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ORIGINAL PAPER

Some dual characterizations of Farkas-type results for fractional programming problems

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Abstract In this paper, we present some Farkas-type results for a fractional programming problem. To this end, by using the properties of dualizing parametrization functions, Lagrangian functions and the epigraph of the conjugate functions, we introduce some new notions of regularity conditions and then obtain some dual forms of Farkas-type results for this fractional programming problem. We also obtain sufficient conditions for alternative type theorems. As an application of these results, we obtain the corresponding results for a convex optimization problem.

Keywords Duality · Farkas-type results · Alternative type theorems · Fractional programming problem

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1 Introduction

The classical Farkas-type results characterizes those linear inequalities which are consequences of a consistent ordinary linear inequality system (i.e., they are satisfied by every solution of the system). This result plays an important role in the development of linear programming and optimization theory. Recently, different types of Farkas-type results and their extensions have been given in the literature with applications to more general nonlinear programming problems and nonsmooth optimization problems, see [1–8] and the references therein. Centered around the celebrated Farkas-type results and its extensions, in this paper, we investigate the following fractional programming problem:

$$(\mathrm{P}) \quad \inf_{x \in X} \frac{f(x)}{g(x)},$$

where X is a locally convex vector space, $f, -g : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ are two proper, lower semicontinuous and convex functions. Moreover, we suppose that $g(x) > 0$ for all $x \in X$.

The study of fractional programming problems is very important since many optimization problems which arise from practical needs turn out to be of fractional type. Many important results have been established for fractional programming problems in the last decades, see [9–17] and the references therein. However, to the best of our knowledge, there are few papers to deal with Farkas-type results for fractional programming problems, see [18–21]. Here, we specially mention the works [18,21] on Farkas-type results for fractional programming via different kinds of regularity conditions. By using an interiority condition, Boţ et al. [18] obtained some sufficient conditions for Farkas-type results and alternative type theorems for (P) in finite-dimensional spaces (see also [19,20]). Since the interiority condition does not always hold for finite-dimensional optimization problems and frequently fail for infinite-dimensional optimization problems arising in applications, Sun et al. [21] obtained some sufficient conditions for Farkas-type results and alternative type theorems for a fractional programming problem by using some closedness condition in infinite-dimensional spaces. It is therefore of interest to investigate Farkas-type results for fractional programming problems. So, the aim of this paper is to introduce some regularity conditions to investigate Farkas-type results and alternative type theorems for (P).

As mentioned above, the purpose of this paper is to establish some Farkas-type results and alternative type theorems for (P). It is important to note that there are some papers to investigate duality or Farkas-type results for other kinds of optimization problems, see [4,22–24], where the Farkas-type results and regularity conditions introduced here are defined in terms of conjugates of the functions involved. Now, in this paper, we will use a new method which is different from the method used in the references. In particular, our dual problems and regularity conditions considered in this paper are defined in terms of conjugates of the Lagrangian functions for the functions involved. It is also important to note that this technique has been employed in the study of the duality theory of convex or nonconvex programming in [25–27].

In order to do so, by using the idea due to Dinkelbach [9], we associate (P) with the following optimization problem:

$$(P_\mu) \quad \inf_{x\in X}\{f(x)-\mu g(x)\},$$

where $\mu \in \mathbb{R}$. Then, by using dualizing parametrization functions and Lagrangian functions, we introduce some new regularity conditions and formulate a type of dual problems of (P_μ). By virtue of the epigraph technique, we establish some sufficient and necessary conditions of the weak and strong dualities for (P_μ). Then, by using the regularity conditions and the duality assertions for (P_μ), we obtain some Farkas-type results and alternative type theorems for (P). Moreover, the results obtained here underline the connections that exist between Farkas-type results and alternative type theorems and, on the other hand, the duality.

The paper is organized as follows. In Sect. 2, we recall some notions and give some preliminary results. In Sect. 3, we first construct the dual problem of (P_μ). Then, we introduce some new regularity conditions, and investigate several relationships among them. By using these new regularity conditions, we characterize the duality results of (P_μ). After that, we obtain some Farkas-type results and alternative type theorems of (P). In Sect. 4, we apply these problems to convex optimization problems.

2 Mathematical preliminaries

Throughout this paper, let X be a real locally convex vector space with its continuous dual space X^*, endowed with the weak* topology $w(X^*, X)$. We always use the notation $\langle\cdot,\cdot\rangle$ for the canonical paring between X and X^*. Let D be a set in X, the interior (resp. closure, convex hull, convex cone hull) of D is denoted by int D (resp. cl D, co D, cone D). Thus if $W \subseteq X^*$, then cl W denotes the weak* closure of W. We shall adopt the convention that cone $D = \{0\}$ when D is an empty set. Let $D^* = \{x^* \in X^* : \langle x^*, x\rangle \geq 0, \forall x \in D\}$ be the dual cone of D.

