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ORIGINAL PAPER

Chance constrained 0–1 quadratic programs using copulas

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Abstract In this paper, we study 0–1 quadratic programs with joint probabilistic constraints. The row vectors of the constraint matrix are assumed to be normally distributed but are not supposed to be independent. We propose a mixed integer linear reformulation and provide an efficient semidefinite relaxation of the original problem. The dependence of the random vectors is handled by the means of copulas. Finally, numerical experiments are conducted to show the strength of our approach.

Keywords Stochastic programming · Joint probabilistic constraints · 0–1 quadratic program · Copula theory · Semidefinite programming

1 Introduction

In this paper, we study the following *0–1 quadratic program with joint probabilistic* (or *chance*) *constraints*, called (QCC) hereafter:

$$\min x^T Qx + c^T x \text{ subject to } \quad \Pr\{Tx \leq d\} \geq p, \quad Ax = b,\ x \in \{0,1\}^n \tag{1}$$

where $c \in \mathbb{R}^n$, $d \in \mathbb{R}^K$, and $b \in \mathbb{R}^m$ are deterministic vectors, $Q \in \mathbb{R}^{n\times n}$ and $A \in \mathbb{R}^{m\times n}$ are deterministic matrices, $T \in \mathbb{R}^{K\times n}$ is a random matrix with rows

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T_k^T, $k = 1, \ldots, K$, and $p \in (0; 1)$ is a prescribed probability level. Chance-constrained problems of such form have been extensively studied in the literature starting with [5]. Excellent surveys on the theory, algorithms and bibliography of probabilistic programming are given by [9,17,18].

In this paper, we consider the nonconvex (QCC) with normally distributed dependent rows. This is an extension of previous results where the row independence was assumed [6–8]. Cheng et al. [8] reformulate a linear program with joint chance constraints as a convex *completely positive problem*, and solve its semidefinite relaxation. A conic approximation is performed using the formulations proposed in [7]. Furthermore, our current approach makes use of *copula theory*, i.e., we explore some favorable properties of Gumbel-Hougaard copula to describe the dependence between rows of the matrix T.

The rest of this paper is organized as follows. In Sect. 2, we present an SOCP and MILP formulations, together with introducing copula theory describing the row dependence. In Sect. 3, we present our semidefinite relaxation. In Sect. 4, we give our computational experiments to illustrate the strength of our SDP relaxation. To simplify the notation, we use indices $k = 1, \ldots, K$, $i, j = 1, \ldots, n$, and $l = 1, \ldots, N$ with these ranges throughout the paper without mentioning it explicitly.

2 SOCP and LP formulations

We start our investigation with an equivalent description of (QCC) and its relaxed and restriction (or conservative) approximations by linearizing second order cone programming (SOCP) constraints. For the sequel we assume that T_k^T are multivariate normally distributed vectors with known mean vectors μ_k, and covariance matrices Σ_k which are assumed to be positive definite. We do not assume independence of the rows; instead, we use the theory of copulas to represent inter-row dependence.

2.1 Row dependence

Copula theory was developed in the fields of probability theory and mathematical statistics to represent general dependence between random variables. To the best of our knowledge, copulas are not commonly used in stochastic optimization. Recently, Henrion et al. [12] used copulas to come up with convexity results for chance constrained problems with dependent random right hand side.

The following notions were taken from the book [15].

Definition 1 A *copula* is the distribution function $C : [0; 1]^K \to [0; 1]$ of some K-dimensional random vector whose marginals are uniformly distributed on [0; 1].

The joint distribution function of a random vector and the dependence of its marginals are closely related by Sklar's Theorem.

Proposition 1 (Sklar's Theorem) *For any K-dimensional distribution function $F : \mathbb{R}^K \to [0; 1]$ with marginals $F_1, \ldots, F_K$, there exists a copula C such that*

$$\forall z \in \mathbb{R}^K \quad F(z) = C(F_1(z_1), \ldots, F_K(z_K)). \tag{2}$$

If, moreover, F_k are continuous, then C is uniquely given by

$$C(u) = F(F_1^{-1}(u_1), \dots, F_K^{-1}(u_K)). \tag{3}$$

Otherwise, C is uniquely determined on range $F_1 \times \cdots \times$ range F_K.

