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Nonlinear separation concerning E-optimal solution of constrained multi-objective optimization problems

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Abstract This paper aims at investigating optimality conditions in terms of E-optimal solution for constrained multi-objective optimization problems in a general scheme, where E is an improvement set with respect to a nontrivial closed convex point cone with apex at the origin. In the case where E is not convex, nonlinear vector regular weak separation functions and scalar weak separation functions are introduced respectively to realize the separation between the two sets in the image space, and Lagrangian-type optimality conditions are established. These results extend and improve the convex ones in the literature.

Keywords E-optimal solution · Image space analysis · Nonlinear vector separation · Saddle point · Optimality condition

1 Introduction

There are several important solution notions of constrained vector optimization problems like efficiency, weak efficiency, proper efficiency, strong efficiency, strict efficiency and ε-efficiency. Chicco et al. [1] unified these classical solution concepts via a more general solution notion called E-optimal solution, where E is an improvement set with respect to a nontrivial closed convex point cone with apex at the origin denoted by K (see Definition 2.1).

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The image space analysis (for short, ISA) was initiated in [2], it was later shown to be a unified scheme for constrained extremum problems, variational inequalities and any other problems which can be expressed under the form of the impossibility of a parametric system by Giannessi [3,5]. And the impossibility of such a system was reduced to the disjunction of two suitable subsets in the image space (for short, IS), where the two sets are usually denoted by $\mathcal{H}$ and $\mathcal{K}_{\bar{x}}$.

K and D are nontrivial closed convex point cones with apex at the origin, and K has nonempty interior. If $\mathcal{H}$ is convex like $\mathcal{H} := \mathbb{R}_{++} \times D$ in [12,13,16] and $\mathcal{H} := (K \backslash \{0\}) \times D$ or $\mathcal{H} := \text{int}\, K \times D$ in [4,14], scholars have constructed separation functions concerning the corresponding problems. But in terms of nonconvex $\mathcal{H}$, it is much difficult to construct a separation function in order to realize the separation between $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$. On one hand, we fail to define the dual cone of E since E may be not a cone such as $E = [0, +\infty) \times [1, +\infty) \cup [1, +\infty) \times [0, +\infty)$, this makes it impossible to use linear scalar separation functions for $\mathcal{H} := E \times D$ and nonlinear ones must be employed. On the other hand, we may apply vector separation functions, even though there have been some results on convex $\mathcal{H}$ by employing linear vector separation functions [4,8,14,15], but how to construct nonlinear vector separation functions for constrained vector optimization problems still deserves discussion and is still an aporia.

In this paper, we investigate general constrained multi-objective optimization problems in the sense of E-optimal solution. We construct the class of vector regular weak separation functions as well as scalar weak separation functions in a general scheme and illustrate some specific expressions of nonlinear ones, respectively. Generalized Lagrangian functions with respect to both nonlinear vector and scalar separation functions are introduced to discuss saddle point properties, namely, the existence of a saddle point of the corresponding generalized Lagrangian function implies that $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ admit a nonlinear separation. Then we use the important properties to derive Lagrangian-type sufficient optimality conditions.

The rest of this paper are organized as follows. In Sect. 2, we present some basic notions about E-optimal solution and analyze the general features of image space approach for constrained multi-objective optimization problems. In Sect. 3, we introduce nonlinear vector regular weak separation functions and establish Lagrangian-type optimality conditions in the sense of vector separation. In Sect. 4, we introduce nonlinear scalar weak separation functions and establish Lagrangian-type optimality conditions in terms of scalar separation.

2 Preliminaries

In this section, we recall some notations and concepts which will be used in the sequel. Let Y be a normed linear space, the closure, the interior, the boundary and the complement of a set $A \subseteq Y$ are denoted by cl A, int A, bd A and A^c, respectively. For arbitrary finite dimensional Euclidean space $\mathbb{R}^m$, we denote by $0_{\mathbb{R}^m}$ the zero element in $\mathbb{R}^m$.

We concentrate on the following multi-objective optimization problems with equality and inequality constraints:

$$\text{(P)} \quad \min\ f(x) \quad \text{s.t.} \quad x \in S := \{x \in X \mid g(x) \in D\},$$

where X is a nonempty set of $\mathbb{R}^n$, $f : X \to \mathbb{R}^l$, $g : X \to \mathbb{R}^p$ and $D =: 0_{\mathbb{R}^m} \times \mathbb{R}_+^{p-m}$.

Let $K \subseteq \mathbb{R}^l$ be a nontrivial closed convex point cone with apex at the origin, we next state the definition of an improvement set with respect to K.

Definition 2.1 [6] A nonempty set $E \subset \mathbb{R}^l$ is said to be an improvement set for $\leq_K$ (or an improvement set with respect to K) if $0_{\mathbb{R}^l} \notin E$ and E is free disposal, i.e., $E + K = E$. We also call it $\leq_K$-i.s. or i.s. for short if there is no confusion.

An order relation has been introduced in [11], moreover, an order relation is called a partial order if it is reflexive, transitive and antisymmetric. So K defines a partial order and E defines an order relation which states

$$a \geq_K b \Leftrightarrow a - b \in K, \quad \forall a, b \in \mathbb{R}^l; \tag{1}$$

$$a \geq_E b \Leftrightarrow a - b \in E, \quad \forall a, b \in \mathbb{R}^l. \tag{2}$$

In fact, the order relation defined by (2) can be regarded as an extension of the partial order defined by (1) since $K \backslash \{0_{\mathbb{R}^l}\}$ is a special improvement set. Next we introduce an important nonlinear scalarization function.

Definition 2.2 [9] Let Y be a normed linear space with norm $\|\cdot\|$. For a set $A \subset Y$, let the function $\Delta_A : Y \to \mathbb{R} \cup \{\pm\infty\}$ be defined as

$$\Delta_A(y) = d_A(y) - d_{Y \backslash A}(y),$$

where $d_A(y) = \inf_{a \in A} \|y - a\|$ is the distance function, and $d_\emptyset(y) = +\infty$.

Some of its main properties are gathered together in the following proposition.

Proposition 2.1 [17] *If the set A is nonempty and $A \neq Y$, then*

(i) Δ_A *is real-valued;*
(ii) *if* $\operatorname{int} A \neq \emptyset$, *then* $\Delta_A(y) < 0$ *for every* $y \in \operatorname{int} A$;
(iii) $\Delta_A(y) = 0$ *for every* $y \in \operatorname{bd} A$;
(iv) *if* $\operatorname{int} A^c \neq \emptyset$, *then* $\Delta_A(y) > 0$ *for every* $y \in \operatorname{int} A^c$.

Now, we introduce a concept of E-optimal solution for (P).

Definition 2.3 A point $\bar{y}$ is called a E-optimal point of a nonempty set $Q \subset \mathbb{R}^l$ if and only if there exists no $y \in Q$ such that

$$\bar{y} \geq_E y.$$

$\bar{x} \in S$ is a E-optimal solution of (P) if and only if $f(\bar{x})$ is a E-optimal point of the set $f(S)$.

From this definition, we obviously observe that $\bar{x} \in S$ is a E-optimal solution of (P) if and only if $(f(S) - f(\bar{x})) \cap (-E) = \emptyset$, which coincides with Definition 4.1 in [6]. We obtain some special kinds of solutions by taking various E.