Let $f : X \to \overline{\mathbb{R}}$ be an extended real valued function. The effective domain and the epigraph are defined by dom $f = \{x \in X : f(x) < +\infty\}$ and epi $f = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$, respectively. We say that f is proper, iff dom $f \neq \emptyset$ and $f(x) > -\infty$. The conjugate function $f^* : X^* \to \overline{\mathbb{R}}$ of f is defined by $f^*(x^*) = \sup_{x\in X}\ \{\langle x^*, x\rangle - f(x)\}$. The biconjugate function of f is the conjugate function $f^{**} : X \to \overline{\mathbb{R}}$ of f^*: $f^{**}(x) = \sup_{x^*\in X^*}\ \{\langle x, x^*\rangle - f^*(x^*)\}$. By [28, Theorem 2.3.4], if f is proper, convex and lower semicontinuous function, then $f^{**} = f$. For details, see [15,28].

In this paper, we endow $X^* \times \mathbb{R}$ with the product topology of $w(X^*, X)$ and the usual Euclidean topology. Now, we give the following important results which will be used in the following section.

Lemma 2.1 [28] *Let I be an index set, and let $\{f_i : i \in I\}$ be a family of functions. Then* epi $(\sup_{i\in I} f_i) = \cap_{i\in I}$ epi f_i.

We conclude this section by recalling the following notions which will be used in this paper. Let $p \in X^*$. Then, p can be regarded as a function on X in such a way that

$p(x) := \langle p, x\rangle$, for any $x \in X$. Thus, for any $\alpha \in \mathbb{R}$ and any function $h : X \to \overline{\mathbb{R}}$, we have

$$(h + p + \alpha)^*(x^*) = h^*(x^* - p) - \alpha, \text{ for each } x^* \in X^*, \quad (1)$$

and

$$\text{epi } (h + p + \alpha)^* = \text{epi } h^* + (p, -\alpha). \quad (2)$$

Here and throughout this paper, following Zălinescu [28], we adapt the convention that

$$(+\infty) + (-\infty) = (+\infty) - (+\infty) = +\infty, \text{ and } 0 \cdot \infty = 0.$$

3 Main results

In this section, we use some new regularity conditions to establish some Farkas-type results for (P). The obtained results are new in two features: firstly, our regularity conditions considered in this paper are defined in terms of conjugates of the Lagrangian functions for the functions involved, while most of similar regularity conditions in the literature are defined in terms of conjugates of the functions involved; secondly, we obtain sufficient conditions for alternative type theorems by using a new method which is different from the method used in the references. To this end, we first construct the dual problems of (P_μ), and present the weak and strong duality assertions. Then, by using the duality assertions, we establish some Farkas-type results for (P). It is obvious that there exists the following relation between the optimal value $val(P)$ of the problem (P) and the optimal value $val(P_\mu)$ of the problem (P_μ).

Lemma 3.1 *The inequality $val(P) \geq \mu$ holds if and only if the inequality $val(P_\mu) \geq 0$ holds.*

Now, consider the problem (P_μ). For any $x \in X$, we introduce the dualizing parametrization functions $F : X \times Z \to \overline{\mathbb{R}}$ and $G : X \times Z \to \overline{\mathbb{R}}$ for f and g respectively by

$$F(x, 0) = f(x), \quad G(x, 0) = -g(x), \quad (3)$$

where Z is also a locally convex Hausdorff topological vector space and Z^* is the dual space. Throughout the paper, we also assume that F and G are proper convex functions, and $F(x, \cdot)$ and $G(x, \cdot)$ are closed. Similar to the methods used on pages 18, 19 in [25], for the problem (P_μ), we define the Lagrangian functions K and L on $X \times Z^*$ by

$$K(x, z^*) = \inf_{z \in Z} \{F(x, z) + z^*(z)\} \quad (4)$$

and

$$L(x, z^*) = \inf_{z \in Z} \{G(x, z) + z^*(z)\}, \tag{5}$$

respectively. Then, the functions $z^* \to K(x, z^*)$ and $z^* \to L(x, z^*)$ are the conjugates in the concave sense of the functions $z \to -F(x, z)$ and $z \to -G(x, z)$, respectively. Since $F(x, \cdot)$ and $G(x, \cdot)$ are closed and convex, the conjugates are reciprocal, i.e., the functions $z \to -F(x, z)$ and $z \to -G(x, z)$ are the conjugates of the functions $z^* \to K(x, z^*)$ and $z^* \to L(x, z^*)$, or

$$F(x, z) = \sup_{z^* \in Z^*} \{K(x, z^*) - z^*(z)\} \tag{6}$$

and

$$G(x, z) = \sup_{z^* \in Z^*} \{L(x, z^*) - z^*(z)\}. \tag{7}$$

Let $z = 0$. Then, it follows from (3), (6), and (7) that

$$f(x) = \sup_{z^* \in Z^*} K(x, z^*), \quad -g(x) = \sup_{z^* \in Z^*} L(x, z^*). \tag{8}$$

One should notice that the Eq. (8) plays an important role in establishing the dual problem of (P_μ). For given $z^* \in Z^*$, denote

$$K^{z^*}(x) = K(x, z^*), \quad L^{z^*}(x) = L(x, z^*). \tag{9}$$

Now, we will construct a dual problem to (P_μ) and completely characterize the weak and strong Lagrange dualities. Since the objective function of the problem (P_μ) depends on the sign of μ, we have to treat two different cases. In the case that μ is a negative value, the objective function of the problem (P_μ) is the difference of two convex functions and therefore we can use a approach inspired from DC programming. In the case that μ is a non-negative value, the objective function of the problem (P_μ) is a convex function and the convex optimization theory can be used.

