices by their vector representations, then this system takes the form

$$\mathcal{A}_{\text{svec}}^{Q_B} \operatorname{svec}\Delta Z + \mathcal{A}^{h_k} = 0_m. \tag{15}$$

Solving the system (15), we obtain

$$\operatorname{svec}\Delta Z = - \left(\mathcal{A}_{\text{svec}}^{Q_B} \right)^{-1} \mathcal{A}^{h_k}. \tag{16}$$

Let us compute $C \bullet \Delta X$. According to (12) the equality

$$C \bullet \Delta X = C^{Q_B} \bullet \Delta Z + C^{h_k}$$

holds. Then, taking into account (11), we get after substitution of the vector $\operatorname{svec}\Delta Z$ from (16)

$$\begin{aligned}
C \bullet \Delta X &= \left\langle \operatorname{svec} C^{Q_B}, \operatorname{svec}\Delta Z \right\rangle + C^{h_{j_k}} = - \left\langle \operatorname{svec} C^{Q_B}, \left(\mathcal{A}_{\text{svec}}^{Q_B} \right)^{-1} \mathcal{A}^{h_k} \right\rangle + C^{h_k} \\
&= - \left\langle \left(\mathcal{A}_{\text{svec}}^{Q_B} \right)^{-T} \operatorname{svec} C^{Q_B}, \mathcal{A}^{h_k} \right\rangle + C^{h_k} = C^{h_k} - \langle u, \mathcal{A}^{h_k} \rangle = \theta^k < 0.
\end{aligned} \tag{17}$$

Hence, the objective function decreases along the direction ΔX. Therefore, in this case the matrix $\mathcal{A}_{\text{svec}}^{Q_B}$ plays the role of basic matrix. Moreover, the matrix Q_B and the vector η_B can be treated as the basic pair of variables, i.e. the basic set consisting from the eigenvectors and eigenvalues.

For passage to the new extreme point (the updated basic pair) we must determine the step-size $\alpha > 0$. Denote $\bar{X}(\alpha) = X + \alpha \Delta X$. Then the following formula

$$\bar{X}(\alpha) = X + \alpha \Delta X = Q_B \left[D(\eta_B) + \alpha \Delta Z \right] Q_B^T + \alpha h_k h_k^T$$

is valid. Since the rank one matrix $h_k h_k^T$ is positive semidefinite, the matrix $\bar{X}(\alpha)$ remains positive semidefinite for α sufficiently small. Therefore, the maximal possible step-size $\bar{\alpha}$ can be find from the condition that the negative eigenvalue appears at first time among all eigenvalues of $M(\alpha) = D(\eta_B) + \alpha \Delta Z$.

Let P be a nonsingular matrix of order r with the help of which both symmetric matrices $D(\eta_B)$ and ΔZ are reduced to the diagonal form, that is $D(\eta_B) = P P^T$,

$\Delta Z = PD(\lambda)P^T$. Therefore, $M(\alpha) = P[D(\bar{e}) + \alpha D(\lambda)]P^T$, where $\bar{e}$ is the r-dimensional vector with all components equal to one. It can be seen from here that the choice of $\bar{\alpha}$ depends on the negative component of the vector λ with maximal absolute value. Presuppose that it is the component λ_*. Then $\bar{\alpha} = -\lambda_*^{-1}$. If $\lambda \geq 0_r$, the problem (1) does not have solution, since $\bar{X}(\alpha) \in \mathcal{F}_P$ for any $\alpha > 0$. Furthermore, due to (17) $C \bullet \bar{X}(\alpha) \to -\infty$, when $\alpha \to +\infty$.

Let the face $\Gamma_{\min}^*(X_k; \mathbb{S}_+^n)$ be dual to the minimal face $\Gamma_{\min}(X_k; \mathbb{S}_+^n)$. If the eigenvector h_k of the matrix $V(u)$ belongs to $\Gamma_{\min}^*(X_k; \mathbb{S}_+^n)$, then the equality $Q_B^T h_k = 0_r$ holds. The vector h_k is in fact a column of the matrix Q_N. Hence, in this case we exclude some vector from the basic set of eigenvectors (the columns of the matrix Q_B) and introduce instead of it the new eigenvector of X, which is a column of Q_N.