(i) If $E = K\backslash\{0_{\mathbb{R}^l}\}$, then E-optimal solution collapses into efficient solution[10];
(ii) If $E = \operatorname{int} K$, then E-optimal solution collapses into weak efficient solution [10];
(iii) If $E = (-K)^c$, then E-optimal solution collapses into strong solution [10];
(iv) If $E = \varepsilon k^0 + K$ with $\varepsilon > 0$ and $k^0 \in K\backslash\{0_{\mathbb{R}^l}\}$, then E-optimal solution collapses into εk^0-efficient solution, i.e., approximate solution (see Definition 2.2 in [7]);
(v) If there exists proper convex pointed cone C with $K\backslash\{0_{\mathbb{R}^l}\} \subset \operatorname{int} C$ such that $(f(S) - f(\bar{x})) \cap (-\operatorname{int} C) = \emptyset$ and take $E = \operatorname{int} C$, then E-optimal solution collapses into GHe-proper solution (see Definition 2.4.4 in [11]).

If we introduce the map $\mathcal{A}_{\bar{x}} : X \to \mathbb{R}^{l+p}$ defined by

$$\mathcal{A}_{\bar{x}}(x) := (f(\bar{x}) - f(x),\ g(x)),$$

and denote

$$\mathcal{K}_{\bar{x}} := \{(u, v) \in \mathbb{R}^{l+p} \mid u = f(\bar{x}) - f(x),\ v = g(x),\ x \in X\} = \mathcal{A}_{\bar{x}}(X),$$
$$\mathcal{H} =: \{(u, v) \in \mathbb{R}^{l+p} \mid u \in E,\ \ v \in D\} = E \times D,$$

where $\mathcal{K}_{\bar{x}}$ is called the image associated with (P), then it is easy to observe that $\bar{x} \in S$ is a E-optimal solution of (P) $\Leftrightarrow \mathcal{K}_{\bar{x}} \cap \mathcal{H} = \emptyset$. We can realize the disjunction between $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ by proving that they lie in two different level sets defined by a separation function, respectively. This motivates us to introduce concepts of separation functions and their level sets.

3 Vector regular weak separation functions

In this section, we investigate nonlinear vector regular weak separation functions concerning E-optimal solution and give concrete examples, then saddle point properties are discussed and Lagrangian-type optimality conditions are derived. By convention, we use the notations:

$$x_1 \not\geq_E x_2 \Leftrightarrow x_1 - x_2 \notin E,\ \forall x_1, x_2 \in \mathbb{R}^l.$$

Definition 3.1 The class of all the functions $\omega : \mathbb{R}^{l+p} \times \Pi \to \mathbb{R}^l$ such that

$$\bigcap_{\pi \in \Pi} lev_E \omega(\cdot; \pi) = \mathcal{H}$$

is called the class of vector regular weak separation functions associated with E-optimal solution, where Π is the parameter set and

$$lev_E \omega(\cdot; \pi) := \{(u, v) \in \mathbb{R}^{l+p} \mid \omega(u, v; \pi) \geq_E 0_{\mathbb{R}^l}\}$$

is E-level set of ω.

Let $\mathcal{T} := \{T = (T_1, T_2, \ldots, T_l) \mid T_i$ is an operator from $\mathbb{R}^p$ to $\mathbb{R}\}$ represent the set containing all the operators from $\mathbb{R}^p$ to $\mathbb{R}^l$, and let O represent the zero operator in $\mathcal{T}$ by convention.

Theorem 3.1 *Take $\{T^\epsilon\}_{\epsilon\in\Omega}$ as a subset of $\mathcal{T}$, where Ω is the parameter set with $\bar{\epsilon} \in \Omega$ such that $T^{\bar{\epsilon}} = O$. Select $e \in \operatorname{int} K$. We define $\omega : \mathbb{R}^{l+p} \times \Omega \to \mathbb{R}^l$ as $\omega(u, v; \epsilon) = u + T^\epsilon v$, where $T^\epsilon = (T_1^\epsilon, T_2^\epsilon, \ldots, T_l^\epsilon) : \mathbb{R}^p \to \mathbb{R}^l$ satisfies*

$$T^\epsilon v \geq_K 0_{\mathbb{R}^l}, \ \forall v \in D, \quad \forall \epsilon \in \Omega; \tag{3}$$

$$\forall \lambda > 0, \forall v \notin D, \ \exists \epsilon_{\lambda,v} \in \Omega \ \text{ s.t. } \ T^{\epsilon_{\lambda,v}} v \in -\lambda e - K. \tag{4}$$

Then ω is a class of vector regular weak separation functions associated with E-optimal solution.

Proof From (3), we get

$$\omega(u, v; \epsilon) = u + T^\epsilon v \in E + K = E, \quad \forall (u, v) \in E \times D, \ \forall \epsilon \in \Omega, \tag{5}$$

then (5) indicates $\bigcap_{\epsilon\in\Omega} lev_E \omega(\cdot; \epsilon) \supseteq \mathcal{H}$. Next we prove the opposite inclusion relation, ab absurdo, suppose that there exists $(\bar{u}, \bar{v}) \notin \mathcal{H}$ such that

$$\omega(\bar{u}, \bar{v}; \epsilon) = \bar{u} + T^\epsilon \bar{v} \geq_E 0_{\mathbb{R}^l}, \quad \forall \epsilon \in \Omega. \tag{6}$$

For $(\bar{u}, \bar{v}) \notin \mathcal{H}$, we discuss the following two cases respectively.

Case 1: $\bar{u} \notin E$, since $T^{\bar{\epsilon}} = O$ for $\bar{\epsilon} \in \Omega$, we get $\omega(\bar{u}, \bar{v}; \bar{\epsilon}) = \bar{u} + O\bar{v} = \bar{u} \notin E$, which is a contradiction with (6).

Case 2: $\bar{u} \in E$, but $\bar{v} \notin D$, from (4), $\forall \lambda > 0$, there exists $\epsilon_{\lambda,\bar{v}} \in \Omega$ such that $T^{\epsilon_{\lambda,\bar{v}}} \bar{v} \in -\lambda e - K$. If (6) holds, since we can find a sufficient large λ' such that $(-\lambda' e - K) \bigcap (-\bar{u} + E) = \emptyset$ and $T^{\epsilon_{\lambda',\bar{v}}} \bar{v} \in -\lambda' e - K$, but there holds $T^{\epsilon_{\lambda',\bar{v}}} \bar{v} \in (-\lambda' e - K) \bigcap (-\bar{u} + E)$ since (6) indicates $T^{\epsilon_{\lambda',\bar{v}}} \bar{v} \in -\bar{u} + E$, we derive a contradiction. The proof is completed. □

The following proposition gives a class of $\{T^\epsilon\}_{\epsilon\in\Omega}$ which satisfy (3) and (4).

Proposition 3.1 *Let Ω be a unbounded set of a finite dimensional Euclidean space $\mathbb{R}^q$ with $0_{\mathbb{R}^q} \in \Omega$. Let $e \in \operatorname{int} K$ be the same with that in Theorem 3.1. If we take $T^\epsilon v = \|\epsilon\| h(v) \cdot e$ such that $h : \mathbb{R}^p \to \mathbb{R}$ satisfies*

$$h(v) \geq 0, \quad \forall v \in D; \tag{7}$$

$$h(v) < 0, \quad \forall v \notin D, \tag{8}$$

then $\{T^\epsilon\}_{\epsilon\in\Omega}$ satisfies (3) and (4), where $\|\cdot\|$ is a norm in $\mathbb{R}^q$.

Proof Since (7) implies $T^\epsilon v = \|\epsilon\| h(v) \cdot e \in K, \ \forall v \in D, \ \forall \epsilon \in \Omega$, we conclude that (3) holds.