3.1 The case that μ is a negative value

For any real number $\mu < 0$, in order to formulate a dual problem for (P_μ), we further assume throughout this subsection that there exist some $z^* \in Z^*$ such that L^{z^*} is proper lower semicontinuous function. As G is proper convex function, and $G(x, \cdot)$ is closed, then, by Theorem 6 in [25], $L^{z^*}(x)$ is convex. So, it follows that

$$L^{z^*}(x) = \left(L^{z^*}\right)^{**}(x) = \sup_{v^* \in \operatorname{dom} \left(L^{z^*}\right)^*} \left\{v^* x - \left(L^{z^*}\right)^*(v^*)\right\}. \tag{10}$$

Then,

$$\begin{aligned}
&\inf_{x\in X} \left\{K(x,z^*)+\mu L(x,z^*)\right\}\\
&=\inf_{x\in X} \left\{K(x,z^*)-(-\mu)L(x,z^*)\right\}\\
&=\inf_{\substack{x\in X,\\ v^*\in \operatorname{dom}\left(L^{z^*}\right)^*}} \left\{K(x,z^*)-(-\mu)\left(v^*x-\left(L^{z^*}\right)^*(v^*)\right)\right\}\\
&=\inf_{\substack{x\in X,\\ v^*\in \operatorname{dom}\left(L^{z^*}\right)^*}} \left\{K(x,z^*)-(-\mu v^*)x+(-\mu)\left(L^{z^*}\right)^*(v^*)\right\}\\
&=\inf_{\substack{x\in X,\\ v^*\in \operatorname{dom}\left(L^{z^*}\right)^*}} \left\{K^{z^*}(x)-(-\mu v^*)x+(-\mu)\left(L^{z^*}\right)^*(v^*)\right\}\\
&=\inf_{v^*\in \operatorname{dom}\left(L^{z^*}\right)^*} \left\{-\left(K^{z^*}\right)^*(-\mu v^*)-\mu\left(L^{z^*}\right)^*(v^*)\right\}.
\end{aligned}$$

Let

$$\phi_\mu(z^*):=\inf_{v^*\in \operatorname{dom}\left(L^{z^*}\right)^*} \left\{-\left(K^{z^*}\right)^*(-\mu v^*)-\mu\left(L^{z^*}\right)^*(v^*)\right\}. \tag{11}$$

Then, we introduce the dual problem of (P_μ) as follows:

$$(D_\mu) \qquad \sup_{z^*\in Z^*} \phi_\mu(z^*). \tag{12}$$

In order to characterize the weak and strong dualities between (P_μ) and (D_μ), we need to introduce some new regularity conditions. To this aim, we will make use of the following characteristic set Ω defined by

$$\Omega:=\bigcup_{z^*\in Z^*}\ \bigcap_{v^*\in \operatorname{dom}\left(L^{z^*}\right)^*} \left\{\operatorname{epi}\left(K^{z^*}\right)^*+\mu\left(v^*,(L^{z^*})^*(v^*)\right)\right\}. \tag{13}$$

The following lemma shows that Ω is equal to $\bigcup_{z^*\in Z^*}\operatorname{epi}\left(K^{z^*}+\mu L^{z^*}\right)^*$.

Lemma 3.2 *The following formula holds:*

$$\Omega=\bigcup_{z^*\in Z^*}\operatorname{epi}\left(K^{z^*}+\mu L^{z^*}\right)^*. \tag{14}$$

Proof It follows from (10) that

$$
\begin{aligned}
&\left(K^{z^*}+\mu L^{z^*}\right)^*(x^*)\\
&=\sup_{x\in X}\left(x^*(x)-K^{z^*}(x)+(-\mu)\sup_{v^*\in \text{ dom }\left(L^{z^*}\right)^*}\left\{v^*x-\left(L^{z^*}\right)^*(v^*)\right\}\right)\\
&=\sup_{\substack{x\in X,\\ v^*\in \text{ dom }\left(L^{z^*}\right)^*}}\left(x^*(x)-K^{z^*}(x)-\mu v^*x+\mu\left(L^{z^*}\right)^*(v^*)\right)\\
&=\sup_{v^*\in \text{ dom }\left(L^{z^*}\right)^*}\left(K^{z^*}+\mu v^*-\mu\left(L^{z^*}\right)^*(v^*)\right)^*(x^*).
\end{aligned}
$$

This, together with (2) and Lemma 2.1, implies that

$$
\begin{aligned}
\text{epi }\left(K^{z^*}+\mu L^{z^*}\right)^* &= \bigcap_{v^*\in \text{ dom }\left(L^{z^*}\right)^*} \text{epi }\left(K^{z^*}+\mu v^*-\mu\left(L^{z^*}\right)^*(v^*)\right)^*\\
&= \bigcap_{v^*\in \text{ dom }\left(L^{z^*}\right)^*}\left\{\text{epi }\left(K^{z^*}\right)^*+\mu\left(v^*,\left(L^{z^*}\right)^*(v^*)\right)\right\}.
\end{aligned}
$$

This completes the proof. □

Comparing with the expressions of problems (P_μ) and (D_μ), it is easy to see that $\Omega\cap(\{0\}\times\mathbb{R})$ is associated with (D_μ) and epi $(f-\mu g)^*\cap(\{0\}\times\mathbb{R})$ is associated with (P_μ). Considering the possible relationships between epi $(f-\mu g)^*$ and Ω, we introduce the following regularity conditions.

