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ORIGINAL PAPER

On nonsmooth V-invexity and vector variational-like inequalities in terms of the Michel–Penot subdifferentials

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Abstract In this paper, we establish some results which exhibit an application for Michel–Penot subdifferential in nonsmooth vector optimization problems and vector variational-like inequalities. We formulate vector variational-like inequalities of Stampacchia and Minty type in terms of the Michel–Penot subdifferentials and use these variational-like inequalities as a tool to solve the vector optimization problem involving nonsmooth V-invex function. We also consider the corresponding weak versions of the vector variational-like inequalities and establish various results for the weak efficient solutions.

Keywords Michel–Penot subdifferential · Generalized convexity · Nonsmooth optimization · Efficient solution · Vector variational inequalities

1 Introduction

Convexity plays an important role to derive the optimality conditions and duality results for various scalar and vector optimization problems, see, e.g. [6,8,18,35]. In order to relax the convexity assumptions imposed on the objective functions involved, a new class of functions containing the class of convex functions was introduced

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in [14] and further termed as invex functions in [10]. The class of invex functions preserves many properties of the class of convex functions and has shown to be useful in a variety of applications, see, e.g. [21]. However, the main difficulty to deal with the vector optimization problems involving invex functions is the requirement of the same kernel function for all the involved objective functions. In order to overcome such restrictions a new class of differentiable vector valued functions called as V-invex functions was introduced in [16] which coincides with the class of invex functions for the scalar case. Further, the concept of V-invexity was extended in [11] for locally Lipschitz vector valued functions using the notion of Clarke subdifferentials. We refer to [22,23] and the references therein for more details related to the vector optimization problems involving V-invex functions.

The concept of vector variational inequalities was introduced in [12] for finite dimensional Euclidean spaces as a generalization of the classical Stampacchia variational inequalities for the vector valued functions. Using the concept of invexity, the Stampacchia vector variational inequalities were extended to Stampacchia vector variational-like inequalities in [38] and further studied in [24,36]. The concept of Minty vector variational inequalities was introduced in [13] and an equivalence with the vector optimization problems involving differentiable convex functions was established. Further, the results were extended in [42] and [43] for differentiable pseudoconvex functions and differentiable pseudoinvex function, respectively, and in [3] for locally Lipschitz invex functions. We refer to the recent results[2,4,25–30] and the references therein for more details related to vector variational inequalities.

The outline of this paper is as follows: in Sect. 2, we give some basic definitions and results which will be used in the sequel. In Sect. 3, we give the concept of V-invariant monotonicity and establish equivalence between the V-invexity of the vector valued function and the V-invariant monotonicity of the corresponding Michel–Penot subdifferential. We also derive relationships between the V-invexity of the vector valued function and the preinvexity of the scalar functions involved. In Sect. 4, we formulate vector variational-like inequalities of Stampacchia and Minty type in terms of the Michel–Penot subdifferentials and establish relationships with the efficient solutions of the vector optimization problem involving V-invex function. In Sect. 5, we formulate weak vector variational-like inequalities of Stampacchia and Minty type in terms of the Michel–Penot subdifferentials and establish relationships with the weak efficient solutions of the vector optimization problem involving V-invex function. In Sect. 6, we conclude the results of this paper and discuss some future research possibilities.

2 Preliminaries

In this section, we give some preliminary definitions and results, which will be used in the sequel.

Let X be a real Banach space endowed with a norm $\|\cdot\|$ and X^* its dual space with a norm $\|\cdot\|_*$. We denote by 2^{X^*}, $\langle\cdot,\cdot\rangle$, $[x, y]$ and (x, y), the family of all nonempty subsets of X^*, the dual pair between X and X^*, the line segment for $x, y \in X$ and the interior of $[x, y]$, respectively. Let S be a nonempty subset of X, let $\eta : X \times X \to X$ be a mapping and let $\alpha_i : X \times X \to \mathbb{R}_+\backslash\{0\}$ are strictly positive scalar valued functions for all $i \in M := \{1, \ldots, m\}$. Let $f := (f_1, \ldots, f_m) : X \to \mathbb{R}^m$ be a vector valued function such that $f_i : X \to \mathbb{R}$ are locally Lipschitz on S for all $i \in M$. We consider the vector optimization problem (VOP) as follows:

$$min \qquad f(x) := (f_1(x), \ldots, f_m(x)), \qquad \text{s.t.} \quad x \in S.$$

The following concept of efficiency was introduced in [34]. For recent developments in the field of vector optimization, we refer to the monograph [1] and the references therein.

Definition 1 *A vector $\bar{x} \in S$ is said to be an efficient solution of the VOP, iff for all $x \in S$, one has*

$$f(x) - f(\bar{x}) := (f_1(x) - f_1(\bar{x}), \ldots, f_m(x) - f_m(\bar{x})) \notin \mathbb{R}^m_+\backslash\{0\}.$$

Definition 2 *A vector $\bar{x} \in S$ is said to be a weak efficient solution of the VOP, iff for all $x \in S$, one has*

$$f(x) - f(\bar{x}) := (f_1(x) - f_1(\bar{x}), \ldots, f_m(x) - f_m(\bar{x})) \notin -int\, \mathbb{R}^m_+.$$

Remark 1 It is clear that every efficient solution is a weak efficient solution, but the converse is not true in general.