$\forall \lambda > 0, \forall v \notin D$, (8) implies $-\frac{\lambda}{h(v)} > 0$, then, since Ω is unbounded, there exists a $\epsilon_{\lambda,v} \in \Omega$ such that $\|\epsilon_{\lambda,v}\| \geq -\frac{\lambda}{h(v)}$. Moreover, $\|\epsilon_{\lambda,v}\| \geq -\frac{\lambda}{h(v)}$ indicates

$\|\epsilon_{\lambda,v}\|h(v) + \lambda \leq 0$ due to (8), then we get $(\|\epsilon_{\lambda,v}\|h(v) + \lambda)e \in -K$. So $T^{\epsilon_{\lambda,v}}v = \|\epsilon_{\lambda,v}\|h(v) \cdot e \in -\lambda e - K$ and (4) holds. □

There are some concrete expressions of the function h in Proposition 3.1.

(i) $h(v) = \min\{-|v_1|, -|v_2|, \ldots, -|v_m|, v_{m+1}, v_{m+2}, \ldots, v_p\}$, where $|\cdot|$ denotes absolute value of a real number in $\mathbb{R}$;
(ii) $h(v) = -\sum_{i=1}^{m}|v_i| + \sum_{j=m+1}^{p}(v_j - |v_j|)$, where $|\cdot|$ is as above;
(iii) $h(v) = -\delta_D(v) + \sum_{i=1}^{p} v_i$, where δ_D represents the indicator function;
(iv) $h(v) = -\Delta_D(v)$, where $\Delta_D(\cdot)$ is the same with that in Definition 2.2.

Remark 3.1 When K satisfies $\mathbb{R}^l_+ \subseteq K$, if we set $\Omega = \prod_{i=1}^{l}\Omega_i$, $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_l) \in \Omega$ with $\epsilon_i \in \Omega_i$, take $\Omega_i = \mathbb{R}_+$, $\forall i \in \{1, 2, \ldots, l\}$, and take $T_i^{\epsilon} : \mathbb{R}^p \to \mathbb{R}$ as $T_i^{\epsilon}v = \epsilon_i P_i(v)$ such that each $P_i(v)$ satisfies (7) and (8), then obviously, $\bar{\epsilon} = (0, 0, \ldots, 0) \in \Omega$ satisfies $T^{\bar{\epsilon}} = O$, $\{T^{\epsilon}\}_{\epsilon\in\Omega}$ also satisfies (3) and (4). Thus, $\omega(u, v; \epsilon) = u + (\epsilon_1 P_1(v), \epsilon_2 P_2(v), \ldots, \epsilon_l P_l(v))$ with $\epsilon \in \Omega = \mathbb{R}^l_+$ is a class of vector regular weak separation functions associated with E-optimal solution.

We now introduce the generalized vector Lagrangian function $\mathcal{L}' : X \times \Omega \to \mathbb{R}^l$ defined by

$$\mathcal{L}'(x, \epsilon) = f(x) - T^{\epsilon}g(x),$$

and a pair $(\bar{x}, \hat{\epsilon}) \in X \times \Omega$ is said to be a saddle point of $\mathcal{L}'(x, \epsilon)$ if and only if

$$\mathcal{L}'(\bar{x}, \epsilon) \not\geq_E \mathcal{L}'(\bar{x}, \hat{\epsilon}) \not\geq_E \mathcal{L}'(x, \hat{\epsilon}), \ \forall x \in X, \quad \forall \epsilon \in \Omega. \tag{9}$$

We first prove a useful inclusion relation which states $E^c - K \subseteq E^c$ before proving the following theorem, ab absurdo, suppose that there exists $x \in E^c$ and $k \in K$ such that $x - k \in E$, then $x \in k + E \subset K + E = E$, which contradicts $x \in E^c$.

Theorem 3.2 *Suppose $\bar{x} \in S$ and there exists $\hat{\epsilon} \in \Omega$ such that*

$$\omega(u, v; \hat{\epsilon}) \not\geq_E 0_{\mathbb{R}^l}, \ \forall (u, v) \in \mathcal{K}_{\bar{x}}. \tag{10}$$

Then $(\bar{x}, \hat{\epsilon})$ is a saddle point of $\mathcal{L}'(x, \epsilon)$.

Proof Suppose that there exists $\hat{\epsilon} \in \Omega$ such that (10) holds, that is

$$f(\bar{x}) - f(x) + T^{\hat{\epsilon}}g(x) \not\geq_E 0_{\mathbb{R}^l}, \ \forall x \in X. \tag{11}$$

Take $x = \bar{x}$ in (11), we obtain $T^{\hat{\epsilon}}g(\bar{x}) \not\geq_E 0_{\mathbb{R}^l}$. By (3), $\bar{x} \in S$ implies $T^{\hat{\epsilon}}g(\bar{x}) \geq_K 0_{\mathbb{R}^l}$. Then,

$$\mathcal{L}'(\bar{x}, \hat{\epsilon}) - \mathcal{L}'(x, \hat{\epsilon}) = f(\bar{x}) - f(x) + T^{\hat{\epsilon}}g(x) - T^{\hat{\epsilon}}g(\bar{x}) \in E^c - K \subseteq E^c,$$

so we proved $\mathcal{L}'(\bar{x}, \hat{\epsilon}) \not\geq_E \mathcal{L}'(x, \hat{\epsilon})$, $\forall x \in X$. On the other hand, again by (3), we get $T^{\epsilon}g(\bar{x}) \geq_K 0_{\mathbb{R}^l}$, $\forall \epsilon \in \Omega$, then it follows from $T^{\hat{\epsilon}}g(\bar{x}) \not\geq_E 0_{\mathbb{R}^l}$ that

$$\mathcal{L}'(\bar{x}, \epsilon) - \mathcal{L}'(\bar{x}, \hat{\epsilon}) = T^{\hat{\epsilon}}g(\bar{x}) - T^{\epsilon}g(\bar{x}) \in E^c - K \subseteq E^c,$$

so we proved $\mathcal{L}'(\bar{x}, \epsilon) \ngeq_E \mathcal{L}'(\bar{x}, \hat{\epsilon}),\ \forall \epsilon \in \Omega$. □

If we want to prove the conclusion in the opposite direction of Theorem 3.2, we need an additional assumption.

Theorem 3.3 *If there exists $\hat{\epsilon} \in \Omega$ such that $(\bar{x}, \hat{\epsilon})$ is a saddle point of $\mathcal{L}'(x, \epsilon)$, suppose that $E^c \cap K = \{0_{\mathbb{R}^l}\}$ or $T^{\hat{\epsilon}} g(\bar{x}) = 0_{\mathbb{R}^l}$ is satisfied, then $\bar{x} \in S$ and (10) holds.*

Proof If there exists $\hat{\epsilon} \in \Omega$ such that $(\bar{x}, \hat{\epsilon})$ is a saddle point of $\mathcal{L}'(x, \epsilon)$, then (9) holds. We prove $\bar{x} \in S$, ab absurdo, suppose $g(\bar{x}) \notin D$, for brevity of notation, we denote $\bar{v} = g(\bar{x})$, then by (4), $\forall \lambda > 0$, there exists $\epsilon_{\lambda,\bar{v}} \in \Omega$ such that $-T^{\epsilon_{\lambda,\bar{v}}} \bar{v} \in \lambda e + K$, so we can find a sufficient large λ' such that $-T^{\epsilon_{\lambda',\bar{v}}} \bar{v} \in -T^{\hat{\epsilon}} g(\bar{x}) + E$, which is a contradiction with the first inequality of (9).

