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ORIGINAL PAPER

Characterizations of improvement sets via quasi interior and applications in vector optimization

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Abstract In this paper, we first give some characterizations of improvement sets via quasi interior. Furthermore, as applications of these characterizations, we establish an alternative theorem via improvement sets and quasi interior, and then obtain a scalarization result of weak E-efficient solutions defined by improvement sets and quasi interior for vector optimization problems with set-valued maps. Moreover, we also present some examples to illustrate the main conditions and results.

Keywords Improvement sets · Quasi interior · Weak E-efficient solutions · Scalarization · Vector optimization problems with set-valued maps

1 Introduction

In recent years, research on the theory of vector optimization has attracted more attentions and has become one of the most important research topics in optimization theory and applications. So far, there are a lot of related research works, see [1,2] and the references therein. In vector optimization, various kinds of solutions including as efficient solutions, weak efficient solutions and proper efficient solutions have been playing an important role. In particular, the classical weak efficient solutions via topological interior possess some very nice properties, see [3,4] and the references therein.

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Approximate solutions also have been defined in several ways in vector optimization. Loridan presented ε-solutions of vector minimization problems in [5]. Rong and Wu introduced weak ε-minimal solutions of vector optimization problems with set-valued maps and established the corresponding linear scalarization theorems and Lagrange multipliers theorems in [6]. Furthermore, Chicco et al. [7] introduced the concept of improvement sets and presented E-efficient solutions via improvement sets in a finite dimensional space. Gutiérrez et al. [8] generalized the concept of improvement sets to a general real locally convex Hausdorff topological vector space. Moreover, Zhao et al. proposed weak E-efficient solutions via improvement sets and topological interior, and then obtained some characterizations for vector optimization problems with set-valued maps in [9]. Zhao and Yang also proposed E-Benson proper efficient solutions and established the corresponding linear scalarization theorems and Lagrange multipliers theorems in [10].

The classical weak efficiency requires the nonemptiness of topological interior of the ordering cones. However, there are many vector optimization problems with the ordering cones having possibly empty topological interior. Hence, some notions of generalized interiors will be essential. Limber and Goodrich introduced the notion of quasi interior and obtained some characterizations in [11]. Borwein and Lewis proposed the notion of quasi relative interior and gave some characterizations in [12]. Furthermore, Bao and Mordukhovich established some existence results of weak efficiency defined by several kinds of generalized interiors in [13]. Further study on characterizations of various kinds of generalized interiors and applications can be found in [14–23].

Weak E-efficiency defined by improvement sets and topological interior unifies some known exact and approximate efficiency in vector optimization. In this paper, we first give some characterizations of improvement sets via quasi interior. Then we establish an alternative theorem and a scalarization result of weak E-efficient solutions via improvement sets and quasi interior for vector optimization problems with set-valued maps.

2 Preliminaries

Let X be a real linear space, Y be a real nontrivial separated locally convex topological vector space and Y^* be the topological dual space of Y. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $\mathbb{R}^n_+$ be the nonnegative orthant, $\mathbb{R}^n_{++}$ be the positive orthant, $\mathbb{N}^+$ be the set of all positive integers and

$$l^p = \left\{ y = (y_n)_{n\in\mathbb{N}^+} \middle| \sum_{n\in\mathbb{N}^+} |y_n|^p < +\infty \right\}, \quad 1 \le p < +\infty$$

endowed with its usual norm. The positive cone of l^p, denoted by l^p_+, is

$$l^p_+ = \{y = (y_n)_{n\in\mathbb{N}^+} \in l^p | y_n \ge 0, n \in \mathbb{N}^+\}, \quad 1 \le p < +\infty.$$

We denote l^p_{++} by

$$l^p_{++} = \{y = (y_n)_{n \in \mathbb{N}^+} \in l^p | y_n > 0, n \in \mathbb{N}^+\}, \quad 1 \leq p < +\infty.$$

For a nonempty subset A in Y, we denote the topological interior and topological closure by int$A$ and cl$A$, respectively. A is said to be proper if $A \neq \emptyset$ and $A \neq Y$. Moreover, the generated cone and the positive dual cone of A are respectively defined as

$$\text{cone}A = \{\alpha a | \alpha \geq 0, a \in A\}, \quad A^+ = \{\mu \in Y^* | \langle \mu, y \rangle \geq 0, \quad \forall y \in A\}.$$