Now, we recall the definitions of the Clarke and Michel–Penot subdifferentials. For more details related to nonsmooth analysis, we refer to the monographs [9,37].

Definition 3 *Let S be a nonempty subset of X and let $g : X \to \mathbb{R}$ be locally Lipschitz at $\bar{x} \in S$. The Clarke directional derivative of g at $\bar{x}$ in the direction $v \in X$, denoted by $g^\circ(\bar{x}; v)$, is given by*

$$g^\circ(\bar{x}; v) := \limsup_{x \to \bar{x}, t \downarrow 0} \frac{g(x + tv) - g(x)}{t},$$

and the Clarke subdifferential of g at $\bar{x}$, denoted by $\partial^\circ g(\bar{x})$, is given by

$$\partial^\circ g(\bar{x}) := \left\{x^* \in X^* : \left\langle x^*, v\right\rangle \leq g^\circ(\bar{x}; v), \forall v \in X\right\}.$$

Definition 4 *Let S be a nonempty subset of X and let $g : S \to \mathbb{R}$ be locally Lipschitz at $\bar{x} \in S$. The Michel–Penot directional derivative of g at $\bar{x}$ in the direction $v \in X$, denoted by $g^\diamond(\bar{x}; v)$, is given by*

$$g^\diamond(\bar{x}; v) := \sup_{w \in X} \limsup_{t \downarrow 0} \frac{g(\bar{x} + tv + tw) - g(\bar{x} + tw)}{t},$$

and the Michel–Penot subdifferential of g at $\bar{x}$, denoted by $\partial^\diamond g(\bar{x})$, is given by

$$\partial^\diamond g(\bar{x}) := \left\{x^* \in X^* : \left\langle x^*, v\right\rangle \leq g^\diamond(\bar{x}; v), \forall v \in X\right\}.$$

Remark 2 It is clear that $g^\diamond(\bar{x}; v) \leq g^\circ(\bar{x}; v), \forall v \in X$, and $\partial^\diamond g(\bar{x}) \subseteq \partial^\circ g(\bar{x})$.

The following example illustrates the fact that the above inequality or inclusion may be strict.

Example 1 Let $g : \mathbb{R} \to \mathbb{R}$ be a real valued function such that $g(x) = x^2 \sin \frac{1}{x}$, when $x \neq 0$ and $g(x) = 0$, when $x = 0$. g is locally Lipschitz near $x = 0$. It is easy to see that $\partial^\diamond g(0) = \{0\}$ and $g^\diamond(0; v) = 0$. However, $\partial^\circ g(0) = [-1, 1]$ and $g^\circ(0; v) = |v|$.

The following concepts of the invex sets and the preinvex functions was given in [33].

Definition 5 *Let S be a nonempty subset of X and let $\eta : X \times X \to X$ be a mapping. The set S is said to be an invex set with respect to η, iff for all $x, y \in S$ and $\lambda \in [0, 1]$, one has $x + \lambda\eta(y, x) \in S$.*

Definition 6 *Let S be a nonempty invex subset of X with respect to η and let $g : X \to \mathbb{R}$ be a scalar valued function. The function g is said to be preinvex at $x \in S$ over S, iff for all $y \in S$ and $\lambda \in [0, 1]$, one has*

$$g(x + \lambda\eta(y, x)) \leq \lambda g(y) + (1 - \lambda)g(x).$$

g is said to be preinvex on S, iff g is preinvex at $x \in S$ over S for every $x \in S$.

Based on the M-P subdifferential, we give the notions of invexity and V-invexity.

Definition 7 *Let S be a nonempty subset of X and let g be locally Lipschitz near $y \in S$. The function g is said to be M-P invex at $y \in S$ over S with respect to η, iff for all $x \in S$ and $y^* \in \partial^\diamond g(y)$, one has*

$$g(x) - g(y) \geq \left\langle y^*, \eta(x, y)\right\rangle.$$

The function g is said to be M-P invex on S with respect to η, iff g is M-P invex at $y \in S$ over S with respect to η for all $y \in S$.

Definition 8 *Let S be a nonempty subset of X and let $f := (f_1, \ldots, f_m) : X \to \mathbb{R}^m$ be a vector valued function such that $f_i : X \to \mathbb{R}$ is locally Lipschitz near $y \in S$ for every $i \in M := \{1, \ldots, m\}$. The function f is said to be M-P V-invex at $y \in S$ over S with respect to η and α_i, $i \in M$, iff for all $i \in M$, $x \in S$ and $y_i^* \in \partial^\diamond f_i(y)$, one has*

$$f_i(x) - f_i(y) \geq \alpha_i(x, y)\left\langle y_i^*, \eta(x, y)\right\rangle.$$

The function f is said to be M-P V-invex on S with respect to η and α_i, $i \in M$, iff f is M-P V-invex at $y \in S$ over S with respect to η and α_i, $i \in M$ for all $y \in S$.

The following assumptions will be used in the sequel.

Condition A Let S be an invex subset of X with respect to η, and let $f := (f_1, \ldots, f_m) : X \to \mathbb{R}^m$ be a vector valued function. Then, for all $x, y \in S$ and for all $i \in M$, one has

$$f_i(x + \eta(y, x)) \leq f_i(y).$$

Condition C Let S be an invex subset of X with respect to η. Then, for all $x, y \in S$ and $\lambda, \lambda_1, \lambda_2 \in [0, 1]$, one has

(a) $\eta(x, x + \lambda\eta(y, x)) = -\lambda\eta(y, x)$,
(b) $\eta(y, x + \lambda\eta(y, x)) = (1 - \lambda)\eta(y, x)$,
(c) $\eta(x + \lambda_1\eta(y, x), x + \lambda_2\eta(y, x)) = (\lambda_1 - \lambda_2)\eta(y, x)$.