Take $T^{\bar{\epsilon}} = O$ in the first inequality of (9), we have $T^{\hat{\epsilon}} g(\bar{x}) \ngeq_E 0_{\mathbb{R}^l}$. If $E^c \cap K = \{0_{\mathbb{R}^l}\}$ is satisfied, since $\bar{x} \in S$, we deduce $T^{\hat{\epsilon}} g(\bar{x}) = 0_{\mathbb{R}^l}$ by (3). And $T^{\hat{\epsilon}} g(\bar{x}) = 0_{\mathbb{R}^l}$ holds naturally if the second assumption of the theorem is satisfied. Moreover, $T^{\hat{\epsilon}} g(\bar{x}) = 0_{\mathbb{R}^l}$ and the second inequality of (9) imply that (10) holds. □

Then, Lagrangian-type optimality conditions for (P) related to E-optimal solution in the sense of vector separation are established.

Theorem 3.4 *If there exists $\hat{\epsilon} \in \Omega$ such that $(\bar{x}, \hat{\epsilon})$ is a saddle point of $\mathcal{L}'(x, \epsilon)$, suppose that $E^c \cap K = \{0_{\mathbb{R}^l}\}$ or $T^{\hat{\epsilon}} g(\bar{x}) = 0_{\mathbb{R}^l}$ is satisfied, then $\bar{x}$ is a E-optimal solution of (P).*

Proof Theorems 3.1 and 3.3 derive $\mathcal{K}_{\bar{x}} \cap \mathcal{H} = \emptyset$ and $\bar{x} \in S$, so we obtain that $\bar{x}$ is a E-optimal solution of (P).

We give an example where E is neither a cone nor a convex set to demonstrate Theorem 3.4.

Example 3.1 In problem (P), take $n = l = 2,\ m = 1,\ p = 2$ and $X = [0, 2] \times [0, 2]$, then $D = 0 \times \mathbb{R}_+$. For $x = (x_1, x_2) \in X$, set $f(x) = x$, $g_1(x) = x_2 - x_1 = 0$, $g_2(x) = x_1 - x_2^2 + 1 \geq 0$. Let $K = \mathbb{R}^2_+$ and $E = [0, +\infty) \times [1, +\infty) \cup [1, +\infty) \times [0, +\infty)$.

By Remark 3.1, we use the separation function $\omega(u, v; \epsilon) = u + T^{\epsilon} v$ with $\epsilon \in \Omega = \mathbb{R}^2_+$, where $T^{\epsilon} v = (\epsilon_1 P_1(v), \epsilon_2 P_2(v))$. We take $P_1(v) = P_2(v) = -\Delta_D(v)$. For $\bar{x} \in \{(a, a) \mid 0 \leq a < 1\}$ and $\hat{\epsilon} = (1, 1)$, let us very that $(\bar{x}, \hat{\epsilon})$ is a saddle point of $\mathcal{L}'(x, \epsilon)$. We calculate $g(\bar{x}) = (0, a - a^2 + 1)$ and $-\Delta_D(g(\bar{x})) = 0$ since $a - a^2 + 1 > 0$ for $0 \leq a < 1$, then we get $T^{\epsilon} g(\bar{x}) = 0_{\mathbb{R}^2},\ \forall \epsilon \in \mathbb{R}^2_+$, so

$$\mathcal{L}'(\bar{x}, \epsilon) - \mathcal{L}'(\bar{x}, \hat{\epsilon}) = 0_{\mathbb{R}^2} \notin E, \quad \forall \epsilon \in \mathbb{R}^2_+. \tag{12}$$

Since $-\Delta_D(v) < 0,\ \forall v \notin D$ and $-\Delta_D(v) = 0,\ \forall v \in D$, we obtain

$$\mathcal{L}'(\bar{x}, \hat{\epsilon}) - \mathcal{L}'(x, \hat{\epsilon}) = (a - x_1 - \Delta_D(g(x)), a - x_2 - \Delta_D(g(x))) \notin E,\ \forall x \in X. \tag{13}$$

(12) and (13) imply that (9) holds. Moreover, we have $T^{\hat{\epsilon}} g(\bar{x}) = 0_{\mathbb{R}^2}$, then by Theorem 3.4, $\{(a, a) \mid 0 \leq a < 1\}$ are E-optimal solutions of this problem. Indeed, $\{(a, a) \mid 0 \leq a < 1\}$ to be E-optimal solutions can be obtained by direct calculation.

Next, we demonstrate that the conditions used in Theorem 3.4 could not be further simplified.

Example 3.2 In problem (P), take $n = l = 2,\ m = 0,\ p = 2$ and $X = (0, 1] \times [0, 1]$, then $D = \mathbb{R}^2_+$. For $x = (x_1, x_2) \in X$, set $f(x) = x$, $g_1(x) = x_1 - x_2 + \frac{1}{2} \geq 0,\ g_2(x) = x_2 - x_1 + \frac{1}{2} \geq 0$. Let $K = \mathbb{R}^2_+$ and $E = [0, +\infty) \times [1, +\infty) \cup [1, +\infty) \times [0, +\infty)$.

By Remark 3.1, we use the separation function $\omega(u, v; \epsilon) = u + T^\epsilon v$ with $\epsilon \in \Omega = \mathbb{R}^2_+$, where $T^\epsilon v = (\epsilon_1 P_1(v), \epsilon_2 P_2(v))$. We take $P_1(v) = P_2(v) = \min\{v_1,\ v_2\}$. Obviously, the feasible point $\bar{x} = (1, 1)$ is not a E-optimal solution. We calculate $f(\bar{x}) = (1, 1)$, $g(\bar{x}) = (\frac{1}{2}, \frac{1}{2})$ and

$$\mathcal{L}'(\bar{x}, \epsilon) - \mathcal{L}'(\bar{x}, \hat{\epsilon}) = \left(\frac{1}{2}(\hat{\epsilon}_1 - \epsilon_1),\ \frac{1}{2}(\hat{\epsilon}_2 - \epsilon_2)\right). \tag{14}$$

Moreover, we calculate

$$f(x) - T^{\hat{\epsilon}} g(x) = \begin{cases} (x_1 - \hat{\epsilon}_1 g_1(x),\ x_2 - \hat{\epsilon}_2 g_1(x)) & \text{if } x_1 \leq x_2, \\ (x_1 - \hat{\epsilon}_1 g_2(x),\ x_2 - \hat{\epsilon}_2 g_2(x)) & \text{if } x_1 > x_2. \end{cases} \tag{15}$$

For every $\hat{\epsilon} \in \mathbb{R}^2_+$ satisfying $\hat{\epsilon}_1 \geq 2$ or $\hat{\epsilon}_2 \geq 2$, $(\bar{x}, \hat{\epsilon})$ could not be a saddle point of $\mathcal{L}'(x, \epsilon)$. Indeed, there exists $\epsilon = (0, 0) \in \mathbb{R}^2_+$ such that $\mathcal{L}'(\bar{x}, \epsilon) - \mathcal{L}'(\bar{x}, \hat{\epsilon}) = (\frac{1}{2}\hat{\epsilon}_1, \frac{1}{2}\hat{\epsilon}_2) \in E$ by (14).

For every $\hat{\epsilon} \in [0, 2) \times \{0\}$, $(\bar{x}, \hat{\epsilon})$ could not be a saddle point of $\mathcal{L}'(x, \epsilon)$. Indeed, there exists $x = (\frac{1}{1+\hat{\epsilon}_1}, 0) \in X$ such that $\mathcal{L}'(\bar{x}, \hat{\epsilon}) - \mathcal{L}'(x, \hat{\epsilon}) = (0, 1) \in E$ by (15).