Sklar's theorem ensures the existence and uniqueness of a copula for any distribution function and all its marginals. In our paper, we restrict the consideration to the following two classes of copulas:

1. *independent (product)* copula, defined by

$$C_\Pi(u) = \prod_{k=1}^{K} u_k.$$

Indeed, the independent copula represents the joint distribution of independent random variables.

2. *Gumbel-Hougaard family* of copulas, given for a $\theta \geq 1$ by

$$C_\theta(u) = \exp\left\{ -\left[\sum_{k=1}^{K} (-\ln u_k)^\theta \right]^{1/\theta} \right\}$$

It is easy to see that the independent copula is a special case of the Gumbel-Hougaard copula with $\theta = 1$. The Gumbel-Hougaard copula is a member of strict Archimedean copulas and shares of course the general properties of this class of copulas. Nelsen [15] showed that Gumbel-Hougaard copula can be considered as the representation of the bivariate extreme value distribution. This copula was used for several applications amongst all multivariate hydrologic frequency analysis for extreme hydrological events [19]. It was also used for trivariate rainfall frequency analysis in the subhumid climate of Southern Louisiana [22].

For $x \neq 0$ and for each k we introduce the transformation

$$\xi_k(x) := \frac{T_k^T x - \mu_k^T x}{\|\Sigma_k^{1/2} x\|}, \qquad g_k(x) := \frac{d_k - \mu_k^T x}{\|\Sigma_k^{1/2} x\|}.$$

The random variable $\xi_k(x)$ has one-dimensional standard normal distribution which is independent of x. Therefore, the probabilistic constraint $\Pr\{Tx \leq d\} \geq p$ can be equivalently rewritten as

$$\Pr\left\{\xi_k(x) \leq g_k(x) \ \forall k\right\} \geq p. \tag{4}$$

Lemma 1 *If the random vector $(\xi_1(x), \dots, \xi_K(x))^T$, where $\xi_k(x)$ has one-dimensional standard normal distribution, has a joint distribution driven by the*

Gumbel-Hougaard copula C_θ with some $\theta \geq 1$ then the constraint $\Pr\{Tx \leq d\} \geq p$ *is equivalent to the set of constraints*

$$\mu_k^T x + \Phi^{-1}\left(p^{z_k^{1/\theta}}\right)\left\|\Sigma_k^{1/2}x\right\| \leq d_k \quad \forall k,$$
$$\sum\nolimits_k z_k = 1,\ z_k \geq 0 \quad \forall k \tag{5}$$

where $\Phi(\cdot)$ is the inverse of the standard normal cumulative distribution function.

Proof Assume that there exists z_k such that (5) holds and $x \neq 0$. Then, the inequality of (5) is equivalent to

$$\Phi^{-1}\left(p^{z_k^{1/\theta}}\right) \leq \frac{d_k - \mu_k^T x}{\|\Sigma_k^{1/2}x\|}, \quad \text{i.e.,} \quad \Phi(g_k(x)) \geq p^{z_k^{1/\theta}}.$$

Let us first show that a vector x feasible in (5) satisfies (4). Indeed, from the definition of the Gumbel-Hougaard copula and Sklar's theorem,

$$\begin{aligned}\Pr\big\{\xi_k(x) \leq g_k(x)\ \forall k\big\} &= C_\theta(\Phi(g_1(x)), \ldots, \Phi(g_K(x)))\\ &\geq C_\theta\left(p^{z_1^{1/\theta}}, \ldots, p^{z_K^{1/\theta}}\right) = \exp\left\{-\left[\sum_k\left(-\ln p^{z_k^{1/\theta}}\right)^\theta\right]^{1/\theta}\right\}\\ &= \exp\left\{-\left[\sum_k\left(-z_k^{1/\theta}\ln p\right)^\theta\right]^{1/\theta}\right\} = \exp\left\{\ln p\left[\sum_k z_k\right]^{1/\theta}\right\} = p.\end{aligned}$$