For the examples of the map η satisfying the Conditions C(a), C(b) and C(c), we refer to [3,40,41].

Condition D Let S be an invex subset of X with respect to η and let $\alpha_i, i \in M$ be the scalar valued mappings. Then, for all $i \in M$, $x, y \in S$ and $\lambda \in [0, 1]$, one has

(a) $\alpha_i(x, x + \lambda\eta(y, x)) \geq \alpha_i(y, x)$,
(b) $\alpha_i(y, x + \lambda\eta(y, x)) \geq \alpha_i(y, x)$,
(c) $\frac{\alpha_i(x, x+\lambda\eta(y,x))}{\alpha_i(x+\lambda\eta(y,x), x)} \geq \alpha_i(y, x)$.

Now, we give example of a map $\alpha_i, i \in M$ which satisfies Condition D.

Example 2 Let S be an invex subset of X with respect to $\eta : X \times X \to X$ such that η satisfies Condition C. Let $\alpha_i : X \times X \to \mathbb{R}_+\backslash\{0\}, i \in M$ be the scalar valued mappings defined as follows

$$\alpha_i(x, y) := \frac{1}{i + \|\eta(x, y)\|}, \forall x, y \in X.$$

Then, it is easy to see that $\alpha_i, i \in M$ satisfies Condition D.

The mapping η is said to be skew on S, iff for all $x, y \in S$, one has, $\eta(x, y) + \eta(y, x) = 0$. The mapping $\alpha_i, i \in M$ is said to be symmetric on S, iff for all $x, y \in S$, one has, $\alpha_i(x, y) = \alpha_i(y, x)$.

The following mean value theorem in terms of the Michel–Penot subdifferential was proved in [7]. We refer to [20,44,45] and the references therein for more applications of the Michel–Penot subdifferentials.

Theorem 1 *Let $x, y \in X$, and suppose that $g : X \to \mathbb{R}$ is locally Lipschitz on an open set containing the line segment $[x, y]$. Then there exists a point $z \in (x, y)$ such that*

$$g(x) - g(y) \in \left\langle \partial^\diamond g(y), x - y \right\rangle.$$

Remark 3 The above mean value theorem in terms of the Michel–Penot subdifferential is stronger than the Lebourg mean value theorem (see, e.g. [9]), the corresponding version for the Clarke subdifferential, since the Michel–Penot subdifferential of a function at a point is contained, and sometimes properly contained, in its Clarke subdifferential at this point, and hence the results obtained by the application of above theorem will be stronger than the results obtained by the application of the Lebourg mean value theorem.

3 V-invexity and V-invariant monotonicity using the Michel–Penot subdifferentials

In this section, we extend the concept of invariant monotonicity (see, e.g. [15,41]) to V-invariant monotonicity. We also establish relationship between the M-P V-invexity of a vector valued function and the preinvexity of the corresponding scalar valued functions.

Definition 9 *Let S be a nonempty subset of X and let $T_i : X \to 2^{X^*}$ be a set-valued mapping for every $i \in M := \{1, \dots, m\}$. The mapping $T : (T_1, \dots, T_m)$ is said to be V-invariant monotone on S with respect to η and α_i, $i \in M$, iff for all $i \in M$, $x, y \in S$, $x_i^* \in T_i(x)$ and $y_i^* \in T_i(y)$, one has*

$$\alpha_i(x, y) \left\langle y_i^*, \eta(x, y) \right\rangle + \alpha_i(y, x) \left\langle x_i^*, \eta(y, x) \right\rangle \leq 0.$$

The following proposition gives the relationship between the M-P V-invexity of the vector valued function and the V-invariant monotonicity of the corresponding Michel–Penot subdifferential.

Proposition 1 *Let S be a nonempty subset of X and let $f := (f_1, \dots, f_m) : X \to \mathbb{R}^m$ be locally Lipschitz on S. If f is M-P V-invex with respect to $\eta : X \times X \to X$ and $\alpha_i : X \times X \to \mathbb{R}_+ \setminus \{0\}$, $i \in M := \{1, \dots, m\}$ on S, then $\partial^\diamond f := \partial^\diamond f_1 \times \cdots \times \partial^\diamond f_m$ is V-invariant monotone with respect to η and α_i, $i \in M$ on S.*

Proof Suppose that f is M-P V-invex with respect to η and α_i, $i \in M$ on S. Then, for every $x, y \in S$, $x_i^* \in \partial^\diamond f_i(x)$, $y_i^* \in \partial^\diamond f_i(y)$, and $i \in M$, one has

$$f(x) - f(y) \geq \alpha_i(x, y) \left\langle y_i^*, \eta(x, y) \right\rangle \text{ and } f(y) - f(x) \geq \alpha_i(y, x) \left\langle x_i^*, \eta(y, x) \right\rangle.$$

Adding the above inequalities, for every $x, y \in S$, $x_i^* \in \partial^\diamond f_i(x)$, $y_i^* \in \partial^\diamond f_i(y)$, and $i \in M$, one has

$$0 \geq \alpha_i(x, y) \left\langle y_i^*, \eta(x, y) \right\rangle + \alpha_i(y, x) \left\langle x_i^*, \eta(y, x) \right\rangle.$$

Hence, $\partial^\diamond f$ is V-invariant monotone with respect to η and α_i, $i \in M$ on S. □

The following proposition gives the converse of above proposition under the assumption that Condition A, Condition C and Condition D hold.