For a nonempty convex subset A in Y, the quasi interior (see [11]) and quasi relative interior (see [12–18]) denoted by qi$A$ and qri$A$ are respectively defined as

$$\text{qi}A = \{y \in A | \text{cl}(\text{cone}(A - y)) = Y\},$$
$$\text{qri}A = \{y \in A | \text{cl}(\text{cone}(A - y)) \text{ is a linear subspace in } Y\}.$$

Zălinescu pointed out the following fact in [18]:

$$\text{qi}A = A \cap \text{qi}(\text{cl}A), \quad \text{qri}A = A \cap \text{qri}(\text{cl}A). \tag{1}$$

For a convex subset A in Y, it is well known that int$A \subset$ qi$A \subset$ qri$A$; If int$A \neq \emptyset$, then int$A$ = qi$A$ = qri$A$; If qi$A \neq \emptyset$, then qi$A$ = qri$A$.

Definition 2.1 [7–9] Let E be a nonempty subset in Y and K be a proper convex cone in Y. If $0 \notin E$ and $E + K = E$, then E is said to be an improvement set with respect to K.

Remark 2.1 There exist some improvement sets with respect to a proper convex cone K. For example, let $K = \mathbb{R}^2_+$ in $\mathbb{R}^2$. Then $\mathbb{R}^2_+ \backslash \{0\}$, $\mathbb{R}^2_{++}$, $(1, 1) + \mathbb{R}^2_+$, $\{(y_1, y_2) | y_1 \geq -1, y_2 \geq 1\}$ and $\mathbb{R}^2_+ \backslash \{(y_1, y_2) | 0 \leq y_1 < 1, 0 \leq y_2 < 1\}$ are improvement sets with respect to K. So far, some applications of improvement sets via topological interior in vector optimization are given, see [7–10] and the references therein.

Lemma 2.1 [14], Theorem 2.1 *Let A and B be two nonempty convex subsets in Y with $qri A \neq \emptyset$, $qri B \neq \emptyset$ and such that $cl(cone(qri A - qri B))$ is not a linear subspace in Y. Then there exists $y^* \in Y^* \backslash \{0_{Y^*}\}$ such that $\langle y^*, a \rangle \leq \langle y^*, b \rangle$ for all $a \in A$ and $b \in B$.*

Lemma 2.2 [17], Lemma 2.6 *Let A be a nonempty subset in Y and K be a proper convex cone with nonempty quasi relative interior in Y. Then*

(i) $cl(cone(A + qri K)) = cl(cone A + qri K)$;
(ii) $cl(cone(A + qri K)) = cl(cone(A + K))$.

Remark 2.2 Lemmas 2.1 and 2.2 still hold for the case of quasi interior since quasi relative interior coincides with quasi interior when quasi interior of a set is nonempty.

Lemma 2.3 [17], Lemma 2.5 *Let K be a proper convex cone with nonempty quasi relative interior in Y and $cl(K-K)=Y$. If $\overline{k}\in qri K$, then we have $\langle k^*,\overline{k}\rangle>0$ for all $k^*\in K^+\backslash\{0\}$.*

Remark 2.3 In Lemma 2.3, if we assume that K be a proper convex cone with nonempty quasi interior in Y, then $\mathrm{cl}(K-K)=Y$ is trivial according to the definition of quasi interior. Hence we have $\langle k^*,\overline{k}\rangle>0$ for all $k^*\in K^+\backslash\{0\}$ and $\overline{k}\in \mathrm{qi}K$.

3 Characterizations of improvement sets via quasi interior

In this section, we mainly give some characterizations of improvement sets via quasi interior. These characterizations will be important in the sequel.

Lemma 3.1 *Let K be a proper convex cone with nonempty quasi interior in Y and E be a convex improvement set with respect to K in Y. Then $E+qiK\subset \mathrm{qi}E$.*

Proof By the fact that $E+K=E$ and then from Proposition 1 in [18], the result is trivial. □

Remark 3.1 Lemma 3.1 implies that $\mathrm{qi}E\neq\emptyset$ if $E\neq\emptyset$ and $\mathrm{qi}K\neq\emptyset$.