For the opposite direction, we have to prove the existence of such z_k. Let x be a feasible solution for (4). Hence, assume $p < 1$ and define

$$\tilde{z}_k := \left(\frac{\ln\Phi(g_k(x))}{\ln p}\right)^\theta \quad \text{for } k = 1, \ldots, K, \quad z_k := \frac{\tilde{z}_k}{\sum_{k=1}^K \tilde{z}_k} \quad \text{for } k = 1, \ldots, K.$$

It is easy to verify that such definition of z_k satisfies $\sum_{k=1}^K z_k = 1,\ z_k \geq 0$. Since $\tilde{z}_k = \left(\frac{\ln\Phi(g_k(x))}{\ln p}\right)^\theta$, then we have $\mu_k^T x + \Phi^{-1}\left(p^{\tilde{z}_k^{1/\theta}}\right)\left\|\Sigma_k^{1/2}x\right\| = d_k\ \forall k$. Moreover, as

$$\begin{aligned}p &\leq \Pr\big\{\xi_k(x) \leq g_k(x)\ \forall k\big\} = C_\theta(\Phi(g_1(x)), \ldots, \Phi(g_K(x)))\\ &= C_\theta\left(p^{\tilde{z}_1^{1/\theta}}, \ldots, p^{\tilde{z}_K^{1/\theta}}\right) = \exp\left\{-\left[\sum_k\left(-\ln p^{\tilde{z}_k^{1/\theta}}\right)^\theta\right]^{1/\theta}\right\} = p^{\left[\sum_{k=1}^K \tilde{z}_k\right]^{1/\theta}}\end{aligned}$$

and $p < 1$, one has $\left[\sum_{k=1}^K \tilde{z}_k\right]^{1/\theta} \leq 1$ and further $\sum_{k=1}^K \tilde{z}_k \leq 1$. Then we have $z_k \geq \tilde{z}_k \forall k$. Therefore, it is attained $\mu_k^T x + \Phi^{-1}\left(p^{z_k^{1/\theta}}\right)\left\|\Sigma_k^{1/2}x\right\| \leq d_k\ \forall k$, which means z_k satisfies (5).

The remaining case, $x = 0$, where (4) does not make sense but $\Pr\{Tx \leq d\} \geq p$ is equivalent to the constraint $d \geq 0$.

Above all, the conclusion follows. □

For the sequel, we will also need the following convexity lemma:

Lemma 2 *If $p \geq \frac{1}{2}$ and $\theta \geq 1$ then $H(z) := \Phi^{-1}\left(p^{z^{1/\theta}}\right)$ is convex on* $[0; 1]$.

Proof H is convex if $\Phi^{-1}(\cdot)$ is convex, nondecreasing, and $z \mapsto p^{z^{1/\theta}}$ is convex. The first assertion is true if $p^{z^{1/\theta}} \geq \frac{1}{2}$, that is if

$$z \leq \left(-\frac{\ln 2}{\ln p}\right)^{\theta} \tag{6}$$

(excluding the case $p = 1$ which is trivial). If $p \geq \frac{1}{2}$ then $-\frac{\ln 2}{\ln p} \geq 1$ hence (6) is valid for $z \in [0; 1]$. The function $z \mapsto p^{z^{1/\theta}}$ is convex if $z \mapsto z^{1/\theta} \ln p$ is convex, i. e., $z \mapsto z^{1/\theta}$ concave. But $\frac{1}{\theta} \in (0; 1]$ hence the last assertion is true for all $z \in [0; 1]$. □

Lemma 2 provides a condition which is crucial for the deduction of our subsequent approximations. Hence, for the rest of the paper, we will assume that $1 > p \geq \frac{1}{2}$.