Definition 3.1 The family (f, g) is said to satisfy

i. the further regularity condition (FRC), iff

$$\text{epi }(f-\mu g)^*\cap(\{0\}\times\mathbb{R})=\Omega\cap(\{0\}\times\mathbb{R});$$

ii. the semi-(FRC) $(SFRC)$, iff

$$\text{epi }(f-\mu g)^*\cap(\{0\}\times\mathbb{R})\supseteq\Omega\cap(\{0\}\times\mathbb{R}).$$

Remark 3.1 It is worth noting that since

$$val(P_\mu)=\inf_{x\in X}\{f(x)-\mu g(x)\}=-(f-\mu g)^*(0),$$

we have

$$(0,\alpha)\in\text{epi }(f-\mu g)^*\Leftrightarrow val(P_\mu)\geq-\alpha. \tag{15}$$

In order to completely characterize the duality results in terms of these regularity conditions, we need the following lemma.

Lemma 3.3 *Let* $\alpha \in \mathbb{R}$. *Then,* $(0, \alpha) \in \Omega$ *if and only if there exists* $z^* \in Z^*$*, such that for any* $v^* \in \operatorname{dom}\left(L^{z^*}\right)^*$, *satisfies*

$$-\left(K^{z^*}\right)^*(-\mu v^*) - \mu\left(L^{z^*}\right)^*(v^*) \geq -\alpha. \tag{16}$$

Proof ($\Rightarrow$) Let $(0, \alpha) \in \Omega$. Then, there exists $z^* \in Z^*$, such that for any $v^* \in \operatorname{dom}\left(L^{z^*}\right)^*$,

$$(0, \alpha) \in \operatorname{epi}\left(K^{z^*}\right)^* + \mu\left(v^*, \left(L^{z^*}\right)^*(v^*)\right).$$

Then, there exists $(x_1^*, \alpha_1) \in \operatorname{epi}\left(K^{z^*}\right)^*$ such that

$$x_1^* = -\mu v^* \tag{17}$$

and

$$\alpha = \alpha_1 + \mu\left(L^{z^*}\right)^*(v^*). \tag{18}$$

Since $\left(K^{z^*}\right)^*(x_1^*) \leq \alpha_1$, it follows from (17) and (18) that

$$-\left(K^{z^*}\right)^*(-\mu v^*) - \mu\left(L^{z^*}\right)^*(v^*) \geq -\alpha_1 - \mu\left(L^{z^*}\right)^*(v^*) = -\alpha.$$

Then, (16) holds.

($\Leftarrow$) Suppose that there exists $z^* \in Z^*$, such that for any $v^* \in \operatorname{dom}\left(L^{z^*}\right)^*$, (16) holds. Then,

$$\left(K^{z^*}\right)^*(-\mu v^*) \leq \alpha - \mu\left(L^{z^*}\right)^*(v^*),$$

which means that

$$\left(-\mu v^*,\ \alpha - \mu\left(L^{z^*}\right)^*(v^*)\right) \in \operatorname{epi}\left(K^{z^*}\right)^*. \tag{19}$$

Then,

$$\begin{aligned}(0, \alpha) &= \left(-\mu v^*,\ \alpha - \mu\left(L^{z^*}\right)^*(v^*)\right) + \mu\left(v^*,\ \left(L^{z^*}\right)^*(v^*)\right)\\ &\in \operatorname{epi}\left(K^{z^*}\right)^* + \mu\left(v^*,\ \left(L^{z^*}\right)^*(v^*)\right).\end{aligned}$$

Since $v^* \in \text{ dom } (L^{z^*})$ is arbitrary, we get

$$(0, \alpha) \in \bigcap_{v^* \in \text{ dom } \left(L^{z^*}\right)^*} \left\{ \text{epi } \left(K^{z^*}\right)^* + \mu \left(v^*, \left(L^{z^*}\right)^* (v^*)\right)\right\}.$$

By (13), we get $(0, \alpha) \in \Omega$ and the proof is complete. □

Now, we use these regularity conditions to completely characterize weak and strong parametrized Lagrange dualities between (P_μ) and (D_μ).

Theorem 3.1 *The family* (f, g) *satisfies the* $(SFRC)$ *if and only if the weak duality between* (P_μ) *and* (D_μ) *holds.*

Proof ($\Rightarrow$) Suppose that the weak duality between (P_μ) and (D_μ) does not hold. Then, there exist $\alpha \in \mathbb{R}$ such that $val(P_\mu) < -\alpha < val(D_\mu)$. By (11) and (12), there exists $z^* \in Z^*$, such that for any $v^* \in \text{ dom } \left(L^{z^*}\right)^*$, we have (16) holds. Then, from Lemma 3.3, $(0, \alpha) \in \Omega$. Since the family (f, g) satisfies the $(SFRC)$, we have that $(0, \alpha) \in \text{epi } (f - \mu g)^*$. Then, from (15), $val(P_\mu) \geq -\alpha$. This contradicts $val(P_\mu) < -\alpha$. Thus, the weak duality between (P_μ) and (D_μ) holds.