For every $\hat{\epsilon} \in [0, 2) \times (0, 2)$, we very that $(\bar{x}, \hat{\epsilon})$ is a saddle point of $\mathcal{L}'(x, \epsilon)$. Indeed, (14) implies $\mathcal{L}'(\bar{x}, \epsilon) - \mathcal{L}'(\bar{x}, \hat{\epsilon}) \notin E,\ \forall \epsilon \in \mathbb{R}^2_+$ since $\frac{1}{2}(\hat{\epsilon}_1 - \epsilon_1) < 1$ and $\frac{1}{2}(\hat{\epsilon}_2 - \epsilon_2) < 1$. Then $\mathcal{L}'(\bar{x}, \epsilon) \not\geq_E \mathcal{L}'(\bar{x}, \hat{\epsilon}),\ \forall \epsilon \in \mathbb{R}^2_+$. On the other hand, for $x \in X$ satisfying $x_1 \leq x_2$, we have $0 < x_1 \leq 1$, $0 < x_2 \leq 1$ and

$$\mathcal{L}'(\bar{x}, \hat{\epsilon}) - \mathcal{L}'(x, \hat{\epsilon}) = (1 - x_1 + \hat{\epsilon}_1(x_1 - x_2),\ 1 - x_2 + \hat{\epsilon}_2(x_1 - x_2)) \notin E.$$

For $x \in X$ satisfying $x_1 > x_2$, we have $0 < x_1 \leq 1$, $0 \leq x_2 < 1$ and

$$\mathcal{L}'(\bar{x}, \hat{\epsilon}) - \mathcal{L}'(x, \hat{\epsilon}) = (1 - x_1 + \hat{\epsilon}_1(x_2 - x_1),\ 1 - x_2 + \hat{\epsilon}_2(x_2 - x_1)) \notin E.$$

Thus, $\mathcal{L}'(\bar{x}, \hat{\epsilon}) \not\geq_E \mathcal{L}'(x, \hat{\epsilon}),\ \forall x \in X$. However, $E^c \cap K \neq \{0_{\mathbb{R}^2}\}$ and $T^{\hat{\epsilon}} g(\bar{x}) \neq 0_{\mathbb{R}^2}$ due to $T^{\hat{\epsilon}} g(\bar{x}) = (\frac{1}{2}\hat{\epsilon}_1, \frac{1}{2}\hat{\epsilon}_2)$ and $\hat{\epsilon} \in [0, 2) \times (0, 2)$. This demonstrates that the conditions used in Theorem 3.4 could not be further simplified to ensure $\bar{x}$ being a E-optimal solution.

4 Scalar weak separation functions

In this section, we introduce nonlinear scalar weak separation functions related to E-optimal solution and discuss Lagrangian-type optimality conditions.

Definition 4.1 The class of all the functions $\omega : \mathbb{R}^{l+p} \times \Pi \to \mathbb{R}$ such that

(i) $lev_{\geq 0}\omega(\cdot;\pi) \supset \mathcal{H}, \ \forall \pi \in \Pi$;
(ii) $\bigcap_{\pi\in\Pi} lev_{>0}\omega(\cdot;\pi) \subset \mathcal{H}$

is called the class of scalar weak separation functions associated with E-optimal solution, where Π is the parameter set, and

$$lev_{\geq 0}\omega(\cdot;\pi) := \{(u,v) \in \mathbb{R}^{l+p} \mid \omega(u,v;\pi) \geq 0\} \tag{16}$$

is nonnegative level set of w, if we substitute $>$ for $\geq$ in (16), it is called positive level set.

We give a special class of nonlinear scalar weak separation functions.

Theorem 4.1 *We select a family of sets $\{E_\xi\}_{\xi\in\Xi}$ such that*

$$\bigcap_{\xi\in\Xi} E_\xi = E;$$
$$E \subseteq E_\xi, \quad \forall \xi \in \Xi. \tag{17}$$

Let the parameter set $\Pi = \Xi\times\Gamma$ with $(\xi,\gamma) \in \Xi\times\Gamma$. We define $\omega : \mathbb{R}^{l+p}\times\Xi\times\Gamma \to \mathbb{R}$ as $\omega(u,v;\xi,\gamma) = -\Delta_{E_\xi}(u) + \underline{\omega}(v;\gamma)$, where $\underline{\omega} : \mathbb{R}^p \times \Gamma \to \mathbb{R}$ satisfies

$$\forall v \in D, \ \exists \gamma_v \in \Gamma \ \text{ s.t. } \ \underline{\omega}(v;\gamma_v) = \min_{\gamma\in\Gamma} \underline{\omega}(v;\gamma) = 0; \tag{18}$$

$$\forall v \notin D, \ \inf_{\gamma\in\Gamma} \underline{\omega}(v;\gamma) = -\infty. \tag{19}$$

Then ω is a class of scalar weak separation functions associated with E-optimal solution.

Proof From (17) and Proposition 2.1, we obtain $-\Delta_{E_\xi}(u) \geq 0, \ \forall u \in E, \forall \xi \in \Xi$. And we get $\underline{\omega}(v;\gamma) \geq 0, \ \forall v \in D, \forall \gamma \in \Gamma$ because of (18). Then it follows that $lev_{\geq 0}\omega(\cdot;\pi) \supset \mathcal{H}, \ \forall \pi \in \Pi$. In order to prove $\bigcap_{\pi\in\Pi} lev_{>0}\omega(\cdot;\pi) \subset \mathcal{H}$, we suppose that there exists $(\bar{u},\bar{v}) \notin \mathcal{H}$ such that

$$\omega(\bar{u},\bar{v};\xi,\gamma) > 0, \ \forall (\xi,\gamma) \in \Xi \times \Gamma. \tag{20}$$

For $(\bar{u},\bar{v}) \notin \mathcal{H}$, we discuss the following two cases respectively.

Case 1: $\bar{u} \notin E$, then $\bar{u} \notin E = \bigcap_{\xi\in\Xi} E_\xi$, this implies that there exists $\xi_0 \in \Xi$ such that $\bar{u} \notin E_{\xi_0}$, so we conclude $-\Delta_{E_{\xi_0}}(\bar{u}) \leq 0$ from Proposition 2.1. If $\bar{v} \in D$, by (18), there exists $\gamma_{\bar{v}} \in \Gamma$ such that $\underline{\omega}(\bar{v};\gamma_{\bar{v}}) = 0$. If $\bar{v} \notin D$, from (19), there exists $\gamma_{\bar{v}} \in \Gamma$ such that $\underline{\omega}(\bar{v};\gamma_{\bar{v}}) < 0$. So we conclude $\omega(\bar{u},\bar{v};\xi_0,\gamma_{\bar{v}}) = -\Delta_{E_{\xi_0}}(\bar{u}) + \underline{\omega}(\bar{v};\gamma_{\bar{v}}) \leq 0$, which contradicts (20).

Case 2: $\bar{u} \in E$, but $\bar{v} \notin D$. (19) implies that for an arbitrary real number $x \geq 0$, we can find a $\gamma_x \in \Gamma$ such that $\underline{\omega}(\bar{v};\gamma_x) < -x$. So for a fixed $\xi_0 \in \Xi$, we can find a $\bar{\gamma} \in \Gamma$ such that $\omega(\bar{u},\bar{v};\xi_0,\bar{\gamma}) = -\Delta_{E_{\xi_0}}(\bar{u}) + \underline{\omega}(\bar{v};\bar{\gamma}) < 0$ since $\bar{u} \in E$ indicates $-\Delta_{E_{\xi_0}}(\bar{u}) \geq 0$, which is a contradiction with (20). We complete the proof. □

The following proposition gives a class of $\underline{\omega}(v;\gamma)$ which satisfy (18) and (19).