5 Pivoting at irregular point

Assume now that the extreme point $X \in \mathcal{F}_P$ is irregular, that is $r_\triangle < m$, and assume also for definiteness $m = r_\triangle + p$, where $p < r$. In this case the system of equations (9) is undetermined and, thus, can have the whole set of solutions.

Let us take the normal solution with minimal norm among all possible solutions

$$u = \left(\mathcal{A}_{\text{svec}}^{Q_B}\right) \left[\left(\mathcal{A}_{\text{svec}}^{Q_B}\right)^T \left(\mathcal{A}_{\text{svec}}^{Q_B}\right)\right]^{-1} \text{svec} C^{Q_B}. \tag{18}$$

Here by (7) the matrix $\mathcal{A}_{\text{svec}}^{Q_B}$ has full rank equal to $r_\triangle$.

Again, we compute the weak dual variable $V = V(u)$. Let θ^k be the negative eigenvalue of the matrix V, and let h_k be the corresponding eigenvector. It is a column of the orthogonal matrix H. As before, the vector h_k does not belong to the subspace $\mathcal{R}(Q_B)$.

Now formula (12) for search direction ΔX is non-applicable, since system (15) for finding the vector $\text{svec} Z$ is overdetermined. This system has a solution only if the vector $\mathcal{A}^{h_k}$ belongs to the null-space of the matrix $(\mathcal{A}_{\text{svec}}^{Q_B})^T$.

In order to eliminate this drawback, we alter the approach for choosing the matrix ΔX. Namely, we define now ΔX in the form

$$\Delta X = [Q_B \, h_k] \begin{bmatrix} \Delta Z & w \\ w^T & 1 \end{bmatrix} [Q_B \, h_k]^T, \tag{19}$$

where $\Delta Z \in \mathbb{S}^r$ and $w \in \mathbb{R}^r$. Observe that this direction ΔX coincides with the direction (12), when $w = 0_r$.

Let us take

$$w = \frac{1}{2}\tilde{W}y, \quad \tilde{W} = [\tilde{w}_1, \ldots, \tilde{w}_p], \tag{20}$$

where all columns $\tilde{w}_j \in \mathbb{R}^r$, $1 \leq j \leq p$, of the matrix $\tilde{W}$ are linearly independent, and $y \in \mathbb{R}^p$. Furthermore, we require that the vectors $q_{w_j} = Q_B \tilde{w}_j$, $1 \leq j \leq p$, be orthogonal to the vector h_k, i.e.

$$\langle h_k, q_{w_j} \rangle = \left\langle Q_B^T h_k, \tilde{w}_j \right\rangle = 0, \quad 1 \le j \le p. \tag{21}$$

All vectors q_{w_j}, $1 \le j \le p$, belong to the subspace $\mathcal{R}(Q_B)$. The vector $q_y = Q_B \tilde{W} y$ also belongs to this subspace, and according to (21) $h_k^T q_y = 0$.

Now we have instead of (14)

$$A_i^{Q_B} \bullet \Delta Z + 2 \left\langle Q_B^T A_i h_k, w \right\rangle + h_k^T A_i h_k = 0, \quad 1 \le i \le m,$$

or after substitution of the vector w from (20)

$$A_i^{Q_B} \bullet \Delta Z + \left\langle Q_B^T A_i h_k, \tilde{W} y \right\rangle + h_k^T A_i h_k = 0, \quad 1 \le i \le m. \tag{22}$$

The second term in (22) can be rewritten also as $\left\langle Q_B^T A_i h_k, \tilde{W} y \right\rangle = \langle h_k, A_i Q_B \tilde{W} y \rangle$.