($\Leftarrow$) Suppose that the weak duality between (P_μ) and (D_μ) holds. Let $(0, \alpha) \in \Omega$. By Lemma 3.3, $val(D_\mu) \geq -\alpha$. Then, $val(P_\mu) \geq val(D_\mu) \geq -\alpha$. which implies that $(0, \alpha) \in \text{epi } (f - \mu g)^*$ in terms of (15). Then, the family (f, g) satisfies the $(SFRC)$. The proof is complete. □

Now, we give an example to explain Theorem 3.1.

Example 3.1 Let $X = Y = Z = \mathbb{R}$ and $\mu = -1$. Define $f, g : \mathbb{R} \to \overline{\mathbb{R}}$ by $f = \delta_{[1,+\infty)}$ and

$$g(x) = \begin{cases} 2x, & \text{if } x \geq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, it is easy to see that

$$(f - \mu g)^*(x^*) = \begin{cases} x^* - 2, & \text{if } x^* \leq 2, \\ +\infty, & \text{otherwise.} \end{cases}$$

So,

$$\text{epi } (f - \mu g)^* = \left\{(x^*, r) \in \mathbb{R}^2 : x^* \leq 2, r \geq x^* - 2\right\}.$$

On the other hand, let

$$F(x, z) = \begin{cases} 0, & \text{if } x \geq 1, z \leq 2x + 0.5, \\ +\infty, & \text{otherwise.} \end{cases}$$

and $G(x,z) = -g(x) + z$. Obviously, $F(x,0) = f(x)$ and $G(x,0) = -g(x)$. Moreover, we have

$$K(x,z^*) = \begin{cases} z^*(2x+0.5), & \text{if } x \geq 1, z^* \leq 0, \\ -\infty, & \text{if } x \geq 1, z^* > 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$L(x,z^*) = \begin{cases} -2x, & \text{if } x \geq 1, z^* = -1, \\ -\infty, & \text{if } x \geq 1, z^* \neq -1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then,

$$\left(K^{z^*} + \mu L^{z^*}\right)^* (x^*) = \begin{cases} x^* + 0.5, & \text{if } x^* \leq 0, z^* = -1, \\ +\infty, & \text{otherwise.} \end{cases}$$

So,

$$\Omega = \left\{(x^*, r) \in \mathbb{R}^2 : x^* \leq 0, r \geq x^* + 0.5\right\}.$$

Obviously, $\Omega \cap (\{0\} \times \mathbb{R}) \subseteq \text{epi } (f - \mu g)^* \cap (\{0\} \times \mathbb{R})$ and then the family (f, g) satisfies the $(SFRC)$. It is also easy to see that $val(P_\mu) = 2$, $val(D_\mu) = -0.5$. Consequently, $val(P_\mu) > val(D_\mu)$ and the weak duality between (P_μ) and (D_μ) holds.

Theorem 3.2 *The family (f, g) satisfies the (FRC) if and only if the strong duality between (P_μ) and (D_μ) holds.*

Proof ($\Rightarrow$) Suppose that the family (f, g) satisfies the (FRC). Then, the family (f, g) satisfies the $(SFRC)$. By Theorem 3.1, $val(P_\mu) \geq val(D_\mu)$. So, it suffices to show that $val(D_\mu) \geq val(P_\mu)$. If $val(P_\mu) = -\infty$, then we get strong duality for (P_μ) and (D_μ) via weak duality. So, we assume that $val(P_\mu) = -\alpha \in \mathbb{R}$. By (15),

$$(0, \alpha) \in \text{epi } (f - \mu g)^*.$$

Since the family (f, g) satisfies the (FRC), we have $(0, \alpha) \in \Omega$. By Lemma 3.3, there exists $z^* \in Z^*$, such that for any $v^* \in \text{ dom } \left(L^{z^*}\right)^*$, satisfies

$$-\left(K^{z^*}\right)^* (-\mu v^*) - \mu \left(L^{z^*}\right)^* (v^*) \geq -\alpha.$$

This follows that $val(D_\mu) \geq -\alpha$. This means that the strong duality between (P_μ) and (D_μ) holds.

(⇐) Assume that the strong duality between (P_μ) and (D_μ) holds. By Theorem 3.1, we only need to prove that

$$\text{epi } (f-\mu g)^* \cap (\{0\}\times\mathbb{R}) \subseteq \Omega \cap (\{0\}\times\mathbb{R}). \tag{20}$$

In fact, let $(0,\alpha)\in$ epi $(f-\mu g)^*$. By (15), $val(P_\mu)\geq -\alpha$. Then, from the strong duality, $val(D_\mu)=val(P_\mu)\geq -\alpha$. By Lemma 3.3, $(0,\alpha)\in\Omega$ and (20) holds. The proof is complete. □

We are now in a position to prove the main results of this section. It serve as the main tools for establishing alternative type theorem for (P) (see Corollary 3.1).