2.2 Mixed integer formulation

According to our assumptions and Lemma 1, we can derive a deterministic reformulation of (QCC) as

$$\min x^T Qx + c^T x \quad \text{subject to} \quad \begin{aligned} \mu_k^T x + \Phi^{-1}\left(p^{z_k^{1/\theta}}\right) \|\Sigma_k^{1/2} x\| \leq d_k \ \forall k, \\ \textstyle\sum_{k=1}^K z_k = 1,\ z_k \geq 0\ \forall k,\ Ax = b,\ x \in \{0, 1\}^n. \end{aligned} \tag{7}$$

The problem (7) is equivalent to

$$\min x^T Qx + c^T x \quad \text{subject to} \quad \begin{aligned} \left(\Phi^{-1}\left(p^{z_k^{1/\theta}}\right)\right)^2 x^T \Sigma_k x \leq (d_k - \mu_k^T x)^2\ \forall k, \\ \mu_k^T x \leq d_k\ \forall k, \\ \textstyle\sum_{k=1}^K z_k = 1,\ z_k \geq 0\ \forall k,\ Ax = b,\ x \in \{0, 1\}^n. \end{aligned} \tag{8}$$

Using the Taylor approximation of $H(z)^2$ at z_k and linearizing the quadratic terms, we provide its piecewise-tangent approximation, and obtain a mixed integer linear program

$$\min\langle X, Q\rangle + c^T x \tag{9}$$

$$\text{subject to} \quad \langle \Sigma_k, Z^k\rangle \le d_k^2 - 2d_k\mu_k^T x + \langle \mu_k\mu_k^T, X\rangle \quad \forall k \tag{10}$$

$$\mu_k^T x \le d_k \quad \forall k \tag{11}$$

$$\hat{F}^k - (1 - X_{ij})U^+ \le Z_{ij}^k \le \hat{F}^k \quad \forall i, j, k \tag{12}$$

$$0 \le Z_{ij}^k \le X_{ij}U^+ \quad \forall i, j, k \tag{13}$$

$$a_l + b_l z_k \le \hat{F}^k \quad \forall k, l \tag{14}$$

$$x_i + x_j - 1 \le X_{ij} \le \min\{x_i, x_j\} \quad \forall i, j, \tag{15}$$

$$X_{ii} = x_i \quad \forall i, \; X_{ij} \ge 0 \quad \forall i, j, \tag{16}$$

$$\textstyle\sum_{k=1}^K z_k = 1, \; z_k \ge 0 \quad \forall k, \; Ax = b, \; x \in \{0, 1\}^n \tag{17}$$

where U^+ is an upper bound of $\hat{F}_k := \max_l\{a_l + b_l z_k\}$, a_l and b_l are coefficients of the piecewise-tangent approximation of $H(z_k)^2$ at N selected tangent points $z_{(l)}$, $l = 1, \ldots, N$. Constraints (10) are obtained by replacing the function $H(z)^2 xx^T$ by Z^k. Constraints (12) and (13) are linearization constraints for the quadratic terms $Z_{ij}^k = \hat{F}^k X_{ij}$ whilst constraints (15) are linearization constraints for the quadratic terms $X_{ij} = x_i x_j$ [10]. The model (9–17) is a relaxation of problem (7) as shown by the following lemma:

Lemma 3 *The optimal value ϕ_N^* of* (9–17) *is a lower bound of* (7)*. Moreover if the trivial solution $x = 0$ is not feasible to* (7) *and tangents points are uniformly selected on the interval* $(0, 1]$*, then* $\lim_{N\to+\infty} \phi_N^* = \phi^*$ *where ϕ^* is the optimal value of* (7).