Proposition 2 *Let S be a nonempty invex subset of X with respect to η such that η satisfies Condition C and let $f := (f_1, \dots, f_m) : X \to \mathbb{R}^m$ be locally Lipschitz on S such that f satisfies Condition A. If $\partial^\diamond f := \partial^\diamond f_1 \times \cdots \times \partial^\diamond f_m$ is V-invariant monotone with respect to η and α_i, $i \in M$ on S such that α_i, $i \in M$ satisfy Condition D, then f is M-P V-invex with respect to η and α_i, $i \in M$ on S.*

Proof Let $x, y \in S$ and let $z(\lambda) := y + \lambda\eta(x, y)$ for every $\lambda \in [0, 1]$. Since S is an invex set with respect to η, it follows that, $z(\lambda) \in S, \forall \lambda \in [0, 1]$. By the mean value theorem, for every $i \in M$ and for any $\hat{\lambda} \in (0, 1)$, there exists $\tilde{\lambda}_i \in (0, \hat{\lambda})$ and

$\bar{\lambda}_i \in (\hat{\lambda}, 1)$ such that, for every $i \in M$ and for some $\tilde{z}_i^* \in \partial^\diamond f_i(z(\tilde{\lambda}_i)), i \in M$ and for some $\bar{z}_i^* \in \partial^\diamond f_i(z(\bar{\lambda}_i)), i \in M$, one has

$$f_i(z(\hat{\lambda})) - f_i(z(0)) = \hat{\lambda} \left\langle \tilde{z}_i^*, \eta(x, y) \right\rangle, \quad \forall i \in M, \tag{1}$$

and

$$f_i(z(1)) - f_i(z(\hat{\lambda})) = (1 - \hat{\lambda}) \left\langle \bar{z}_i^*, \eta(x, y) \right\rangle, \quad \forall i \in M. \tag{2}$$

By the V-invariant monotonicity of $\partial^\diamond f$ with respect to η and $\alpha_i, i \in M$ on S, Condition C for η and Condition D for $\alpha_i, i \in M$, for any $i \in M$ and $y_i^* \in \partial^\diamond f_i(y)$, one has

$$\left\langle \tilde{z}_i^*, \eta(x, y)) \right\rangle \geq \alpha_i(x, y) \left\langle y_i^*, \eta(x, y)) \right\rangle, \tag{3}$$
$$\left\langle \bar{z}_i^*, \eta(x, y)) \right\rangle \geq \alpha_i(x, y) \left\langle y_i^*, \eta(x, y)) \right\rangle. \tag{4}$$

From (1), (2), (3) and (4), for any $i \in M$, one has

$$f_i(z(\hat{\lambda})) - f_i(z(0)) \geq \hat{\lambda}\alpha_i(x, y) \left\langle y_i^*, \eta(x, y)) \right\rangle, \tag{5}$$

and

$$f_i(z(1)) - f_i(z(\hat{\lambda})) \geq (1 - \hat{\lambda})\alpha_i(x, y) \left\langle y_i^*, \eta(x, y)) \right\rangle. \tag{6}$$

Adding the above inequalities and using Condition A for f, it follows that

$$f_i(x) - f_i(y) \geq \alpha_i(x, y) \left\langle y_i^*, \eta(x, y)) \right\rangle, \quad \forall y_i^* \in \partial^\diamond f_i(y), \forall i \in M.$$

Since $x, y \in S$ are arbitrary, it implies that, f is M-P V-invex with respect to η and $\alpha_i, i \in M$ on S and hence the result.

The following theorem is a direct consequence of Proposition 1 and 2. □

Theorem 2 *Let S be a nonempty invex subset of X with respect to η such that η satisfies Condition C and let $f := (f_1, \ldots, f_m) : X \to \mathbb{R}^m$ be locally Lipschitz on S such that f satisfies Condition A. Let $\alpha_i : X \times X \to \mathbb{R}_+ \backslash \{0\}, i \in M := \{1, \ldots, m\}$ be such that $\alpha_i, i \in M$ satisfy Condition D. Then, f is M-P V-invex with respect to η and $\alpha_i, i \in M$ on S, iff $\partial^\diamond f := \partial^\diamond f_1 \times \cdots \times \partial^\diamond f_m$ is V-invariant monotone with respect to η and $\alpha_i, i \in M$ on S.*

The following proposition gives the relationship between the M-P V-invexity of the vector valued function and the preinvexity of the corresponding scalar valued functions.