Theorem 3.3 *If the family* (f,g) *satisfies the* (FRC)*, then the following statements are equivalent:*

i. $x\in X \Longrightarrow \frac{f(x)}{g(x)}\geq\mu$.
ii. *There exists* $z^*\in Z^*$*, such that for any* $v^*\in$ dom $\left(L^{z^*}\right)^*$*, satisfies*

$$-\left(K^{z^*}\right)^*(-\mu v^*)-\mu\left(L^{z^*}\right)^*(v^*)\geq 0. \tag{21}$$

Proof Suppose that the family (f,g) satisfies the (FRC). If (i) holds, then, $val(P)\geq\mu$. By Lemma 3.1, $val(P_\mu)\geq 0$. It follows from Theorem 3.2 that $val(D_\mu)=val(P_\mu)\geq 0$, which means that (ii) holds.

Conversely, assume that (ii) holds. Then, there exists $z^*\in Z^*$, such that for any $v^*\in$ dom $(L^{z^*})^*$, we have (21) holds. Therefore, it comes that

$$\sup_{z^*\in Z^*}\ \inf_{v^*\in \text{ dom }\left(L^{z^*}\right)^*}\left\{-\left(K^{z^*}\right)^*(-\mu v^*)-\mu\left(L^{z^*}\right)^*(v^*)\right\}\geq 0.$$

This means that $val(D_\mu)\geq 0$. By Theorem 3.2, we obtain that $val(P_\mu)\geq 0$. By Lemma 3.1, we have $val(P)\geq\mu$, and then (i) holds. The proof is complete. □

The previous result can be reformulated as a theorem of the alternative in the following way.

Corollary 3.1 *If the family* (f,g) *satisfies the* (FRC)*, then precisely one of the following statements is true*

i. $\exists x\in X$*, such that* $\frac{f(x)}{g(x)}<\mu$.
ii. *There exists* $z^*\in Z^*$*, such that for any* $v^*\in$ dom $\left(L^{z^*}\right)^*$*, satisfies*

$$-\left(K^{z^*}\right)^*(-\mu v^*)-\mu\left(L^{z^*}\right)^*(v^*)\geq 0.$$

3.2 The case that μ is a non-negative value

For any real number $\mu \geq 0$, it is easy to see that the objective function of the problem (P_μ) is a convex function. Then, the problem (P_μ) is a convex programming problem. For given $z^* \in Z^*$, denote $(K + \mu L)^{z^*}(x) = K(x, z^*) + \mu L(x, z^*)$. It is easy to see that

$$\begin{aligned}\inf_{x\in X} \left\{K(x, z^*) + \mu L(x, z^*)\right\} &= \inf_{x\in X} \left\{(K + \mu L)^{z^*}(x)\right\} \\ &= -\sup_{x\in X} \left\{-(K + \mu L)^{z^*}(x)\right\} \\ &= -\left(K^{z^*} + \mu L^{z^*}\right)^*(0).\end{aligned}$$

Let

$$\psi_\mu(z^*) = -\left(K^{z^*} + \mu L^{z^*}\right)^*(0). \tag{22}$$

Then, we define the following dual problem of (P_μ) as follows:

$$(D_\mu) \qquad \sup_{z^*\in Z^*} \psi_\mu(z^*). \tag{23}$$

In order to characterize the weak and strong dualities between (P_μ) and (D_μ), we need to introduce some new regularity conditions. To this aim, we will make use of the following characteristic set Λ defined by

$$\Lambda := \bigcup_{z^*\in Z^*} \text{epi}\left(K^{z^*} + \mu L^{z^*}\right)^*.$$

Comparing with the expressions of problems (P_μ) and (D_μ), it is easy to see that $\Lambda \cap (\{0\} \times \mathbb{R})$ is associated with (D_μ) and epi $(f - \mu g)^* \cap (\{0\} \times \mathbb{R})$ is associated with (P_μ). Considering the possible relationships between epi $(f - \mu g)^*$ and Λ, we introduce the following regularity conditions.

Definition 3.2 The family (f, g) is said to satisfy

i. the further regularity condition $(\overline{FRC})$, iff

$$\text{epi}\,(f - \mu g)^* \cap (\{0\} \times \mathbb{R}) = \Lambda \cap (\{0\} \times \mathbb{R});$$

ii. the semi-$(\overline{FRC})$ $(\overline{SFRC})$, iff

$$\text{epi}\,(f - \mu g)^* \cap (\{0\} \times \mathbb{R}) \supseteq \Lambda \cap (\{0\} \times \mathbb{R}).$$

Remark 3.2 One should notice that when $\mu < 0$, Lemma 3.2 shows that Ω is equal to $\bigcup_{z^*\in Z^*} \text{epi}\left(K^{z^*} + \mu L^{z^*}\right)^*$. So, if we do not consider the sign of μ, the notions

$(\overline{FRC})$ and $(\overline{SFRC})$ of Definition 3.2 coincide with corresponding ones of Definition 3.1.

Now, we use these regularity conditions to completely characterize weak and strong parametrized Lagrange dualities between (P_μ) and (D_μ).