Let $\mathcal{B}$ be a matrix of the size $m \times p$ with the (i, j) element equal to $h_k^T A_i q_{w_j}$, $1 \le i \le m$, $1 \le j \le p$, that is

$$\mathcal{B} = \begin{bmatrix} h_k^T A_1 Q_B \tilde{w}_1 & \dots & h_k^T A_1 Q_B \tilde{w}_p \\ & \vdots & \\ h_k^T A_m Q_B \tilde{w}_1 & \dots & h_k^T A_m Q_B \tilde{w}_p \end{bmatrix}.$$

Note that $\mathcal{B} y = \left[A_1^{h_k q_y}; \dots; A_m^{h_k q_y} \right]$, where $A_i^{h_k q_y} = h_k^T A_i q_y$, $1 \le i \le m$.

Putting together all equations (22), we obtain

$$\mathcal{A}_{\text{svec}}^{Q_B} \text{svec} \Delta Z + \mathcal{B} y + \mathcal{A}^{h_k} = 0_m. \tag{23}$$

The system (23) is a system of m equations with respect to m unknowns, namely, components of the vectors $\text{svec} \Delta Z \in \mathbb{R}^{r_\triangle}$ and $y \in \mathbb{R}^p$.

We take the $(p \times m)$ matrix $\tilde{U}$, whose rows are linearly independent vectors $\tilde{u}_1, \dots, \tilde{u}_p$ from the null-space of the matrix $(\mathcal{A}_{\text{svec}}^{Q_B})^T$. Further, we compose from $(\mathcal{A}_{\text{svec}}^{Q_B})^T$ and $\tilde{U}$ the square matrix $\Omega = \left[(\mathcal{A}_{\text{svec}}^{Q_B})^T; \tilde{U} \right]$ of order m. The matrix Ω is non-singular, and their rows generate the space $\mathbb{R}^m$.

Let Φ denote the square non-singular matrix $\Phi = (\mathcal{A}_{\text{svec}}^{Q_B})^T \mathcal{A}_{\text{svec}}^{Q_B}$ of order $r_\triangle$. After multiplying the system (23) from the left on the matrix Ω we get the equivalent system which is decomposed in two subsystems

$$\Phi \, \text{svec} \Delta Z + (\mathcal{A}_{\text{svec}}^{Q_B})^T \left[\mathcal{B} y + \mathcal{A}^{h_k} \right] = 0_{r_\triangle}, \quad \tilde{U} \left[\mathcal{B} y + \mathcal{A}^{h_k} \right] = 0_p. \tag{24}$$

We find from the first subsystem (24)

$$\text{svec} \Delta Z = -\Phi^{-1} (\mathcal{A}_{\text{svec}}^{Q_B})^T \left[\mathcal{B} y + \mathcal{A}^{h_k} \right]. \tag{25}$$

Proposition 2 *Let $\mathcal{R}(\mathcal{B})$ and $\mathcal{R}(\mathcal{A}_{\text{svec}}^{Q_B})$ be columns spaces of the matrices $\mathcal{B}$ and $\mathcal{A}_{\text{svec}}^{Q_B}$, respectively. Let also $\mathcal{R}(\mathcal{B}) \cap \mathcal{R}(\mathcal{A}_{\text{svec}}^{Q_B}) = \emptyset$. Then the matrix $\tilde{U}\mathcal{B}$ is nonsingular.*

Proof We will show that the homogeneous system of equations $\tilde{U}\mathcal{B}y = 0_p$ has only the trivial solution $y = 0_p$. By contradiction, assume that this is not a case. Then there exists the nonzero vector $y \in \mathbb{R}^p$, satisfying this system. Since $\mathcal{B}$ is a full rank matrix, the nonzero vector $z = \mathcal{B}y$ must belong to the null-space of the matrix $\tilde{U}$, which coincides with the column space of the matrix $\mathcal{A}_{\text{svec}}^{Q_B}$. We come to contradiction. The assertion is proven.

Solving the second subsystem (24) and taking into consideration the assertion 2, we get

$$y = -\left(\tilde{U}\mathcal{B}\right)^{-1}\tilde{U}\mathcal{A}^{h_k}. \tag{26}$$

Denote $\tilde{\mathcal{B}} = \mathcal{B}\left(\tilde{U}\mathcal{B}\right)^{-1}\tilde{U}$. Then after substituting the vector y into expression (25) for $\text{svec}\Delta Z$ we get

$$\text{svec}\Delta Z = \Phi^{-1}(\mathcal{A}_{\text{svec}}^{Q_B})^T\left[\tilde{\mathcal{B}} - I_m\right]\mathcal{A}^{h_k}. \tag{27}$$

Compute now the derivative of the objective function along ΔX.