Proof According to Lemma 2, $H(z)$ is convex and so is $H(z)^2$ as the square function is non-decreasing and convex on the interval $[0, \infty)$. We approximate this function at tangent points $z_{(l)}$, $l = 1, \ldots, N$ by the first-order Taylor polynomial. We obtain N lines $z_k \mapsto a_l + b_l z_k$. By applying the linearization technique and due to the convexity, we obtain an outer approximation of the feasible set of (8). Hence, the optimal value ϕ_N^* is a lower bound of ϕ^*. Further, as $x = 0$ is an infeasible solution, $x \in \{0, 1\}^n$ and Σ_k is positive definite, then $x^T \Sigma_k x$ is lower bounded by a constant $L > 0$. Thus, $\frac{(d_k - \mu_k^T x)^2}{x^T \Sigma_k x}$ is bounded above by $\frac{d_k^2}{L}$ and further $H(z)^2$ is bounded above by $\frac{d_k^2}{L}$. Therefore, z_k is bounded below by $\left(\frac{\ln \Phi(\frac{d_k^2}{L})}{\ln p}\right)^\theta$, which implies that the derivative of $H(z)^2$ is bounded. Moreover the tangents points are uniformly selected on the interval $(0, 1]$, the convergence can be proved directly by applying the results of [21]. □

Note that if $x = 0$ is a feasible solution to (7), we can add the constraint $\sum x_i \ge 1$ to the problem and solve it. We can then compare the optimal objective value of the new problem with 0 to find the optimal solution of (7).

When we approximate $H(z)^2$ by using the piecewise-linear technique, then we have another mixed integer linear program which is a restriction of the problem (7) as shown by the following lemma:

Lemma 4 *The optimal value ϕ_N^* of the restriction problem is an upper bound of* (7)*. Moreover if the trivial solution $x = 0$ is not feasible to* (7) *and the interpolation points are uniformly selected on the interval* $(0, 1]$*, then* $\lim_{N \to +\infty} \phi_N^* = \phi^*$ *where ϕ^* is the optimal value of* (7)*.*

Proof Similar to the proof of Lemma 3.

3 SDP relaxation

Semidefinite programming is a subfield of convex optimization which provides strong modeling capabilities using polynomial solving methods. More precisely, a semidefinite program is a linear program over the cone of positive semidefinite matrices. We refer the reader to [2] for a various applications of semidefinite programming. Since the seminal works of Lovász [14] and Goemans and Williamson [11], several authors proposed approximation algorithms for NP-hard combinatorial problems based on the semidefinite relaxation. They show that SDP relaxation provides generally tightened bounds than LP relaxations. In this paper, we use the inner approximations proposed by [16], namely the first cone $(\mathbb{S}_+ \cap \mathbb{N})$ where $\mathbb{S}_+$ is the cone of positive semidefinite matrices and $\mathbb{N}$ is the cone of non-negative matrices. Here, we give a semidefinite programming approximation by using the piecewise tangent method [7] whose objective value is a lower bound of (QCC).

By using a piecewise tangent approximation of $H(z_k)$, we have an approximation of (7) as follows:

$$\min x^T Qx + c^T x \tag{18}$$

$$\text{subject to} \quad \mu_k^T x + \|\Sigma_k^{1/2} \tilde{z}_k\| \le d_k \quad \forall k \tag{19}$$

$$\tilde{z}_{ki} \ge a_l x_i + b_l z_{ki} \quad \forall i, l \tag{20}$$

$$\textstyle\sum_{k=1}^K z_{ki} = x_i, \; z_{ki} \ge 0 \quad \forall i, k \tag{21}$$

$$\tilde{F}_k - (1 - x_i)M^+ \le \tilde{z}_{ki} \le M^+ x_i \quad \forall i, k \tag{22}$$

$$0 \le \tilde{z}_{ki} \le \tilde{F}_k \quad \forall k \tag{23}$$

$$a_l + b_l z_k \le \tilde{F}_k \quad \forall k, l \tag{24}$$

$$\textstyle\sum_{k=1}^K z_k = 1, \; z_k \ge 0 \quad \forall k, \; Ax = b, \; x \in \{0, 1\}^n \tag{25}$$

where $\tilde{z}_k = (\tilde{z}_{k1}, \ldots, \tilde{z}_{kn})$ and M^+ is an upper bound of $\tilde{F}_k = \max_l\{a_l + b_l z_k\}$, the piecewise tangent approximation of $H(z_k)$.