Theorem 3.4 *The family* (f, g) *satisfies the* $(\overline{SFRC})$ *if and only if the weak duality between* (P_μ) *and* (D_μ) *holds.*

Proof (⇒) Suppose that the weak duality between (P_μ) and (D_μ) does not hold. Then, there exist $\alpha \in \mathbb{R}$ such that $val(P_\mu) < -\alpha < val(D_\mu)$. By (22) and (23), there exists $z^* \in Z^*$, such that $-\left(K^{z^*} + \mu L^{z^*}\right)^*(0) \geq -\alpha$. This means that $(0, \alpha) \in \Lambda$. Since the family (f, g) satisfies the $(\overline{SFRC})$, we get $(0, \alpha) \in \text{epi } (f - \mu g)^*$. By (15), $val(P_\mu) \geq -\alpha$. This contradicts $val(P_\mu) < -\alpha$. Thus, the weak duality between (P_μ) and (D_μ) holds.

(⇐) Suppose that the weak duality between (P_μ) and (D_μ) holds. Let $(0, \alpha) \in \Lambda$. Then, there exists $z^* \in Z^*$, such that

$$(0, \alpha) \in \text{ epi } \left(K^{z^*} + \mu L^{z^*}\right)^*,$$

which means that

$$-\left(K^{z^*} + \mu L^{z^*}\right)^*(0) \geq -\alpha.$$

Then, $val(D_\mu) \geq -\alpha$. So, $val(P_\mu) \geq val(D_\mu) \geq -\alpha$. which implies that $(0, \alpha) \in \text{epi } (f - \mu g)^*$ in terms of (15). Thus, the family (f, g) satisfies the $(\overline{SFRC})$. The proof is complete. □

Theorem 3.5 *The family* (f, g) *satisfies the* $(\overline{FRC})$ *if and only if the strong duality between* (P_μ) *and* (D_μ) *holds.*

Proof (⇒) Suppose that the family (f, g) satisfies the $(\overline{FRC})$. Then, the family (f, g) satisfies the $(\overline{SFRC})$. By Theorem 3.4, $val(P_\mu) \geq val(D_\mu)$. So, it suffices to show that $val(D_\mu) \geq val(P_\mu)$. If $val(P_\mu) = -\infty$, then we get strong duality for (P_μ) and (D_μ) via weak duality. So, we assume that $val(P_\mu) = -\alpha \in \mathbb{R}$. By (15), we get $(0, \alpha) \in \text{epi } (f - \mu g)^*$. Since the family (f, g) satisfies the $(\overline{FRC})$, we have $(0, \alpha) \in \Lambda$. Then, there exists $z^* \in Z^*$, such that $-\left(K^{z^*} + \mu L^{z^*}\right)^*(0) \geq -\alpha$. This follows that $val(D_\mu) \geq -\alpha$. This means that the strong duality between (P_μ) and (D_μ) holds.

(⇐) Assume that the strong duality between (P_μ) and (D_μ) holds. By Theorem 3.4, we only need to prove that

$$\text{epi } (f - \mu g)^* \cap (\{0\} \times \mathbb{R}) \subseteq \Lambda \cap (\{0\} \times \mathbb{R}). \tag{24}$$

In fact, let $(0, \alpha) \in \text{epi } (f - \mu g)^*$. By (15), $val(P_\mu) \geq -\alpha$. Then, from the strong duality, $val(D_\mu) = val(P_\mu) \geq -\alpha$. This follows that there exists $z^* \in Z^*$, such that

$-\left(K^{z^*}+\mu L^{z^*}\right)^*(0) \geq -\alpha$, which means that $(0,\alpha) \in \Lambda$ and (24) holds. The proof is complete. □

Now, we give an example to explain Theorem 3.5.

Example 3.2 Let $X = Y = Z = \mathbb{R}$ and $\mu = 1$. Define $f, g : \mathbb{R} \to \overline{\mathbb{R}}$ by $f = \delta_{[1,+\infty)}$ and

$$g(x) = \begin{cases} -2x, & \text{if } x \geq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, it is easy to see that

$$(f-\mu g)^*(x^*) = \begin{cases} x^*-2, & \text{if } x^* \leq 2, \\ +\infty, & \text{otherwise.} \end{cases}$$

So,

$$\text{epi } (f-\mu g)^* = \left\{(x^*, r) \in R^2 : x^* \leq 2, r \geq x^*-2\right\}.$$

On the other hand, let $F(x,z) = f(x) + z$ and $G(x,z) = -g(x) + z$. Obviously, $F(x,0) = f(x)$ and $G(x,0) = -g(x)$. Moreover, we have

$$K(x, z^*) = \begin{cases} 0, & \text{if } x \geq 1, z^* = -1, \\ -\infty, & \text{if } x \geq 1, z^* \neq -1, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$L(x, z^*) = \begin{cases} 2x, & \text{if } x \geq 1, z^* = -1, \\ -\infty, & \text{if } x \geq 1, z^* \neq -1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then,

$$\left(K^{z^*}+\mu L^{z^*}\right)^*(x^*) = \begin{cases} x^*-2, & \text{if } x^* \leq 2, z^* = -1, \\ +\infty, & \text{otherwise.} \end{cases}$$

So,

$$\Lambda = \left\{(x^*, r) \in \mathbb{R}^2 : x^* \leq 2, r \geq x^*-2\right\}.$$

Obviously,

$$\text{epi } (f-\mu g)^* \cap (\{0\} \times \mathbb{R}) = \Lambda \cap (\{0\} \times \mathbb{R}),$$

and then the further regularity condition $(\overline{FRC})$ holds. It is also easy to see that $val(P_\mu) = val(D_\mu) = 2$. Consequently, the strong duality between (P_μ) and (D_μ) holds.