Remark 3.2 In [10], Zhao and Yang proved the fact that $\mathrm{int}E=E+\mathrm{int}K$ when $\mathrm{int}K\neq\emptyset$. However, we can not expect that $\mathrm{qi}E=E+\mathrm{qi}K$ even if $\mathrm{qi}K\neq\emptyset$. Following the idea of Example 2 proposed by Zălinescu in [18], we present the following example to illustrate it.

Example 3.1 Let $Y=l^2$, $\overline{y}=(1,\frac{1}{2},\frac{1}{3},\ldots)$, $E=\mathbb{R}_{++}\overline{y}+l^1_+$ and $K=l^1_+$. Clearly, $\overline{y}\in l^2$, $\overline{y}\notin l^1$, K is a proper convex cone and E is a convex improvement set with respect to K. Furthermore, $\mathrm{qi}K=l^1_{++}$ and $\mathrm{qi}l^2_+=l^2_{++}$. Therefore, we have

$$\overline{y}\in E\cap \mathrm{qi}l^2_+. \tag{2}$$

Moreover, it follows that

$$\mathrm{cl}E=\mathrm{cl}(\mathbb{R}_{++}\overline{y}+l^1_+)=\mathrm{cl}(\mathbb{R}_{+}\overline{y}+l^1_+).$$

Therefore,

$$l^1_+\subset\mathbb{R}_+\overline{y}+l^1_+\subset\mathrm{cl}(\mathbb{R}_+\overline{y}+l^1_+)=\mathrm{cl}E\subset\mathrm{cl}l^2_+=l^2_+.$$

Hence from the fact that $\mathrm{cl}l^1_+=l^2_+$, we have $\mathrm{cl}E=l^2_+$. So, by (1) and (2), we can obtain that $\overline{y}\in\mathrm{qi}E$. However, we can verify that $\overline{y}\notin E+\mathrm{qi}K$. On the contrary, assume that $\overline{y}\in E+\mathrm{qi}K$. Then we have

$$\overline{y}\in\mathbb{R}_{++}\overline{y}+l^1_++\mathrm{qi}l^1_+=\mathbb{R}_{++}\overline{y}+\mathrm{qi}l^1_+.$$

So, there exists $t\in\mathbb{R}_{++}$ such that

$$(1-t)\overline{y}\in\mathrm{qi}l^1_+\subset l^1.$$

Then from the fact that $\overline{y} \notin l^1$ and l^1 is a linear subspace of l^2, we have $t = 1$. This means that $0 \in \mathrm{qi}K$, which is a contradiction.

Remark 3.3 Under the assumption conditions of Lemma 3.1, we can obtain that for any given set A in Y,

$$\mathrm{cl}(\mathrm{cone}(A + \mathrm{qi}E)) = \mathrm{cl}(\mathrm{cone}(A + E)).$$

In fact, we only need to prove $\mathrm{cl}(\mathrm{cone}(A+E)) \subset \mathrm{cl}(\mathrm{cone}(A+\mathrm{qi}E))$. From $E = E+K$, Lemmas 2.2 and 3.1, we can obtain that

$$\begin{aligned}\mathrm{cl}(\mathrm{cone}(A + E)) &= \mathrm{cl}(\mathrm{cone}(A + E + K)) \\ &= \mathrm{cl}(\mathrm{cone}(A + E + \mathrm{qi}K)) \subset \mathrm{cl}(\mathrm{cone}(A + \mathrm{qi}E)).\end{aligned}$$

Remark 3.4 The relation $E + \mathrm{qi}K \subset \mathrm{qi}E$ is only a necessary condition for improvement sets. For example, let $K = \mathbb{R}^2_+$ in $\mathbb{R}^2$ and $E = \{(y_1, y_2)|y_2 \geq 0\}\backslash\{(y_1, y_2)|y_1 \geq 0, y_2 = 0\}$. We can easily obtain that K is a proper convex cone with nonempty quasi interior, $0 \notin E$, E is a convex set, $E + \mathrm{qi}K \subset \mathrm{qi}E$ and E is not an improvement set with respect to K. However, if K is a proper convex cone with nonempty quasi interior in Y, $0 \notin E$, E is a closed convex set in Y and $E + \mathrm{qi}K \subset \mathrm{qi}E$, we can verify that E is an improvement set with respect to K. In fact, it is clear that $E \subset E + K$ and

$$E + K \subset \mathrm{cl}E + \mathrm{cl}(\mathrm{qi}K) \subset \mathrm{cl}(E + \mathrm{qi}K) \subset \mathrm{cl}(\mathrm{qi}E) = E.$$