Proposition 3 *The matrix ΔX is the decreasing direction for the objective function in problem (1). Moreover, $C \bullet \Delta X = \theta^k$.*

Proof We have

$$C \bullet \Delta X = \begin{bmatrix} Q_B^T C Q_B & Q_B^T C h_k \\ h_k^T C Q_B & h_k^T C h_k \end{bmatrix} \bullet \begin{bmatrix} \Delta Z & w \\ w^T & 1 \end{bmatrix}.$$

Hence,

$$C \bullet \Delta X = \langle \text{svec}C^{Q_B}, \text{svec}\Delta Z \rangle + 2\langle C^{Q_B h_k}, w \rangle + C^{h_k}, \tag{28}$$

where $C^{Q_B h_k} = Q_B^T C h_k$.

We will compute separately the first and the second terms in the right hand side of (28). For the first term we obtain

$$\langle \text{svec}C^{Q_B}, \text{svec}\Delta Z \rangle = \langle \text{svec}C^{Q_B}, \Phi^{-1}(\mathcal{A}_{\text{svec}}^{Q_B})^T\left[\tilde{\mathcal{B}} - I_m\right]\mathcal{A}^{h_k} \rangle$$

$$= \langle \mathcal{A}_{\text{svec}}^{Q_B}\Phi^{-1}\text{svec}C^{Q_B}, \left[\tilde{\mathcal{B}} - I_m\right]\mathcal{A}^{h_k} \rangle$$

$$= \langle u, \left[\tilde{\mathcal{B}} - I_m\right]\mathcal{A}^{h_k} \rangle = \langle u, \tilde{\mathcal{B}}\mathcal{A}^{h_k} \rangle - \langle u, \mathcal{A}^{h_k} \rangle.$$

Moreover, due to (26) $\langle u, \tilde{\mathcal{B}}\mathcal{A}^{h_k} \rangle = -\langle u, \mathcal{B}y \rangle = -\sum_{i=1}^m u^i A_i^{h_k q_y}$.

For the second term in (28) the following equality

$$2\langle C^{Q_B h_k}, w \rangle = \langle C^{Q_B h_k}, \tilde{W}y \rangle = \langle Ch_k, Q^B\tilde{W}y \rangle = \langle h_k, Cq_y \rangle = C^{h_k q_y},$$

holds, where $C^{h_k q_y} = h_k^T C q_y$.

According to (11) $C^{h_k} - \langle u, \mathcal{A}^{h_k} \rangle = h_k^T V h_k = \theta^k$. Taking also into account that $V h_k = \theta^k h_k$ and that the vector $q_y = Q_B \tilde{W} y$ is orthogonal to the vector h_k, we obtain $C^{h_k q_y} - \sum_{i=1}^{m} u^i A_i^{h_k q_y} = h_k^T V q_y = \theta^k h_k^T q_y = 0$. Therefore,

$$C \bullet \Delta X = \theta^k < 0.$$

Thus, the objective function in the problem (1) decreases along the direction ΔX. The assertion is proven.

The choice of the maximal possible step-size α is carried out similar to the regular case. The matrix X is positive semidefinite. It follows from (19) that the symmetric matrix $\bar{X}(\alpha) = X + \alpha \Delta X$ will remain positive semidefinite for α sufficiently small. The maximal possible $\bar{\alpha}$ can be found from the condition that this matrix preserves its positive semi-definiteness. Therefore, $\bar{\alpha}$ must be the minimal positive number such that

$$\det \begin{bmatrix} D(\eta_B) + \alpha \Delta Z & \alpha w \\ \alpha w^T & \alpha \end{bmatrix} = 0.$$

Under $\alpha > 0$ we have

$$\det \begin{bmatrix} D(\eta_B) + \alpha \Delta Z & \alpha w \\ \alpha w^T & \alpha \end{bmatrix} = \alpha \det \left[D(\eta_B) + \alpha \left(\Delta Z - w w^T \right) \right].$$