Proposition 4.1 *Let Γ be a unbounded set of a finite dimensional Euclidean space $\mathbb{R}^q$ with $0_{\mathbb{R}^q} \in \Gamma$. Take $\underline{\omega}(v;\gamma) = \|\gamma\|h(v)$, where $h(v)$ is the same with that in Proposition 3.1 and $\|\cdot\|$ is a norm in $\mathbb{R}^q$. Then $\underline{\omega}(v;\gamma)$ satisfies (18) and (19).*

Proof From (7), we have $\underline{\omega}(v;\gamma) = \|\gamma\|h(v) \geq 0, \ \forall v \in D, \ \forall \gamma \in \Gamma$. Moreover, $\forall v \in D$, there exists $\gamma_v = 0_{\mathbb{R}^q} \in \Gamma$ such that $\underline{\omega}(v;\gamma_v) = \|0_{\mathbb{R}^q}\|h(v) = 0$. So (18) holds.

Since Γ is unbounded, there exists a sequence $\{\gamma_i\}_{i=1}^{\infty}$ of Γ such that $\|\gamma_i\| \to +\infty$ as $i \to \infty$. Then we get from (8) that $\forall v \notin D$, $\underline{\omega}(v;\gamma_i) = \|\gamma_i\|h(v) \to -\infty$ as $i \to \infty$. So (19) holds. □

Moreover, we give some other examples of $\underline{\omega}(v;\gamma)$ satisfying (18) and (19).

(i) $\underline{\omega}(v;\gamma) = \langle \gamma, v\rangle$ with $\gamma \in D^*$, where $D^* := \{z \in \mathbb{R}^p \mid \langle z, y\rangle \geq 0, \ \forall y \in D\}$ is the dual cone or positive polar cone of D;
(ii) $\underline{\omega}(v;\gamma) = -\Delta_{\mathbb{R}_+}(\langle \gamma, v\rangle)$ with $\gamma \in D^*$, where D^* is as above;
(iii) $\underline{\omega}(v;\gamma) = \sup_{z\in v-D}[\langle \gamma, z\rangle - c\sigma(z)]$ with $\gamma \in \mathbb{R}^p$, where $c > 0$ is a real constant and the function $\sigma : \mathbb{R}^p \to \mathbb{R}$ is such that $\arg\min_{z\in\mathbb{R}^p} \sigma(z) = 0_{\mathbb{R}^p}$, and $\sigma(0_{\mathbb{R}^p}) = 0$.

The following concept of separation is useful to establish optimality conditions.

Definition 4.2 The sets $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ admit a nonlinear separation if and only if there exists $(\xi_0, \gamma_0) \in \Xi \times \Gamma$ such that

$$\omega(u, v; \xi_0, \gamma_0) = -\Delta_{E_{\xi_0}}(u) + \underline{\omega}(v;\gamma_0) \leq 0, \quad \forall (u, v) \in \mathcal{K}_{\bar{x}}. \tag{21}$$

Now, we establish a preliminary sufficient optimality condition for (P).

Theorem 4.2 *If there exists $(\xi_0, \gamma_0) \in \Xi \times \Gamma$ such that $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ admit a nonlinear separation, suppose $E \subseteq \operatorname{int} E_{\xi_0}$, then $\bar{x} \in S$ is a E-optimal solution of (P).*

Proof If there exists $(\xi_0, \gamma_0) \in \Xi \times \Gamma$ such that $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ admit a nonlinear separation, then (21) holds. From Theorem 4.1, we know that ω is a class of scalar weak separation functions associated with E-optimal solution, so we have

$$\omega(u, v; \xi_0, \gamma_0) \geq 0, \quad \forall (u, v) \in \mathcal{H}. \tag{22}$$

Since $E \subseteq \operatorname{int} E_{\xi_0}$, we get $-\Delta_{E_{\xi_0}}(u) > 0, \ \forall u \in E$ due to Proposition 2.1, then we obtain that (22) holds for strict inequality. This combined with (21) implies $\mathcal{K}_{\bar{x}} \cap \mathcal{H} = \emptyset$, which derives that $\bar{x} \in S$ is a E-optimal solution of (P).

For a given $\xi_0 \in \Xi$, we introduce the generalized scalar Lagrangian function $\mathcal{L} : X \times \Gamma \to \mathbb{R}$ defined by

$$\mathcal{L}(\bar{x}, x; \xi_0, \gamma) = \Delta_{E_{\xi_0}}(f(\bar{x}) - f(x)) - \underline{\omega}(g(x);\gamma),$$

and a pair $(\bar{x}, \gamma_0) \in X \times \Gamma$ is called a saddle point of $\mathcal{L}(\bar{x}, x; \xi_0, \gamma)$ if and only if

$$\mathcal{L}(\bar{x}, \bar{x}; \xi_0, \gamma) \leq \mathcal{L}(\bar{x}, \bar{x}; \xi_0, \gamma_0) \leq \mathcal{L}(\bar{x}, x; \xi_0, \gamma_0), \ \forall x \in X, \quad \forall \gamma \in \Gamma. \tag{23}$$

Theorem 4.3 *If there exists* $(\xi_0, \gamma_0) \in \Xi \times \Gamma$ *such that* $(\bar{x}, \gamma_0)$ *is a saddle point of* $\mathcal{L}(\bar{x}, x; \xi_0, \gamma)$*, suppose* $\Delta_{E_{\xi_0}}(0_{\mathbb{R}^l}) \geq 0$*, then* $\mathcal{K}_{\bar{x}}$ *and* $\mathcal{H}$ *admit a nonlinear separation and* $\bar{x} \in S$.

Proof If there exists $(\xi_0, \gamma_0) \in \Xi \times \Gamma$ such that $(\bar{x}, \gamma_0)$ is a saddle point of $\mathcal{L}(\bar{x}, x; \xi_0, \gamma)$, then (23) holds. We state $\bar{x} \in S$, ab absurdo, suppose $g(\bar{x}) \notin D$, then (19) indicates that there exists $\bar{\gamma} \in \Gamma$ such that $\underline{\omega}(g(\bar{x}); \bar{\gamma}) \to -\infty$, which is a contradiction with the first inequality of (23).

For simplicity, we denote $\bar{v} = g(\bar{x})$. Since $\bar{x} \in S$, we get from (18) that $\underline{\omega}(g(\bar{x}); \gamma_0) \geq 0$ and that there exists $\gamma_{\bar{v}} \in \Gamma$ such that $\underline{\omega}(g(\bar{x}); \gamma_{\bar{v}}) = 0$. Take $\gamma = \gamma_{\bar{v}}$ in the first inequality of (23), we obtain $\underline{\omega}(g(\bar{x}); \gamma_0) \leq 0$. So we conclude $\underline{\omega}(g(\bar{x}); \gamma_0) = 0$. Combining $\underline{\omega}(g(\bar{x}); \gamma_0) = 0$, $\Delta_{E_{\xi_0}}(0_{\mathbb{R}^l}) \geq 0$ and the second inequality of (23), we obtain (21), the proof is completed. □

Then, Theorems 4.2 and 4.3 deduce Lagrangian-type sufficient optimality conditions for (P) related to E-optimal solution in terms of scalar separation.