Constraints (19) are obtained by approximating the term $H(z_k)x_i$ by $\tilde{z}_{ki} = \tilde{F}_k x_i$. The variables z_{ki} are defined by $z_{ki} = z_k x_i$. Constraints (22) and (23) are linearization constraints for the quadratic terms $\tilde{z}_{ki} = \tilde{F}_k x_i$. Constraints (20) and (21) strengthen our SDP relaxation though they are redundant when x is a binary variable. These constraints are deduced from constraints (24) and (25). Further, as constraint $z_{ki} = z_k x_i$ are difficult to consider explicitly, they are not taken into account in our relaxed model.

Theorem 1 *The optimal value of* (18–25) *is a lower bound of* (*QCC*)*. Moreover, if the trivial solution* $x = 0$ *is not feasible to* (7)*, and the interpolation points are uniformly selected on the interval* (0; 1]*, then it converges to the optimal value of* (*QCC*)*, as the number of segments N goes to infinity.*

Proof The proof is similar to Lemma 3 proof. □

A semidefinite relaxation of (18–25) can be written as

$$\min \quad \langle X, Q\rangle + c^T x \tag{26}$$

$$\text{subject to} \quad \begin{pmatrix} (d_k - \mu_k^T x)I & \Sigma_k^{1/2}\tilde{z}_k \\ \tilde{z}_k{}^T(\Sigma_k^{1/2})^T & d_k - \mu_k^T x \end{pmatrix} \succeq 0 \quad \forall k \tag{27}$$

$$\tilde{z}_{ki} \geq a_l x_i + b_l z_{ki} \quad \forall i, l \tag{28}$$

$$\textstyle\sum_{k=1}^K z_{ki} = x_i, \ z_{ki} \geq 0 \quad \forall i, k \tag{29}$$

$$\tilde{F}_k - (1 - x_i)M^+ \leq \tilde{z}_{ki} \leq M^+ x_i \quad \forall i, k \tag{30}$$

$$0 \leq \tilde{z}_{ki} \leq \tilde{F}_k \tag{31}$$

$$a_l + b_l z_k \leq \tilde{F}_k \quad \forall l \tag{32}$$

$$A_t^T x = b_t, \ A_t^T X A_t = b_t^2, \quad \forall t = 1, \ldots, m \tag{33}$$

$$\textstyle\sum_{k=1}^K z_k = 1, \ z_k \geq 0 \quad \forall k \tag{34}$$

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0, \ X \geq 0 \tag{35}$$

where I is the identity matrix of appropriate dimension. Linear matrix inequality Constraints (27) are derived from constraints (19) using Schur complement [3]. Constraints (33) are used to strength our SDP relaxation; they were used by [4] for copositive programs. Constraints (35) are a relaxation of constraints $X = x^T x$ obtained using the Schur complement.

4 Computational study

We study two MILP approximations and one SDP relaxation on stochastic multi-dimensional quadratic knapsack problems (SMQKP for short) from the OR-library [1]. Three different instances are considered, their sizes are defined by $(n, m, K) =$ (14, 5, 5), (28, 10, 5), (50, 5, 10) where n is number of variables, m number of deterministic constraints and K denotes the number of rows of the joint chance constraints. The parameters of the instances are generated as follows: the elements of Q are randomly generated on the interval [10, 20], the means μ_k are drawn from the uniform distribution on the interval [0, 5], the covariance matrices Σ_k are generated by MATLAB function `gallery('randcorr',n)*2` and the capacity d is independently chosen from the interval [10, 20]. The confidence parameter is set to $p = 0.9$.

We solve and compare four approximations. The first one is SDP relaxation (26–35) whose solution objective value is designed hereafter by V^{SDP}. The second and third ones solve (9–17) and the restriction problem abovementioned which are mixed integer linear problems designed by $V^{MILP}(R)$ and $V^{MILP}(C)$ respectively.