Lemma 3.2 *Let K be a proper cone in Y. Then for any given $y \in Y$,*

$$cl(cone(K + y)) = cl(K + \mathbb{R}_{++}y).$$

Proof Clearly, $\mathrm{cl}(\mathrm{cone}(K + y)) \subset \mathrm{cl}(K + \mathbb{R}_{++}y)$. On the other hand, let $\overline{y} \in K + \mathbb{R}_{++}y$. Then there exist $\overline{k} \in K$ and $\overline{\alpha} > 0$ such that $\overline{y} = \overline{k} + \overline{\alpha}y$. Since K is a cone, then $\overline{\alpha}^{-1}\overline{k} \in K$. Therefore,

$$\overline{y} = \overline{\alpha}(\overline{\alpha}^{-1}\overline{k} + y) \in \overline{\alpha}(K + y) \subset \mathrm{cone}(K + y),$$

which completes the proof. □

Theorem 3.1 *Let A be a nonempty subset in Y, K be a proper convex cone with nonempty quasi interior in Y and E be an improvement set with respect to K in Y. If $cl(cone(A + E))$ is a convex set, then*

$$qi(cl(cone(A + E))) \neq \emptyset.$$

Proof From $\mathrm{qi}K \neq \emptyset$, there exists $\overline{k} \in \mathrm{qi}K$, which means that

$$\mathrm{cl}(\mathrm{cone}(K - \overline{k})) = Y. \tag{3}$$

Since E is an improvement set with respect to K, then for any given $e \in E$ and $y \in A$, we have $y + e + \bar{k} \in \text{cone}(A + E)$ and so $y + e + \bar{k} \in \text{cl}(\text{cone}(A + E))$. Moreover, from Proposition 3.1 in [3], we have

$$\text{cl}(\text{cone}(\text{cl}(\text{cone}(A + E)) - y - e - \bar{k})) = \text{cl}(\text{cone}(\text{cone}(A + E) - y - e - \bar{k})).$$

It follows from $E = E + K$, Lemmas 2.2, 3.2 and (3) that

$$\begin{aligned} \text{cl}(\text{cone}(\text{cl}(\text{cone}(A + E)) - y - e - \bar{k})) &= \text{cl}(\text{cone}(A + E) + \mathbb{R}_{++}(-y - e - \bar{k})) \\ &= \text{cl}(\text{cone}(A + E + \mathbb{R}_{++}(-y - e - \bar{k}))) \\ &\supset \text{cl}(\text{cone}(A + E - y - e - \bar{k})) \\ &= \text{cl}(\text{cone}(A + E + K - y - e - \bar{k})) \\ &\supset \text{cl}(\text{cone}(K - \bar{k})) = Y. \end{aligned} \tag{4}$$

Then from (4) and the definition of quasi interior that

$$y + e + \bar{k} \in \text{qi}(\text{cl}(\text{cone}(A + E))),$$

which completes the proof. □

Remark 3.5 If E is not an improvement set with respect to K, then Theorem 3.1 may not be valid. The following example illustrates it.

Example 3.2 Let $Y = l^2$, $K = l^2_+$, $E = \{y = (y_n)_{n \in \mathbb{N}^+} \in l^2_+ | y_{2n} = y_{2n-1}, n \in \mathbb{N}^+\} \backslash \{0\}$ and

$$A = \{y = (y_n)_{n \in \mathbb{N}^+} \in l^2 | y_{2n} = y_{2n-1}, n \in \mathbb{N}^+\}.$$

Clearly, E is not an improvement set with respect to K and other conditions of Theorem 3.1 are satisfied. Since $A + E = A$ and A is a closed linear subspace in Y, then $\text{qi}(\text{cl}(\text{cone}(A + E))) = \text{qi}A = \emptyset$.