Theorem 4.4 *If there exists* $(\xi_0, \gamma_0) \in \Xi \times \Gamma$ *such that* $(\bar{x}, \gamma_0)$ *is a saddle point of* $\mathcal{L}(\bar{x}, x; \xi_0, \gamma)$*, suppose* $E \subseteq \operatorname{int} E_{\xi_0}$ *and* $\Delta_{E_{\xi_0}}(0_{\mathbb{R}^l}) \geq 0$*, then* $\bar{x}$ *is a* E*-optimal solution of (P).*

We give an example where E is neither a cone nor a convex set to demonstrate Theorem 4.4.

Example 4.1 In problem (P), take $n = l = 2$, $m = 0$, $p = 2$ and $X = [-1, 1] \times [0, 1]$, then $D = \mathbb{R}^2_+$. For $x = (x_1, x_2) \in X$, set $f(x) = x$, $g_1(x) = x_2 + x_1 \geq 0$, $g_2(x) = x_2 - x_1 + \frac{1}{2} \geq 0$. Let $K = \mathbb{R}^2_+$ and $E = (-1, +\infty) \times (0, +\infty) \cup \{(x_1, x_2) \mid -x_1 < x_2 \leq 0\}$.

We use the separation function in Theorem 4.1. Let $\Xi = \{\xi\}$ and $E_\xi = E$. According to Proposition 4.1, we take $\underline{\omega}(v; \gamma) = |\gamma| h(v)$ with $\gamma \in \Gamma = \mathbb{R}_+$, where $h(v) = v_1 - |v_1| + v_2 - |v_2|$. For $\bar{x} = (0, 0)$, we have $g(\bar{x}) = (0, \frac{1}{2})$ and $h(g(\bar{x})) = 0$.

For every $\gamma_0 \geq \frac{\sqrt{2}}{4}$, let us very that $(\bar{x}, \gamma_0)$ is a saddle point of $\mathcal{L}(\bar{x}, x; \xi, \gamma)$. By $h(g(\bar{x})) = 0$, we obtain $\mathcal{L}(\bar{x}, \bar{x}; \xi, \gamma) \leq \mathcal{L}(\bar{x}, \bar{x}; \xi, \gamma_0)$, $\forall \gamma \in \Gamma$. When $x \in X$ satisfying $g_1(x) = x_2 + x_1 \geq 0$, then $f(\bar{x}) - f(x) \in (E_\xi)^c$. Since $h(v) < 0$, $\forall v \notin D$ and $h(v) = 0$, $\forall v \in D$, we get

$$\mathcal{L}(\bar{x}, x; \xi, \gamma_0) = \Delta_{E_\xi}(f(\bar{x}) - f(x)) - |\gamma_0| h(g(x)) \geq 0 = \mathcal{L}(\bar{x}, \bar{x}; \xi, \gamma_0). \quad (24)$$

When $x \in X$ satisfying $g_1(x) = x_2 + x_1 < 0$, then $g_2(x) = x_2 - x_1 + \frac{1}{2} > 0$. So $\gamma_0 \geq \frac{\sqrt{2}}{4}$ implies

$$\begin{aligned}\mathcal{L}(\bar{x}, x; \xi, \gamma_0) &= \Delta_{E_\xi}(f(\bar{x}) - f(x)) - |\gamma_0| h(g(x)) = \frac{\sqrt{2}}{2}(x_1 + x_2) - 2\gamma_0(x_1 + x_2) \\ &= \left(\frac{\sqrt{2}}{2} - 2\gamma_0\right)(x_1 + x_2) \geq 0 = \mathcal{L}(\bar{x}, \bar{x}; \xi, \gamma_0). \quad (25)\end{aligned}$$

(24) and (25) imply $\mathcal{L}(\bar{x}, \bar{x}; \xi, \gamma_0) \le \mathcal{L}(\bar{x}, x; \xi, \gamma_0),\ \forall x \in X$. Moreover, $E \subseteq \operatorname{int} E = \operatorname{int} E_\xi$ and $\Delta_{E_\xi}(0_{\mathbb{R}^2}) = 0 \ge 0$, then by Theorem 4.4, $\bar{x} = (0, 0)$ is a E-optimal solution of this problem. Indeed, $\bar{x} = (0, 0)$ to be a E-optimal solution can be obtained by direct calculation.

Next, we demonstrate that the condition $E \subseteq \operatorname{int} E_{\xi_0}$ in Theorem 4.4 is not redundant.

Example 4.2 In problem (P), take $n = l = 2,\ m = 0,\ p = 2$ and $X = [-1, 1] \times [0, 1]$, then $D = \mathbb{R}^2_+$. For $x = (x_1, x_2) \in X$, set $f(x) = x$, $g_1(x) = x_2 + x_1 \ge 0$, $g_2(x) = x_2 - x_1 + \frac{1}{2} \ge 0$. Let $K = \mathbb{R}^2_+$ and $E = (-1, +\infty) \times (0, +\infty) \cup \{(x_1, x_2) \mid -x_1 \le x_2 \le 0\} \backslash \{0_{\mathbb{R}^2}\}$.

We use the separation function in Theorem 4.1. Let $\Xi = \{\xi\}$ and $E_\xi = E$. According to Proposition 4.1, we take $\underline{\omega}(v; \gamma) = |\gamma| h(v)$ with $\gamma \in \Gamma = \mathbb{R}_+$, where $h(v) = v_1 - |v_1| + v_2 - |v_2|$. Obviously, the feasible point $\bar{x} = (0, 0)$ is not a E-optimal solution.

For every $\gamma_0 \in [0, \frac{\sqrt{2}}{4})$, $(\bar{x}, \gamma_0)$ could not be a saddle point of $\mathcal{L}(\bar{x}, x; \xi, \gamma)$. Indeed, there exists $x = (-1, 0) \in X$ such that $\mathcal{L}(\bar{x}, x; \xi, \gamma_0) = -\frac{\sqrt{2}}{2} + 2\gamma_0 < 0 = \mathcal{L}(\bar{x}, \bar{x}; \xi, \gamma_0)$.

For every $\gamma_0 \ge \frac{\sqrt{2}}{4}$, by using the same technique with that in Example 4.1, we very that $(\bar{x}, \gamma_0)$ is a saddle point of $\mathcal{L}(\bar{x}, x; \xi, \gamma)$. Also, $\Delta_{E_\xi}(0_{\mathbb{R}^2}) = 0 \ge 0$. However, $E \nsubseteq \operatorname{int} E = \operatorname{int} E_\xi$ since E is not open. This demonstrates that the condition $E \subseteq \operatorname{int} E_{\xi_0}$ in Theorem 4.4 is not redundant to ensure $\bar{x}$ being a E-optimal solution.

At last, we demonstrate that the condition $\Delta_{E_{\xi_0}}(0_{\mathbb{R}^l}) \ge 0$ in Theorem 4.4 is not redundant.

Example 4.3 In problem (P), take $n = l = 2,\ m = 0,\ p = 2$, then $D = \mathbb{R}^2_+$. Take $X = \mathbb{B}((0, 1), 1) := \{(x_1, x_2) \mid x_1^2 + (x_2 - 1)^2 \le 1\}$. For $x = (x_1, x_2) \in X$, set $f(x) = x$, $g_1(x) = x_1 - x_2 + 1 \ge 0$, $g_2(x) = -2x_1 - x_2 + 2 \ge 0$. Let $K = \mathbb{R}^2_+$ and $E = \{(x_1, x_2) \mid x_2 \ge 0.1\} \cup \{(x_1, x_2) \mid x_2 \ge -x_1 + 0.1\}$.