Proposition 3 *Let S be a nonempty invex subset of X with respect to η such that η satisfies Condition C. If f is M-P V-invex with respect to η and $\alpha_i, i \in M$ on S, then f_i is preinvex with respect to η on S for all $i \in M$.*

Proof Let $x, y \in S$ for any $\lambda \in (0, 1)$ and let $z := y + \lambda\eta(x, y)$. Since S is invex with respect to η, $z \in S$. By the M-P V-invexity of f on S with respect to η and α_i, $i \in M$, for all $i \in M$ and $z_i^* \in \partial^\diamond f_i(z)$, one has

$$f_i(x) - f_i(z) \geq \alpha_i(x, z)\left\langle z_i^*, \eta(x, z)\right\rangle, \quad \text{and} \quad f_i(y) - f_i(z) \geq \alpha_i(y, z)\left\langle z_i^*, \eta(y, z)\right\rangle.$$

Since η satisfies Condition C and α_i, $i \in M$ satisfy Condition D, it follows that

$$f_i(x) - f_i(z) \geq (1 - \lambda)\alpha_i(x, y)\left\langle z_i^*, \eta(x, y)\right\rangle, \tag{7}$$

and

$$f_i(y) - f_i(z) \geq -\lambda\alpha_i(x, y)\left\langle z_i^*, \eta(x, y)\right\rangle. \tag{8}$$

□

Multiplying (7) by λ, (8) by (1-λ) and adding the inequalities, it follows that

$$\lambda f_i(x) + (1 - \lambda) f_i(y) \geq f_i(y + \lambda\eta(x, y)), \quad \forall i \in M.$$

Since $x, y \in S$ and $\lambda \in (0, 1)$ are arbitrary, f_i is preinvex with respect to η on S for all $i \in M$ and hence the result.

4 Vector variational-like inequalities using the Michel–Penot subdifferentials

In this section, we consider the vector variational-like inequalities of Stampacchia type in terms of the Michel–Penot subdifferentials, denoted by MP-SVVLI (f, S), as follows:

(MP-SVVLI)To find $\bar{x} \in S$ such that, for all $x \in S$, there exists $\bar{x}_i^* \in \partial^\diamond f_i(\bar{x})$, $i \in M := \{1, \ldots, m\}$ such that,$(\left\langle \bar{x}_1^*, \eta(x, \bar{x})\right\rangle, \ldots, \left\langle \bar{x}_m^*, \eta(x, \bar{x})\right\rangle) - R_+^m \backslash \{0\}$.

The following proposition gives the condition under which a solution of the MP-SVVLI (f, S) is also an efficient solution of the VOP (f, S).

Proposition 4 *Let S be a nonempty subset of X and let $f := (f_1, \ldots, f_m) : X \to \mathbb{R}^m$ be locally Lipschitz and M-P V-invex with respect to η and α_i, $i \in M$ at $\bar{x} \in S$ over S. If $\bar{x}$ solves the MP-SVVLI (f, S) with respect to η, then $\bar{x}$ is an efficient solution of the VOP (f, S).*

Proof Suppose that $\bar{x}$ is not an efficient solution of the VOP (f, S). Then, there exists $\widetilde{x} \in S$ such that, $f_i(\widetilde{x}) - f_i(\bar{x}) \leq 0, \forall i \in M$, with strict inequality for at least one $i \in M$. By the M-P V-invexity of f at $\bar{x}$ over S with respect to η and α_i, $i \in M$, it follows that, $\alpha_i(\widetilde{x}, \bar{x})\left\langle \bar{x}_i^*, \eta(\widetilde{x}, \bar{x})\right\rangle \leq 0, \forall \bar{x}_i^* \in \partial^\diamond f_i(\bar{x}), \forall i \in M$, with strict inequality for at least one $i \in M$. Since $\alpha_i(\widetilde{x}, \bar{x}) > 0$ for all $i \in M$, it implies that, $\left\langle \bar{x}_i^*, \eta(\widetilde{x}, \bar{x})\right\rangle \leq 0, \forall \bar{x}_i^* \in \partial^\diamond f_i(\bar{x}), \forall i \in M$, with strict inequality for at least one $i \in M$, a contradiction to the fact that $\bar{x}$ solves MP-SVVLI (f, S) and hence the result.

Example 3 Consider the VOP as follows:

$$\min f(x) := (f_1(x), f_2(x)) \text{ s.t. } x \in S \subseteq \mathbb{R}^2,$$

where $f_1(x) := x_1/x_2$, $f_2(x) := x_2/x_1$ and $S := \{x := (x_1, x_2)|x_1 \geq 1, x_2 \geq 1\}$. It is easy to see that f is M-P V-invex with respect to $\alpha_1(x, \bar{x}) := \bar{x}_2/x_2, \alpha_2(x, \bar{x}) := \bar{x}_1/x_1$ and $\eta(x, \bar{x}) := x - \bar{x}$ on S. Let $\bar{x} := (1, 1) \in S$. Then, $\partial^\diamond f_1(\bar{x}) := \{(1, -1)\}$ and $\partial^\diamond f_2(\bar{x}) := \{(-1, 1)\}$. Now, for any $x \in S, \bar{x}_1^* \in \partial^\diamond f_1(\bar{x})$ and $\bar{x}_2^* \in \partial^\diamond f_2(\bar{x})$, one has, $\langle \bar{x}_1^*, \eta(x, \bar{x}) \rangle = x_1 - x_2$, and $\langle \bar{x}_2^*, \eta(x, \bar{x}) \rangle = x_2 - x_1$, which implies that,

$$\left(\langle \bar{x}_1^*, \eta(x, \bar{x}) \rangle, \langle \bar{x}_2^*, \eta(x, \bar{x}) \rangle\right) - \mathbb{R}_+^2 \backslash \{0\}.$$

Hence, $\bar{x} := (1, 1)$solves MP-SVVLI (f, S) with respect to η. By Proposition 4, it follows that, $\bar{x}$ is also an efficient solution of the VOP (f, S).