Hence, determination of $\bar{\alpha}$ reduces to determination of minimal positive α such that $\det \left[D(\eta_B) + \alpha \left(\Delta Z - w w^T \right) \right] = 0$. Both matrices $D(\eta_B)$ and $G = \Delta Z - w w^T$ are symmetric, the matrix $D(\eta_B)$ is positive definite. Therefore, these matrices can be reduced by congruent transformation to the diagonal form with the help of a non-singular matrix P, i.e. $D(\eta_B) = P P^T$, $G = P D(\lambda) P^T$.

If among all components of the vector $\lambda = [\lambda^1, \ldots, \lambda^r]^T$ there exists at least one negative component, then the value $\bar{\alpha}$ is finite and it is equal to $\bar{\alpha} = -\lambda_*^{-1}$, where λ_* is the maximal by absolute value negative component of the vector λ. Otherwise, problem (1) has not solution.

Consider the question about convergence of the method. For simplicity, let us assume that at each iteration we take the negative eigenvalue θ^k such that its absolute value is maximal among all other negative eigenvalues.

Theorem 1 *Let the problem* (1) *be quasi-regular. Let also the starting extreme point* $X_0 \in \mathcal{F}_P$ *be such, that the set* $\mathcal{F}_P(X_0) = \{X \in \mathcal{F}_P : C \bullet X \leq C \bullet X_0\}$ *is bounded. Then either the sequence of extreme points* $\{X_k\} \subset \mathcal{F}_P(X_0)$, *generated by simplex-method, is finite and in this case the last extreme point is the solution of problem* (1), *or the sequence* $\{X_k\}$ *is infinite and any limit point of* $\{X_k\}$ *is the solution of* (1) *too.*

Proof If the sequence $\{X_k\}$ is finite, then method is stopped at some iteration. This event may occur only if all eigenvalues of the matrix $V_k = V(u_k)$ involved in the determination procedure of u_k are non-negative. Hence, the pair $[u_k, V_k]$ is the feasible

point in the dual problem (2). But in this situation all optimality conditions (3) are fulfilled, therefore, X_k is the solution of the problem (1).

Let us consider now the case, where the sequence $\{X_k\}$ is infinite. Since this sequence is bounded, there exist the limits points of $\{X_k\}$. Let $X_{k_s} \to \bar{X}$. Because of the done assumptions and the rule for the step-size α_k choice, regardless the extreme point X_k with rank r is regular or irregular, the next point X_{k+1} will be an extreme point with the same rank r. Therefore, the point $\bar{X}$ is an extreme point too.

All points X_{k_s} from the sequence $\{X_{k_s}\}$ are such that corresponding matrices Q_B from factorization (8) have the same Frobenius norm, to be exact, $\|Q_B\|_F = \left(\operatorname{tr} Q_B^T Q_B\right)^{1/2} = \sqrt{r}$. Consequently, these matrices Q_B belong to the compact set, and it is possible to extract from $\{X_{k_s}\}$ the subsequence, for which the corresponding matrices Q_B converge to certain matrix $\bar{Q}_B$ such that $\bar{Q}_B^T \bar{Q}_B = I_r$. Without loss of generality, we regard that the sequence $\{X_{k_s}\}$ possesses this property.

Consider the matrix $(\mathcal{A}_{\text{svec}}^{\bar{Q}_B})^T$, entering in the system (9) for finding the vector of dual variables $\bar{u}$ at the point $\bar{X}$. Since $\bar{X}$ is an extreme point, this matrix has full rank coinciding with the row rank. From here, taking into consideration the continuity of vectors $\operatorname{svec}C^{Q_B}$, we conclude that solutions u_{k_s} defined either by formula (10) or by formula (18) converge to $\bar{u}$.