Theorem 3.2 *Let A be a nonempty subset in Y, K be a proper closed convex cone with nonempty quasi interior in Y and E be a convex improvement set with respect to K in Y. If $A \cap (-\text{qi}E) = \emptyset$, then cone$(A + E) \cap (-qiK) = \emptyset$.*

Proof By Lemma 3.1 and $A \cap (-\text{qi}E) = \emptyset$, we can easily obtain that

$$(A + E) \cap (-\text{qi}K) = \emptyset. \tag{5}$$

Assume that there exists $d \in -\text{qi}K$ such that $d \in \text{cone}(A + E)$. Clearly, $d \neq 0$ since $K \neq Y$. Hence from $d \in \text{cone}(A + E)$, there exist $\alpha > 0$, $\overline{y'} \in A$ and $\overline{e'} \in E$ such that $d = \alpha(\overline{y'} + \overline{e'})$. It follows from $d \in -\text{qi}K$ and Proposition 2.5 (iv) in [15] that $\overline{y'} + \overline{e'} \in -\text{qi}K$, which contradicts to (5) and the proof is completed. □

Theorem 3.3 *Let A be a nonempty subset in Y, K be a proper convex cone with nonempty quasi interior in Y and E be an improvement set with respect to K in Y. If $cl(cone(A+E))$ is a convex set and $qi(cl(cone(A+E))) \cap (-qiK) = \emptyset$, then $cl(cone(A+E))$ is not a linear subspace in Y.*

Proof On the contrary, we assume that $\mathrm{cl}(\mathrm{cone}(A+E))$ is a linear subspace in Y. Then it follows from $0 \in \mathrm{cl}(\mathrm{cone}(A+E))$ that $0 \in \mathrm{qri}(\mathrm{cl}(\mathrm{cone}(A+E)))$. By making use of Theorem 3.1, we can obtain that $0 \in \mathrm{qi}(\mathrm{cl}(\mathrm{cone}(A+E)))$ and so $\mathrm{cl}(\mathrm{cone}(A+E)) = Y$. It follows from the definition of quasi interior that $\mathrm{qi}(\mathrm{cl}(\mathrm{cone}(A+E))) = Y$. Therefore,

$$\mathrm{qi}(\mathrm{cl}(\mathrm{cone}(A+E))) \cap (-\mathrm{qi}K) = -\mathrm{qi}K \neq \emptyset,$$

which is a contradiction. □

Remark 3.6 If E is not an improvement set with respect to K and other conditions of Theorem 3.3 are satisfied, then Theorem 3.3 may not be valid. For example, consider the sets A, E and K in Example 3.2 and we can verify that $\mathrm{cl}(\mathrm{cone}(A+E)) = A$ is a linear subspace in Y.

Remark 3.7 If $\mathrm{qi}(\mathrm{cl}(\mathrm{cone}(A+E))) \cap (-\mathrm{qi}K) \neq \emptyset$ and other conditions of Theorem 3.3 are satisfied, then Theorem 3.3 may not be valid. The following example illustrates it.

Example 3.3 Let $Y = l^2$, $K = l^2_+$, $E = K \backslash \{0\}$ and $A = -K$. It is clear that $\mathrm{qi}(\mathrm{cl}(\mathrm{cone}(A+E))) \cap (-\mathrm{qi}K) \neq \emptyset$ and other conditions of Theorem 3.3 are satisfied. However, we can verify that $\mathrm{cl}(\mathrm{cone}(A+E)) = Y$.

4 Applications in vector optimization

In this section, we first introduce the concept of weak E-efficient solutions via improvement sets and quasi interior. Then we establish an alternative theorem via improvement sets and quasi interior and then obtain a scalarization result of weak E-efficient solutions of vector optimization problems with set-valued maps.

Consider the following vector optimization problem:

$$\text{(VP)} \;\; \min F(x) \text{ subject to } \; x \in S,$$

where $F : S \rightrightarrows Y$, $S \subset X$ and $S \neq \emptyset$.

In this section, we assume that K be a proper closed convex cone with nonempty quasi interior in Y.

Definition 4.1 Let E be a convex improvement set with respect to K in Y. A point pair $(\overline{x}, \overline{y})$ is called a weak E-efficient point of (VP) if $\overline{x} \in S$, $\overline{y} \in F(\overline{x})$ such that

$$(\overline{y} - \mathrm{qi}E) \cap F(S) = \emptyset.$$

Remark 4.1 (i) If $\mathrm{int}E \neq \emptyset$, then $\mathrm{qi}E = \mathrm{int}E$, which means that Definition 4.1 coincides with the weak E-optimal point introduced by Zhao et al. in [9];

(ii) If F is a vector-valued map on S, K is pointed and $E = K\backslash\{0\}$ with nonempty quasi interior, then $\mathrm{qi}E = \mathrm{qri}K$, which means that Definition 4.1 coincides with the quasi relative minimal point of (VP) introduced by Bao and Mordukhovich in [13].