The following result is a direct consequence of the fact that every efficient solution is also a weak efficient solution of the VOP (f, S).

Corollary 1 *Let S be a nonempty subset of X and let f be locally Lipschitz and M-P V-invex with respect to η and α_i, $i \in M$ at $\bar{x} \in S$ over S. If $\bar{x}$ solves the MP-SVVLI (f, S) with respect to η, then $\bar{x}$ is a weak efficient solution of the VOP (f, S).*

Now, We consider the vector variational-like inequalities of Minty type in terms of the Michel–Penot subdifferentials, denoted by MP-MVVLI (f, S), as follows:

(MP-MVVLI) To find $\bar{x} \in S$ such that, for all x in S, and for all $x_i^* \in \partial^\diamond f_i(x), i \in M := \{1, \ldots, m\}$, one has,$(\langle x_1^*, \eta(x, \bar{x}) \rangle, \ldots, \langle x_m^*, \eta(x, \bar{x}) \rangle) \notin -\mathbb{R}_+^m \backslash \{0\}$.

The following result gives the condition under which an efficient solution of the VOP (f, S) also solves the MP-MVVLI (f, S).

Proposition 5 *Let S be a nonempty subset of X and let $f := (f_1, \ldots, f_m) : X \to \mathbb{R}^m$ be locally Lipschitz and M-P V-invex on S with respect to η and α_i, $i \in M$ such that η is skew and $\alpha_i, i \in M$ is symmetric. If $\bar{x}$ is an efficient solution of the VOP (f, S), then $\bar{x}$ solves the MP-MVVLI (f, S) with respect to η.*

Proof Suppose to the contrary that $\bar{x}$ does not solve the MP-MVVLI (f, S) with respect to η. Then, there exists $\tilde{x} \in S$ and $\tilde{x}_i^* \in \partial^\diamond f_i(\tilde{x}), i \in M$ such that, $\langle \tilde{x}_i^*, \eta(\tilde{x}, \bar{x}) \rangle \leq 0, \forall i \in M$, with strict inequality for at least one $i \in M$. Since, η is skew, $\alpha_i, i \in M$ is symmetric and $\alpha_i(\tilde{x}, \bar{x}) > 0, i \in M$, it follows that $\alpha_i(\bar{x}, \tilde{x}) \langle \tilde{x}_i^*, \eta(\bar{x}, \tilde{x}) \rangle \geq 0, \forall i \in M$, with strict inequality for at least one $i \in M$. By the M-P V-invexity of f on S, it implies that $f_i(\tilde{x}) - f_i(\bar{x}) \leq 0, \forall i \in M$, with strict inequality for at least one $i \in M$, a contradiction to the fact that $\bar{x}$ is an efficient solution of the VOP (f, S) and hence the result. □

Example 4 Consider the VOP as follows:

$$\min f(x) := (f_1(x), f_2(x)) \text{ s.t. } x \in S \subseteq \mathbb{R}^2,$$

where $f_1(x) := |\frac{2x_1-x_2}{x_1+x_2}|$, $f_2(x) := \frac{x_1-2x_2}{x_1+x_2}$ and $S := \{x_1 \leq x_2, x_1 \geq 1, x_2 \geq 1\}$. It is easy to see that f is M-P V-invex with respect to $\eta(x, y) := (\frac{3(x_1-1)}{x_1+x_2}, \frac{3(x_2-2)}{x_1+x_2})$ and $\alpha_1(x, y) = \alpha_2(x, y) = 1$. Let $\bar{x} := (1, 2) \in S$. Then, for any $x \in S\backslash\{\bar{x}\}$, one has, $f_1(x) - f_1(\bar{x}) = |\frac{2x_1-x_2}{x_1+x_2}| > 0$, which implies that, $(f_1(x) - f_1(\bar{x}), f_2(x) - f_2(\bar{x})) \notin -\mathbb{R}^2_+\backslash\{0\}$. Hence, $\bar{x} := (1, 2)$ is an efficient solution of the VOP (f, S). By Proposition 5, it follows that, $\bar{x}$ also solves MP-MVVLI (f, S) with respect to η.

The following result gives the condition under which converse of the above proposition holds.

Proposition 6 *Let S be a nonempty invex subset of X with respect to η such that η is skew and satisfies Condition C and let f be locally Lipschitz and M-P V-invex with respect to η and $\alpha_i, i \in M$ on S such that $\alpha_i, i \in M$ satisfy Condition D. If $\bar{x} \in S$ solves MP-MVVLI (f, S) with respect to η, then $\bar{x}$ is an efficient solution of the VOP (f, S).*