Determine the matrix $\bar{V} = V(\bar{u})$. This matrix $\bar{V}$ must be positive semi-definite. Indeed otherwise $\bar{V}$ would have a negative eigenvalue. However, eigenvalues of matrices are Lipschitz continuous. Therefore, matrices V_{k_s} sufficiently close to $\bar{V}$ also have negative eigenvalues. Hence, these iterations correspond to the passage from the points X_{k_s} to the next points X_{k_s+1} with the step-size $\alpha_{k_s} > 0$. Moreover, the value of the objective function decreases by $\alpha_{k_s}\bar{\theta}^{k_s}$, where $\bar{\theta}^{k_s}$ is a negative component of θ_{k_s} with a maximum absolute value. However, it follows from (16) or (27) that matrices ΔX_{k_s} are bounded by norm at $\mathcal{F}_P(X_0)$. Therefore, we have $C \bullet X_{k_s+1} < C \bullet \bar{X}$ at certain k_s iteration. In view of monotone decrease of the values of the objective function along the trajectory, this contradicts to convergence of $\{X_{k_s}\}$ to $\bar{X}$. The theorem is proven. □

It is possible to show that, if the problems (1) and (2) are nondegenerate and their solutions are strictly complementary, then the bounded set $\mathcal{F}_P(X_0)$ from conditions of the theorem (1) exists.

6 Initial stage of the method

Suppose that we have a point $X \in \mathcal{F}_P$, which is not an extreme point of $\mathcal{F}_P$. Suppose also that rank of X is equal to $r \leq n$, and for X the Cholesky factorization (5) holds. As before, we assume, that the positive eigenvalues of X are at the beginning of the vector $\eta = [\eta_B; \eta_N]$, and Q_B is the $n \times r$ left submatrix of an orthogonal matrix Q.

We replace the basic problem (1) by its reduction on the minimal face $\mathcal{G}_{\min}(X; \mathbb{S}_+^n)$:

$$\min\ C \bullet X, \qquad A^i \bullet X = b^i, \quad 1 \leq i \leq m, \quad X \in \mathcal{G}_{\min}(X; \mathbb{S}_+^n). \tag{29}$$

Taking into account that points from $\mathcal{G}_{\min}(X; \mathbb{S}^n_+)$ can be represented as $X = Q_B Z Q_B^T$, where $Z \in \mathbb{S}^r_+$, we rewrite the problem (29) in the following form

$$\min \ C^{Q_B} \bullet Z,$$
$$A_i^{Q_B} \bullet Z = b^i, \quad 1 \le i \le m, \qquad Z \succeq 0. \tag{30}$$

The optimality conditions (3) for semidefinite programming problem (30) and for dual problem to it can be written also as

$$Z \circ V^{Q_B} = 0, \quad V^{Q_B} = C^{Q_B} - \sum_{i=1}^m u^i A_i^{Q_B},$$
$$A_i^{Q_B} \bullet Z = b^i, \quad 1 \le i \le m, \qquad Z \succeq 0, \ V^{Q_B} \succeq 0, \tag{31}$$

where $M_1 \circ M_2 = [M_1 M_2 + M_2 M_1]/2$ is the symmetrized product of matrices M_1 and M_2. Using the well-known formula $\operatorname{vec} M_1 M_2 M_3 = \left(M_3^T \otimes M_1 \right) \operatorname{vec} M_2$, where the symbol $\otimes$ denotes the Kronecker product of matrices, we obtain, that the equalities from (31) can be represented as

$$\tilde{Z}^\otimes \operatorname{svec} V^{Q_B} = 0_{r_\triangle}, \quad \mathcal{A}^{Q_B}_{\text{svec}} \operatorname{svec} Z = b, \quad \operatorname{svec} V^{Q_B} = \operatorname{svec} C^{Q_B} - \left(\mathcal{A}^{Q_B}_{\text{svec}} \right)^T u. \tag{32}$$

Here and in what follows $\tilde{Z}^\otimes = \tilde{\mathcal{L}}_r Z^\otimes \tilde{\mathcal{D}}_r$, and $Z^\otimes = \frac{1}{2}[Z \otimes I_r + I_r \otimes Z]$ is the Kronecker product of Z.