Now, we prove the main results of this subsection. It serve as the main tools for establishing alternative type theorem for (P) (see Corollary 3.2).

Theorem 3.6 *If the family (f, g) satisfies the $(\overline{FRC})$, then the following statements are equivalent:*

i. $x \in X \Longrightarrow \frac{f(x)}{g(x)} \geq \mu$.
ii. *There exists $z^* \in Z^*$, such that*

$$-\left(K^{z^*} + \mu L^{z^*}\right)^*(0) \geq 0. \tag{25}$$

Proof Suppose that the family (f, g) satisfies the $(\overline{FRC})$. If (i) holds, then, $val(P) \geq \mu$. By Lemma 3.1, $val(P_\mu) \geq 0$. It follows from Theorem 3.5 that $val(D_\mu) = val(P_\mu) \geq 0$, which means that (ii) holds.

Conversely, assume that (ii) holds. Then, there exists $z^* \in Z^*$, satisfies (25). Therefore, it comes that $\sup_{z^* \in Z^*} \left\{-\left(K^{z^*} + \mu L^{z^*}\right)^*(0)\right\} \geq 0$. This means that $val(D_\mu) \geq 0$. By Theorem 3.5, we obtain that $val(P_\mu) \geq 0$. By Lemma 3.1, we get $val(P) \geq \mu$, and then (i) holds. The proof is complete. □

The previous result can be reformulated as a theorem of the alternative in the following way.

Corollary 3.2 *If the family (f, g) satisfies the $(\overline{FRC})$, then precisely one of the following statements is true*

i. $\exists x \in X$, *such that* $\frac{f(x)}{g(x)} < \mu$.
ii. *There exists $z^* \in Z^*$, such that* $-\left(K^{z^*} + \mu L^{z^*}\right)^*(0) \geq 0$.

4 Applications

In this section, we apply the approaches of the previous sections to a special case of our general results, which has been treated in the previous papers.

Let $g(x) = 1$. (P) becomes the following convex optimization problem:

$$(P_1) \qquad \inf_{x \in X} f(x).$$

By using the similar approach presented in Sect. 3.2, we have the following Lagrange dual problem

$$(D_L) \qquad \sup_{z^* \in Z^*} \inf_{x \in X} K(x, z^*).$$

Remark 4.1 One should note that since $K(x, z^*) = \inf_{z \in Z} \{F(x, z) + z^*(z)\}$, we can easily get

$$\inf_{x \in X} K(x, z^*) = - \sup_{x \in X, z \in Z} \{\langle -z^*, z\rangle - F(x, z)\} = -F^*(0, -z^*).$$

And the Lagrange dual problem (D_L) becomes

$$\sup_{z^* \in Z^*} \{-F^*(0, z^*)\},$$

which is called Fenchel conjugate dual problem and has been investigated in [15,27].

Similarly, we introduce the following regularity conditions in order to investigate the duality and Farkas-type results for (P_1).

Definition 4.1 We say that

i. the further regularity condition $(\overline{FRC})_1$ holds, iff

$$\text{epi } f^* \cap (\{0\} \times \mathbb{R}) = \bigcup_{z^* \in Z^*} \text{epi } \left(K^{z^*}\right)^* \cap (\{0\} \times \mathbb{R});$$

ii. the semi-$(\overline{FRC})_1$ $(\overline{SFRC})_1$ holds, iff

$$\text{epi } f^* \cap (\{0\} \times \mathbb{R}) \supseteq \bigcup_{z^* \in Z^*} \text{epi } \left(K^{z^*}\right)^* \cap (\{0\} \times \mathbb{R}).$$

As some consequences of the results which have been treated in Sect. 3.2, we obtain the following results for (P_1).

Theorem 4.1 *The $(\overline{SFRC})_1$ holds if and only if the weak duality between (P_1) and (D_L) holds.*

Theorem 4.2 *The $(\overline{FRC})_1$ holds if and only if the strong duality between (P_1) and (D_L) holds.*

Theorem 4.3 *If the $(\overline{FRC})_1$ holds, then the following statements are equivalent:*

i. $x \in X \Longrightarrow f(x) \geq 0.$
ii. *There exist $z^* \in Z^*$ and $x \in X$, such that $K(x, z^*) \geq 0$.*

Corollary 4.1 *If the $(\overline{FRC})$ holds, then precisely one of the following statements is true*

i. *$\exists x \in X$, such that $f(x) < 0$.*
ii. *There exist $z^* \in Z^*$ and $x \in X$, such that $K(x, z^*) \geq 0$.*

5 Conclusions

In this paper, we present some sufficient conditions, which ensure that the optimal objective value of a fractional programming problem is greater than or equal to a given real constant. The desired results are obtained using the parametrized Lagrange duality approach applied to an optimization problem with convex or DC objective functions. Our results are new and different from some existing ones in the literature. Moreover, the results obtained here underline the connections that exist between Farkas-type results and alternative type theorems and, on the other hand, the duality.

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