Proof Suppose to the contrary that $\bar{x}$ is not an efficient solution of the VOP (f, S). Then, there exists $\tilde{x} \in S$ such that

$$f_i(\tilde{x}) - f_i(\bar{x}) \leq 0, \quad \forall i \in M, \tag{9}$$

with strict inequality for at least one $i \in M$. Set $x(\lambda) := \bar{x} + \lambda\eta(\tilde{x}, \bar{x})$ for any $\lambda \in [0, 1]$. Since S is an invex set with respect to η, for any $\lambda \in [0, 1]$, $x(\lambda) \in S$. Since f is M-P V-invex with respect to η and $\alpha_i, i \in M$ on S, by Proposition 3, f_i is preinvex with respect to η for all $i \in M$ on S and hence, for all $i \in M$ and $\lambda \in [0, 1]$, one has

$$f_i(\bar{x} + \lambda\eta(\tilde{x}, \bar{x})) - f_i(\bar{x}) \leq \lambda \left[f_i(\tilde{x}) - f_i(\bar{x})\right].$$

In particular, for $\lambda = 1$, it follows that

$$f_i(\bar{x} + \eta(\tilde{x}, \bar{x})) - f_i(\bar{x}) \leq f_i(\tilde{x}) - f_i(\bar{x}), \quad \forall i \in M. \tag{10}$$

By the mean value theorem, for every $i \in M$, there exists $\hat{\lambda}_i \in (0, 1)$ and $\hat{x}_i^* \in \partial^\diamond f_i(x(\hat{\lambda}_i))$ such that

$$f_i(\bar{x} + \eta(\tilde{x}, \bar{x})) - f_i(\bar{x}) = \left\langle \hat{x}_i^*, \eta(\tilde{x}, \bar{x}) \right\rangle. \tag{11}$$

From (10) and (11), it follows that

$$\left\langle \hat{x}_i^*, \eta(\tilde{x}, \bar{x}) \right\rangle \leq f_i(\tilde{x}) - f_i(\bar{x}), \quad \forall i \in M. \tag{12}$$

Suppose that $\hat{\lambda}_1 = \hat{\lambda}_2 = \cdots = \hat{\lambda}_m = \hat{\lambda}$. Multiplying both the sides of the above inequalities by $-\hat{\lambda}$, and using skewness and Condition C for η, it follows that

$$\left\langle \hat{x}_i^*, \eta(x(\hat{\lambda}), \bar{x}) \right\rangle \leq \hat{\lambda} \left[f_i(\tilde{x}) - f_i(\bar{x})\right], \quad \forall i \in M.$$

From (9), it follows that, for every $i \in M$, there exists $x(\hat{\lambda}) \in S$ and $\hat{x}_i^* \in \partial^\diamond f_i(x(\hat{\lambda}))$ such that

$$\left\langle \hat{x}_i^*, \eta(x(\hat{\lambda}), \bar{x}) \right\rangle \leq 0, \quad \forall i \in M,$$

with strict inequality for at least one $i \in M$, a contradiction to the fact that $\bar{x} \in S$ solves the MP-MVVLI (f, S) and hence the result.

Consider the cases when $\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_m$ are not all equal. Without loss of generality, we may assume that $\hat{\lambda}_1 \neq \hat{\lambda}_2$. Then, from (12), one has

$$\left\langle \hat{x}_1^*, \eta(\tilde{x}, \bar{x}) \right\rangle \leq f_1(\tilde{x}) - f_1(\bar{x}), \tag{13}$$

and

$$\left\langle \hat{x}_2^*, \eta(\tilde{x}, \bar{x}) \right\rangle \leq f_2(\tilde{x}) - f_2(\bar{x}). \tag{14}$$

Since f is M-P V-invex with respect to η and α_i, $i \in M$ on S, by Proposition 1, $\partial^\diamond f$ is V-invariant monotone with respect to η and α_i, $i \in M$ on S, and hence, by Condition C for η and by Condition D for α_i, $i \in M$, it follows that, for all $\hat{x}_{12}^* \in \partial^\diamond f_1(x(\hat{\lambda}_2))$ and $\hat{x}_{21}^* \in \partial^\diamond f_2(x(\hat{\lambda}_1))$, one has

$$\alpha_1(\tilde{x}, \bar{x}) \left\langle \hat{x}_{12}^*, (\hat{\lambda}_1 - \hat{\lambda}_2)\eta(\tilde{x}, \bar{x}) \right\rangle + \alpha_1(\tilde{x}, \bar{x}) \left\langle \hat{x}_1^*, (\hat{\lambda}_2 - \hat{\lambda}_1)\eta(\tilde{x}, \bar{x}) \right\rangle \leq 0, \tag{15}$$

and

$$\alpha_2(\tilde{x}, \bar{x}) \left\langle \hat{x}_2^*, (\hat{\lambda}_1 - \hat{\lambda}_2)\eta(\tilde{x}, \bar{x}) \right\rangle + \alpha_2(\tilde{x}, \bar{x}) \left\langle \hat{x}_{21}^*, (\hat{\lambda}_2 - \hat{\lambda}_1)\eta(\tilde{x}, \bar{x}) \right\rangle \leq 0. \tag{16}$$

If $\hat{\lambda}_1 - \hat{\lambda}_2 > 0$, dividing (15) by $\alpha_1(\tilde{x}, \bar{x})(\hat{\lambda}_1 - \hat{\lambda}_2)$, and using (13), for all $\hat{x}_{12}^* \in \partial^\diamond f_1(x(\hat{\lambda}_2))$, one has

$$\left\langle \hat{x}_{12}^*, \eta(\tilde{x}, \bar{x}) \right\rangle \leq f_1(\tilde{x}) - f_1(\bar{x}).$$