Table 1 Computational results when $p = 0.9$

	$V^{MILP}(C)$e+03	CPU (s)	$V^{MILP}(R)$e+03	CPU (s)	Gap (%)	V^{LP}e+03	CPU (s)	Gap (%)	V^{SDP}e+03	CPU (s)	GAP (%)
$\theta = 1$											
(14, 5, 5)	2.731	0.80	2.911	0.39	6.59	3.668	0.03	34.31	3.156	0.56	15.56
(28, 10, 5)	6.734	29.31	6.899	16.15	2.45	9.365	0.44	39.07	7.535	3.13	11.89
(50, 5, 10)	8.025	312.61	8.025	251.79	0.00	11.19	9.31	39.56	9.271	92.75	15.53
$\theta = 2$											
(14, 5, 5)	2.731	0.76	2.911	0.36	6.59	3.668	0.03	34.31	3.178	0.60	16.37
(28, 10, 5)	7.361	15.73	7.361	16.60	0.00	9.369	0.45	27.28	7.725	3.64	4.94
(50, 5, 10)	8.025	253.88	8.025	228.29	0.00	11.20	7.35	39.56	9.513	92.32	18.54
$\theta = 5$											
(14, 5, 5)	2.731	0.47	2.731	0.39	0.00	3.668	0.03	34.31	3.167	0.64	15.96
(28, 10, 5)	7.361	10.76	7.361	14.87	0.00	9.366	0.43	27.24	7.830	3.14	6.37
(50, 5, 10)	8.025	269.26	8.025	103.62	0.00	11.19	5.21	39.44	9.594	85.47	19.55

Table 2 Computational results for (28, 10, 5) with different N and p

	$V^{MILP}(C)$e+03	CPU (s)	$V^{MILP}(R)$e+03	CPU (s)	Gap (%)	V^{LP}e+03	CPU (s)	Gap (%)	V^{SDP}e+03	CPU (s)	GAP (%)
$p = 0.85$											
$\theta = 1$											
$N = 3$	7.361	10.00	7.361	15.77	0.00	9.371	0.40	27.31	7.734	3.33	5.07
$N = 10$	7.361	12.76	7.361	11.50	0.00	9.371	0.42	27.31	7.700	8.38	4.61
$N = 20$	7.361	9.47	7.361	11.26	0.00	9.371	0.46	27.31	7.696	11.72	4.55
$\theta = 2$											
$N = 3$	7.361	12.43	7.416	12.51	0.75	9.371	0.37	27.31	7.957	3.35	8.10
$N = 10$	7.361	15.02	7.416	12.62	0.75	9.371	0.47	27.31	7.926	8.33	7.68
$N = 20$	7.416	13.04	7.416	10.72	0.00	9.371	0.47	26.36	7.924	11.40	6.85
$\theta = 5$											
$N = 3$	7.416	12.61	7.416	13.59	0.00	9.371	0.38	26.36	8.090	3.40	9.09
$N = 10$	7.416	16.05	7.416	12.60	0.00	9.371	0.45	26.36	8.069	8.07	8.81
$N = 20$	7.416	12.73	7.416	11.00	0.00	9.371	0.46	26.36	8.068	10.90	8.79
$p = 0.90$											
$\theta = 1$											
$N = 3$	6.734	29.69	6.899	16.01	2.45	9.365	0.50	39.07	7.534	3.21	11.88
$N = 10$	6.899	21.28	6.899	15.12	0.00	9.363	0.51	35.72	7.494	8.32	8.62
$N = 20$	6.899	16.16	6.899	16.83	0.00	9.363	0.48	35.72	7.492	10.88	8.60
$\theta = 2$											
$N = 3$	7.361	15.73	7.361	16.24	0.00	9.369	0.46	27.28	7.725	3.61	4.94
$N = 10$	7.361	11.75	7.361	12.74	0.00	9.367	0.44	27.25	7.687	8.16	4.43
$N = 20$	7.361	12.73	7.361	14.79	0.00	9.367	0.43	27.25	7.686	11.44	4.42