If E is a convex improvement set with respect to K in Y, Lemma 3.1 shows that $E + \mathrm{qi}K \subset \mathrm{qi}E$. In order to establish a scalarization theorem of weak E-efficient solutions of (VP), following the assumption proposed by Grad and Pop in [23, Remark 4], we propose the following similar condition named as Assumption (Q) for improvement sets.

Assumption (Q) $\mathrm{qi}E \subset E + \mathrm{qi}K$.

Remark 4.2 Example 2 given by Zălinescu in [18] has shown that Assumption (Q) is not always fulfilled. Moreover, Example 3.1 in Sect. 3 has indicated that Assumption (Q) is not always fulfilled even if E is an improvement set with respect to K. Of course, also there exist some improvement sets satisfying Assumption (Q). For example, if we take $Y = l^p$ and $K = l^p_+$, then the following sets are improvement sets with respect to K in Y and satisfy Assumption (Q):

(i) $l^p_+\backslash\{0\}$, l^p_{++};
(ii) $\overline{y} + l^p_+$, where $\overline{y} \in l^p$ satisfies that there exist at least one positive component;
(iii) $E = (0, 1]\overline{y} + l^p_+$, where $\overline{y} \in l^p_+\backslash\{0\}$.

Let the support functional of the set A be defined as $\sigma_A(y^*) = \sup_{y\in A}\{\langle y^*, y\rangle\}$, $\forall y^* \in Y^*$.

Theorem 4.1 *Let E be a convex improvement set with respect to K in Y. If E satisfies Assumption (Q) and $\mathrm{cone}(F(S) + E)$ is a closed convex set, then one and only one of the following statements is true:*

(i) $\exists x \in S,\ F(x) \cap (-qiE) \neq \emptyset$;
(ii) $\exists \mu \in K^+\backslash\{0_{Y^*}\},\ \langle \mu, y\rangle \geq \sigma_{-E}(\mu),\ \forall y \in F(S)$.

Proof Assume that both (i) and (ii) hold. Then there exists $x \in S$ such that $F(x) \cap (-\mathrm{qi}E) \neq \emptyset$. By Assumption (Q), there exist $y \in F(x)$ and $e \in E$ such that $y + e \in -\mathrm{qi}K$. Hence from Lemma 2.3, we have $\langle \mu, y\rangle < \langle \mu, -e\rangle \leq \sigma_{-E}(\mu)$, which contradicts to (ii).

Assume that (i) does not hold. Then by Theorem 3.2, we have

$$\mathrm{cone}(F(S) + E) \cap (-\mathrm{qi}K) = \emptyset.$$

Hence, from the fact that $\mathrm{cone}(F(S) + E)$ is closed and Theorem 3.1, we have $\mathrm{qi}(\mathrm{cone}(F(S) + E)) \neq \emptyset$. It follows from $\mathrm{qi}K \neq \emptyset$, Lemma 2.2 and $E + K = E$ that

$$\begin{aligned}\mathrm{cl}(\mathrm{cone}(\mathrm{qi}(\mathrm{cone}(F(S) + E)) + \mathrm{qi}K)) &= \mathrm{cl}(\mathrm{cone}(\mathrm{cone}(F(S) + E) + \mathrm{qi}K)) \\ &= \mathrm{cl}(\mathrm{cone}(F(S) + E) + \mathrm{qi}K) \\ &= \mathrm{cl}(\mathrm{cone}(F(S) + E + \mathrm{qi}K)) \\ &= \mathrm{cone}(F(S) + E).\end{aligned} \tag{6}$$

Moreover, from Theorem 3.3, cone$(F(S)+E)$ is not a linear subspace in Y and hence by (6), we can obtain that $\mathrm{cl}(\mathrm{cone}(\mathrm{qi}(\mathrm{cone}(F(S)+E))+\mathrm{qi}K))$ is not a linear subspace in Y. It follows from Lemma 2.1 that there exists $\mu \in Y^*\backslash\{0_{Y^*}\}$ such that

$$\langle \mu, y+e+\varepsilon k\rangle \geq 0, \quad \forall y \in F(S), \quad \forall e \in E, \quad \forall k \in K, \quad \forall \varepsilon > 0. \tag{7}$$