We substitute the vector $\operatorname{svec} V^{Q_B}$ from the third equality (32) to the first one. Then after multiplying both sides of this equality by the matrix $\mathcal{A}^{Q_B}_{\text{svec}}$ we obtain the equation

$$\Gamma_{Q_B}(\tilde{Z}^\otimes) u = \mathcal{A}^{Q_B}_{\text{svec}} \tilde{Z}^\otimes \operatorname{svec} C^{Q_B}, \tag{33}$$

in which $\Gamma_{Q_B}(\tilde{Z}^\otimes) = \mathcal{A}^{Q_B}_{\text{svec}} \tilde{Z}^\otimes (\mathcal{A}^{Q_B}_{\text{svec}})^T$. The matrix $\Gamma_{Q_B}(\tilde{Z}^\otimes)$ is nonsingular, if matrices $A_i^{Q_B}$, $1 \le i \le m$, are linearly independent. In the case, where $r_\triangle \ge m$, necessarily the matrix $\Gamma_{Q_B}(\tilde{Z}^\otimes)$ is nonsingular. It follows from assumption that the point X is nondegenerate.

Solving the system (33), we get $u(Z) = \Gamma_{Q_B}^{-1}(\tilde{Z}^\otimes) \mathcal{A}^{Q_B}_{\text{svec}} \tilde{Z}^\otimes \operatorname{svec} C^{Q_B}$. Thus, $\operatorname{svec} V^{Q_B} = \mathcal{P}(\tilde{Z}^\otimes) \operatorname{svec} C^{Q_B}$, where $\mathcal{P}(\tilde{Z}^\otimes) = I_{r_\triangle} - (\mathcal{A}^{Q_B}_{\text{svec}})^T \Gamma_{Q_B}^{-1}(\tilde{Z}^\otimes) \mathcal{A}^{Q_B}_{\text{svec}} \tilde{Z}^\otimes$.

After substitution of the vector $\operatorname{svec} V^{Q_B}$ in the first equality (31) we derive the system of nonlinear equations

$$\tilde{Z}^\otimes \mathcal{P}(\tilde{Z}^\otimes) \operatorname{svec} C^{Q_B} = 0_{r_\triangle} \tag{34}$$

with respect to the matrix Z.

Using the left hand side of equation (34), it is possible to pass from Z to another point $\bar{Z}$ with decreasing of the value of the objective function in problem (30), namely, $\bar{Z} = Z - \alpha \Delta(Z)$, where $\alpha > 0$, and $\Delta(Z)$ is a symmetric matrix for which $\operatorname{svec} \Delta(Z) = \tilde{Z}^\otimes \mathcal{P}(\tilde{Z}^\otimes) \operatorname{svec} C^{Q_B}$. The direction ΔZ is such that equalities $A_i^{Q_B} \bullet \Delta(Z) = 0$, $1 \le i \le m$, hold, and $C^{Q_B} \bullet \Delta(Z) \ge 0$.

Denote $\bar{X}(\alpha) = Q_B [D(\eta_B) - \alpha \Delta(Z)] Q_B^T$. Since the matrix $D(\eta_B)$ is positive definite, the matrix $D(\eta_B) - \alpha \Delta(Z)$ for sufficiently small α remains positive definite too. The maximal possible step-size α_* can be found from the condition that some eigenvalue of the matrix $D(\eta_B) - \alpha \Delta(Z)$ for the first time becomes zero. In this case the feasible point $\bar{X}(\alpha_*)$ is such that the rank of the matrix $\bar{X}(\alpha_*)$ is less than the rank r of the matrix X. If $\bar{X}(\alpha_*)$ is not an extreme point, it is necessary to repeat the procedure.

7 Conclusion

We have proposed the numerical simplex-type method for linear semidefinite programming problems. The method can be regarded as the specific way for solving the system of optimality conditions (3). There is also the dual simplex-type method for solving the linear semidefinite programming problems [19]. In this method another approach is used for solving the system of optimality conditions. The pivoting procedures for both methods are quite analogous to the procedures used in primal and dual simplex methods for linear programming. Simplex-type methods for semidefinite and second order cone programming are particular important in the case when one must solve a sequence of problems whose structures are similar to one another.

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