Table 2 continued

	$V^{MILP}(C)$e+03	CPU (s)	$V^{MILP}(R)$e+03	CPU (s)	Gap (%)	V^{LP}e+03	CPU (s)	Gap (%)	V^{SDP}e+03	CPU (s)	GAP (%)
$\theta = 5$											
$N = 3$	7.361	10.64	7.361	15.12	0.00	9.366	0.44	27.24	7.830	3.35	6.37
$N = 10$	7.361	13.09	7.361	12.17	0.00	9.363	0.48	27.20	7.807	8.38	6.06
$N = 20$	7.361	16.04	7.361	15.20	0.00	9.363	0.47	27.20	7.806	10.70	6.05
$p = 0.95$											
$\theta = 1$											
$N = 3$	6.314	25.68	6.734	22.36	6.65	9.349	0.41	48.07	7.242	3.00	14.70
$N = 10$	6.734	31.73	6.734	27.57	0.00	9.348	0.42	38.82	7.191	8.65	6.79
$N = 20$	6.734	25.96	6.734	26.90	0.00	9.347	0.45	38.80	7.189	11.61	6.76
$\theta = 2$											
$N = 3$	6.734	25.37	6.734	30.39	0.00	9.351	0.45	38.86	7.384	3.25	9.65
$N = 10$	6.734	31.22	6.734	26.22	0.00	9.349	0.44	38.83	7.344	8.34	9.06
$N = 20$	6.734	31.76	6.734	28.91	0.00	9.349	0.46	38.83	7.343	10.22	9.04
$\theta = 5$											
$N = 3$	6.734	21.12	6.734	22.78	0.00	9.340	0.43	38.70	7.453	3.18	10.68
$N = 10$	6.734	23.87	6.734	21.50	0.00	9.336	0.50	38.64	7.435	8.23	10.41
$N = 20$	6.734	17.92	6.734	18.41	0.00	9.336	0.47	38.64	7.433	10.68	10.38

We also implement a continuous linear relaxation of (9–17) called hereafter V^{LP}. For all the approximations, we choose $N = 3$ tangent points $z_{(1)} = 0.01$, $z_{(2)} = 0.15$ and $z_{(3)} = 0.45$ in Table 1, and vary the number of tangent points from 3 up to 20 in Table 2 . The dependence parameter θ is set to 1 (independence), 2 (moderate dependence), and 5 (high dependence), respectively. All the considered models are generated using MATLAB environment and solved either by IBM CPLEX $v12$ [13] on an Intel(R)D @ 2.00 GHz with 4.0 GB RAM, or by Sedumi [20] with default parameters. The BigM number M^+ used in the SDP relaxation is chosen as the maximum of the parameters a_l, $\forall l$.

The numerical results for the three instances and for different values of θ are given by Table 1 where column one gives the name of the instance, columns two and three show the MILP optimal values and the corresponding CPU time respectively. Columns four, five and six give the relaxed MILP objective values. Columns seven, eight and nine show the LP relaxation objective value, the CPU time and the corresponding gap respectively. The last three columns present the SDP relaxation objective value, the CPU time and the corresponding gap respectively. The gap is defined by $Gap = \frac{UB-Opt}{Opt} \cdot 100\ \%$ where UB is the upper bound and Opt is the optimal solution of the restriction problem, i.e., $V^{MILP}(C)$, which provides a feasible solution.

We observe that for all instances, the SDP relaxation of our formulation outperforms the LP relaxation in terms of the quality of the solution. The SDP gaps range from 5 to 20 % while the LP gaps range from 27 to 40 %. The CPU time for our approach is within 100 s. However, both MILP formulations, i.e., the relaxation and the restriction of the original problem (7), give better solutions than SDP but within a larger CPU time.

Table 2 gives the same results as Table 1 for one instance (28, 10, 5) but for three different values of p, namely $p = 0.85$; 0.9; 0.95, and for different number of tangent points, i.e., $N = 3, 10, 20$. Table 2 shows the same performances as before for different values of p and N. Our SDP approach outperforms LP approach for different values of p. Moreover, the gaps decrease when the number of tangent points increase. In addition, both tables show that our two MILP approximations provide good candidate solutions for small size instances for two reasons: first, their CPU time is within 320 s; and their solutions are optimal when $N = 20$.

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