Letting $\varepsilon \to +\infty$ in (7), we have $\langle \mu, k\rangle \geq 0$ for all $k \in K$. Therefore, $\mu \in K^+\backslash\{0_{Y^*}\}$. Letting $\varepsilon \to 0$ in (7), we can obtain that $\langle \mu, y\rangle \geq \langle \mu, -e\rangle$ for all $y \in F(S)$ and $e \in E$. Then

$$\langle \mu, y\rangle \geq \sup_{e' \in -E} \langle \mu, e'\rangle = \sigma_{-E}(\mu), \quad \forall y \in F(S).$$

This implies that (ii) does hold. □

Remark 4.3 Let Y be a real Banach space, A be a nonempty set in Y. Then the asymptotic cone of A is defined by $A^\infty = \{u \in Y | \exists (t_n) \downarrow 0, \exists (a_n) \subset A, t_n a_n \to u\}$. If $0 \in A$, the Bouligand tangent cone to A at 0 is defined by $A^0 = \{u \in Y | \exists (t_n) \to +\infty, \exists (a_n) \subset A, t_n a_n \to u\}$. According to [24], we can obtain that if A is a nontrivial cone, then $A^\infty = A^0 = \mathrm{cl}A$. By means of Corollary 2.1 in [24], we can obtain some sufficient conditions ensuring the closedness of cone$(F(S)+E)$ as follows:

(i) If $0 \notin \mathrm{cl}(F(S)+E)$, $(F(S)+E)^\infty \subset \mathrm{cone}(F(S)+E)$ and $\mathrm{cone}(F(S)+E) = \mathrm{cone}(\mathrm{cl}(F(S)+E))$, then cone$(F(S)+E)$ is closed;
(ii) If $0 \in \mathrm{cl}(F(S)+E)$, $(F(S)+E)^\infty \cup (F(S)+E)^0 \subset \mathrm{cone}(F(S)+E)$ and $\mathrm{cone}(F(S)+E) = \mathrm{cone}(\mathrm{cl}(F(S)+E))$, then cone$(F(S)+E)$ is closed.

Consider the scalar optimization problem:

$$(\mathrm{VP})_\mu \min_{x \in S} \langle \mu, F(x)\rangle, \quad \mu \in Y^*\backslash\{0_{Y^*}\},$$

where $\langle \mu, F(x)\rangle = \{\langle \mu, y\rangle | y \in F(x)\}$.

Definition 4.2 [9] A point $\overline{x} \in S$ is called an optimal solution of $(\mathrm{VP})_\mu$ with respect to E if there exists $\overline{y} \in F(\overline{x})$ such that $\langle \mu, y-\overline{y}\rangle \geq \sigma_{-E}(\mu), \forall x \in S, \forall y \in F(x)$. The point pair $(\overline{x}, \overline{y})$ is called an optimal point of $(\mathrm{VP})_\mu$ with respect to E.

In the following, we establish a scalarization result of weak E-efficient solutions of (VP).

Theorem 4.2 *Let $\overline{x} \in S$, $\overline{y} \in F(\overline{x})$ and E be a convex improvement set with respect to K in Y. If E satisfies Assumption (Q) and cone$(F(S)-\overline{y}+E)$ is a closed convex set, then $(\overline{x}, \overline{y})$ is a weak E-efficient point of (VP) if and only if there exists $\mu \in K^+\backslash\{0_{Y^*}\}$ such that $(\overline{x}, \overline{y})$ is an optimal point of $(\mathrm{VP})_\mu$ with respect to E.*

Proof Let $(\overline{x}, \overline{y})$ be a weak E-efficient point of (VP). Then $(F(S)-\overline{y}) \cap (-\mathrm{qi}E) = \emptyset$. From Theorem 4.1, there exists $\mu \in K^+\backslash\{0_{Y^*}\}$ such that

$$\langle \mu, y\rangle - \sigma_{-E}(\mu) \geq 0, \quad \forall y \in F(S) - \overline{y}.$$

Then for any given $x \in S$ and $y \in F(x)$, $\langle \mu, y - \overline{y} \rangle \geq \sigma_{-E}(\mu)$, which implies that $(\overline{x}, \overline{y})$ is an optimal point of $(\mathrm{VP})_\mu$ with respect to E.