We use the separation function in Theorem 4.1. Let $\{E_{\xi_n}\}_{\xi_n \in \Xi}$ with $\Xi = \{\xi_n\}_{n=1}^\infty$ satisfy

$$E_{\xi_n} = \left\{(x_1, x_2) \mid x_2 > 0.1 - \frac{2}{n}\right\} \cup \left\{(x_1, x_2) \mid x_2 > -x_1 + 0.1 - \frac{2}{n}\right\}.$$

According to Proposition 4.1, we take $\underline{\omega}(v; \gamma) = |\gamma| h(v)$ with $\gamma \in \Gamma = \mathbb{R}_+$, where $h(v) = v_1 - |v_1| + v_2 - |v_2|$. Obviously, the feasible point $\bar{x} = (0, 0)$ is not a E-optimal solution. We calculate $g(\bar{x}) = (1, 2)$ and $h(g(\bar{x})) = 0$.

For every $n \ge 20$ and every $\gamma_0 \in \Gamma$, $(\bar{x}, \gamma_0)$ could not be a saddle point of $\mathcal{L}(\bar{x}, x; \xi_n, \gamma)$. Indeed, there exists $x = (-\frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2}) \in X$ satisfying $-x \in \operatorname{int} E_{\xi_n}$ due to $0.1 - \frac{2}{n} < \sqrt{2} - 1$. But $0_{\mathbb{R}^2} \in (E_{\xi_n})^c$ because of $0.1 - \frac{2}{n} \ge 0$. So, we obtain

$$\mathcal{L}(\bar{x}, \bar{x}; \xi_n, \gamma_0) = \Delta_{E_{\xi_n}}(0_{\mathbb{R}^2}) \ge 0 > \Delta_{E_{\xi_n}}(-x) = \mathcal{L}(\bar{x}, x; \xi_n, \gamma_0).$$

For every $2 \leq n < 20$ and every $\gamma_0 \in \Gamma$, $(\bar{x}, \gamma_0)$ could not be a saddle point of $\mathcal{L}(\bar{x}, x; \xi_n, \gamma)$. Indeed, since $0 > 0.1 - \frac{2}{n} > -1$, there exists $x = (-\frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2}) \in X$ such that

$$\mathcal{L}(\bar{x}, \bar{x}; \xi_n, \gamma_0) = \Delta_{E_{\xi_n}}(0_{\mathbb{R}^2}) = 0.1 - \frac{2}{n},$$
$$\mathcal{L}(\bar{x}, x; \xi_n, \gamma_0) = \Delta_{E_{\xi_n}}(-x) = -1 + \frac{\sqrt{2}}{2}\left(1.1 - \frac{2}{n}\right).$$

Since $2 \leq n$, we have $\mathcal{L}(\bar{x}, \bar{x}; \xi_n, \gamma_0) - \mathcal{L}(\bar{x}, x; \xi_n, \gamma_0) = (1.1 - \frac{2}{n})(1 - \frac{\sqrt{2}}{2}) > 0$.

For $n = 1$ and every $\gamma_0 \in \Gamma$, we very that $(\bar{x}, \gamma_0)$ is a saddle point of $\mathcal{L}(\bar{x}, x; \xi_1, \gamma)$. Indeed, since $0.1 - \frac{2}{1} = -1.9 < -1$ and $h(g(x)) \leq 0, \ \forall x \in X$, we get $\forall x \in X$,

$$\mathcal{L}(\bar{x}, x; \xi_1, \gamma_0) = \Delta_{E_{\xi_1}}(-x) - |\gamma_0| h(g(x)) \geq -1.9 = \Delta_{E_{\xi_1}}(0_{\mathbb{R}^2}) = \mathcal{L}(\bar{x}, \bar{x}; \xi_1, \gamma_0).$$

From $h(g(\bar{x})) = 0$, we obtain $\mathcal{L}(\bar{x}, \bar{x}; \xi_1, \gamma) \leq \mathcal{L}(\bar{x}, \bar{x}; \xi_1, \gamma_0), \ \forall \gamma \in \Gamma$.

Also, $E \subseteq \operatorname{int} E_{\xi_1}$. However, $\Delta_{E_{\xi_1}}(0_{\mathbb{R}^2}) = -1.9 < 0$. This demonstrates that the condition $\Delta_{E_{\xi_0}}(0_{\mathbb{R}^l}) \geq 0$ in Theorem 4.4 is not redundant to ensure $\bar{x}$ being a E-optimal solution.

Examples 4.2 and 4.3 together demonstrate that the conditions used in Theorem 4.4 could not be further simplified.

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References

1. Chicco, M., Mignanego, F., Pusillo, L., Tijs, S.: Vector optimization problems via improvement sets. J. Optim. Theory Appl. **150**, 516–529 (2011)
2. Castellani, G., Giannessi, F.: Decomposition of mathematical programs by means of theorems of alternative for linear and nonlinear systems. In: Proceedings of Ninth Internatational Mathematical Programming Symposium, Budapest. Survey of Mathematical Programming, pp. 423–439. North-Holland, Amsterdam (1979)
3. Giannessi, F.: Theorems of the alternative and optimality conditions. J. Optim. Theory Appl. **42**, 331–365 (1984)
4. Giannessi, F., Mastroeni, G., Pellegrini, L.: On the Theory of Vector Optimization and Variational Inequalities: Image Space Analysis and Separation, Vector Variational Inequalities and Vector Equilibria, Mathematical Theories. F. Giannessi, Kluwer, Dordrecht (2000)
5. Giannessi, F.: Constrained Optimization and Image Space Analysis. Volume 1: Separation of Sets and Optimality Conditions, 1st edn. Mathematical Concepts and Methods in Science and Engineering. Springer, Berlin (2005)
6. Gutirrez, C., Jimnez, B., Novo, V.: Improvement sets and vector optimization. Eur. J. Oper. Res. **223**, 304–311 (2012)
7. Gupta, D., Mehra, A.: Two types of approximate saddle points. Numer. Funct. Anal. Optim. **29**, 532–550 (2008)
8. Guu, S.-M., Li, J.: Vector quasi-equilibrium problems: separation, saddle points and error bounds for the solution set. J. Glob. Optim. **58**, 751–767 (2014)
9. Hiriart-Urruty, J.B.: Tangent cone, generalized gradients and mathematical programming in Banach spaces. Math. Oper. Res. **4**, 79–97 (1979)

10. Jahn, J.: Vector Optimization: Theory, Applications, and Extensions. Springer, Berlin (2004)
11. Khan, A.A., Tammer, C., Zalinescu, C.: Set-Valued Optimization. Springer, Berlin (2015)
12. Li, S.J., Xu, Y.D., Zhu, S.K.: Nonlinear separation approach to constrained extremum problems. J. Optim. Theory Appl. **154**, 842–856 (2012)
13. Mastroeni, G.: Some applications of the image space analysis to the duality theory for constrained extremum problems. J. Glob. Optim. **46**, 603–614 (2010)
14. Mastroeni, G.: Optimality conditions and image space analysis for vector optimization problems. In: Ansari, Q.H., Yao, J.-C. (eds.) Recent Developments in Vector Optimization, Vector Optimization, vol. 1, pp. 169–220. Springer, Dordrecht (2012)
15. Mastroeni, G., Panicucci, B., Passacantando, M., Yao, J.C.: A separation approach to vector quasi-equilibrium problems: saddle point and gap function. Taiwan. J. Math. **13**, 657–673 (2009)
16. Xu, Y.D., Li, S.J.: Nonlinear separation functions and constrained extremum problems. Optim. Lett. **8**, 1149–1160 (2014)
17. Zaffaroni, A.: Degrees of efficiency and degrees of minimality. SIAM J. Control Optim. **42**, 1071–1086 (2003)