Conversely, assume that $(\overline{x}, \overline{y})$ is not a weak E-efficient point of (VP). Then from Assumption (Q), we have $(\overline{y} - E - \mathrm{qi}K) \cap F(S) \neq \emptyset$. Hence, there exist $\overline{x'} \in S$, $\overline{y'} \in F(\overline{x'})$ and $\overline{e'} \in E$ such that $\overline{y'} - \overline{y} + \overline{e'} \in -\mathrm{qi}K$. From $\mu \in K^+ \backslash \{0_{Y^*}\}$ and Lemma 2.3, we have

$$\langle \mu, \overline{y'} - \overline{y} \rangle - \sigma_{-E}(\mu) \leq \langle \mu, \overline{y'} - \overline{y} + \overline{e'} \rangle < 0,$$

which contradicts to the fact that $(\overline{x}, \overline{y})$ is an optimal point of $(\mathrm{VP})_\mu$ with respect to E. □

In the end of this section, we present an example to illustrate Theorem 4.2.

Example 4.1 Let $X = Y = l^2$, $S = l^2_+$, $F(x) = [0, 2]x$, $K = l^2_+$ and $E = (\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots) + l^2_+$. We take

$$\overline{x} = \left(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right) \in S, \quad \overline{y} = \left(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right) \in F(\overline{x}) = [0, 2]\overline{x}.$$

Clearly, K is a proper closed convex cone with nonempty quasi interior, E is a convex improvement set with respect to K. Moreover,

$$\begin{aligned} &\mathrm{qi}K = l^2_{++}, \quad \mathrm{qi}E = \overline{y} + \mathrm{qi}K, \\ &\mathrm{cone}(F(S) - \overline{y} + E) = \mathrm{cone}(l^2_+ - \overline{y} + \overline{y} + l^2_+) = l^2_+. \end{aligned}$$

We can verify that E satisfies Assumption (Q) and $\mathrm{cone}(F(S) - \overline{y} + E)$ is a closed convex set. Since $(\overline{y} - \mathrm{qi}E) \cap F(S) = \emptyset$, then $(\overline{x}, \overline{y})$ is a weak E-efficient point of (VP). Since $K^+ = l^2_+$, then we can take

$$\mu = \left(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right) \in K^+ \backslash \{0_{Y^*}\}.$$

So for any $y \in F(S)$,

$$\langle \mu, y - \overline{y} \rangle = \langle \mu, y \rangle - \langle \mu, \overline{y} \rangle \geq -\langle \mu, \overline{y} \rangle = \sigma_{-E}(\mu),$$

which means that $(\overline{x}, \overline{y})$ is an optimal point of $(\mathrm{VP})_\mu$ with respect to E.

Remark 4.4 Zhao et al. established an alternative theorem (see Theorem 3.1) and a scalarization theorem (see Theorem 4.1) via improvement sets with nonempty topological interior in [9]. These results could not be applicable for those cases with possibly empty topological interior. In this paper, Theorems 4.1 and 4.2 generalize the corresponding results to the generalized interior case to some degree.

Remark 4.5 About the assumption condition that cone$(F(S) - \overline{y} + E)$ is a closed convex set in Theorem 4.2, $\overline{y}$ is a given point previously and it is only used to prove the necessity. We mainly follow the similar ideas of the references [3,24–27] and set this assumption condition. See Theorems 5.1 and 6.1 in [3]; Theorems 2.2 and 2.3 in [24]; Theorems 4.1 and 5.1 in [25]; Theorems 4.1, 5.1 and 5.3 in [26]; Theorems 3.2 and 3.8 in [27] etc. However, we also notice that the assumption condition is revelent with $\bar{y}$. Hence, it remains an interesting problem that how to propose a more appropriate assumption condition which does not depend on $\bar{y}$.

5 Concluding remarks

In this paper, we first obtain some characterizations of improvement sets via quasi interior. Furthermore, we apply these characterizations to establish an alternative theorem, and then obtain a scalarization result of weak E-efficient solutions via improvement sets and quasi interior. It is meaningful to generalize the corresponding results to the case of quasi relative interior. Moreover, if we remove the assumption condition that cone$(F(S) + E)$ is closed, can Theorem 4.1 hold? It also remains one open question.

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