If $\hat{\lambda}_2 - \hat{\lambda}_1 > 0$, dividing (16) by $\alpha_2(\tilde{x}, \bar{x})(\hat{\lambda}_2 - \hat{\lambda}_1)$, and using (14), for all $\hat{x}_{21}^* \in \partial^\diamond f_2(x(\hat{\lambda}_1))$, one has

$$\left\langle \hat{x}_{21}^*, \eta(\tilde{x}, \bar{x}) \right\rangle \leq f_2(\tilde{x}) - f_2(\bar{x}).$$

Therefore, for the case $\hat{\lambda}_1 \neq \hat{\lambda}_2$, setting $\hat{\lambda} := min \left\{\hat{\lambda}_1, \hat{\lambda}_2\right\}$, there exists $\bar{x}_i^* \in \partial^\diamond f_i(x(\bar{\lambda}))$ such that

$$\left\langle \bar{x}_i^*, \eta(\tilde{x}, \bar{x}) \right\rangle \leq f_i(\tilde{x}) - f_i(\bar{x}), \quad \forall i = 1, 2.$$

By continuation of this process, we can find $\bar{\bar{\lambda}} \in (0, 1)$ and $\bar{\bar{x}}_i^* \in \partial^\diamond f_i(x(\bar{\bar{\lambda}}))$ such that, $\bar{\bar{\lambda}} := min \left\{\hat{\lambda}_1, \ldots, \hat{\lambda}_m\right\}$ and

$$\left\langle \bar{\bar{x}}_i^*, \eta(\tilde{x}, \bar{x}) \right\rangle \leq f_i(\tilde{x}) - f_i(\bar{x}), \quad \forall i \in M.$$

Multiplying the above inequalities by $-\bar{\bar{\lambda}}$ and using skewness and Condition C for η, it follows that

$$\left\langle \bar{\bar{x}}_i^*, \eta(x(\bar{\bar{\lambda}}), \bar{x}) \right\rangle \leq \bar{\bar{\lambda}} \left[f_i(\tilde{x}) - f_i(\bar{x}) \right], \quad \forall i \in M.$$

From (9), there exists $x(\bar{\bar{\lambda}}) \in S$ and $\bar{\bar{x}}_i^* \in \partial^\diamond f_i(x(\bar{\bar{\lambda}})), i \in M$ such that, for all $i \in M$, one has

$$\left\langle \bar{\bar{x}}_i^*, \eta(x(\bar{\bar{\lambda}}), \bar{x}) \right\rangle \leq 0,$$

with strict inequality for at least one $i \in M$, a contradiction to the fact that $\bar{x} \in S$ solves MP-MVVLI (f, S) and hence the result. □

The following theorem is a direct consequence of Propositions 5 and 6.

Theorem 3 *Let S be a nonempty invex subset of X with respect to η such that η is skew and satisfies Condition C and let f be locally Lipschitz and M-P V-invex with respect to η and $\alpha_i, i \in M$ such that $\alpha_i, i \in M$ satisfy Condition D. Then, $\bar{x} \in S$ solves MP-MVVLI (f, S) with respect to η, iff $\bar{x}$ is an efficient solution of the NVOP $(f, S.)$*

The following corollary is a direct consequence of the fact that every efficient solution is also a weak efficient solution.

Corollary 2 *Let S be a nonempty invex subset of X with respect to η such that η is skew and satisfies Condition C and let f be locally Lipschitz and M-P V-invex with respect to η and $\alpha_i, i \in M$ such that $\alpha_i, i \in M$ satisfy Condition D. If $\bar{x} \in S$ solves MP-MVVLI (f, S) with respect to η, then $\bar{x}$ is a weak efficient solution of the NVOP $(f, S.)$*

5 Weak vector variational-like inequalities using the Michel–Penot subdifferentials

Now, we consider the weak formulation of the vector variational-like inequalities of Stampacchia type in terms of the Michel–Penot subdifferentials, denoted by MP-SWVVLI (f, S), as follows:

(MP-SWVVLI) To find $\bar{x} \in S$ such that, for all $x \in S$, there exists $\bar{x}_i^* \in \partial^\diamond f_i(\bar{x}), i \in M := \{1, \ldots, m\}$ such that, $(\langle \bar{x}_1^*, \eta(x, \bar{x}) \rangle, \ldots, \langle \bar{x}_m^*, \eta(x, \bar{x}) \rangle) \notin -int\mathbb{R}_+^m$.

Remark 4 It is clear that every solution of the MP-SVVLI (f, S) is also a solution of MP-SWVVLI (f, S), but the converse is not true in general.

The following result gives the condition under which a solution of the MP-SWVVLI (f, S) is also a weak efficient solution of the VOP (f, S).

Proposition 7 *Let S be a nonempty subset of X and let f be locally Lipschitz and M-P V-invex with respect to η and α_i, $i \in M$ at $\bar{x} \in S$ over S. If $\bar{x}$ solves the MP-SWVVLI (f, S) with respect to η, then $\bar{x}$ is a weak efficient solution of the VOP (f, S).*

Proof Suppose to the contrary that $\bar{x}$ is not a weak efficient solution of the VOP (f, S). Then, there exists $\widetilde{x} \in S$ such that

$$f_i(\widetilde{x}) - f_i(\bar{x}) < 0, \quad \forall i \in M. \tag{17}$$

By the M-P V-invexity of f at $\bar{x}$ over S with respect to η and α_i, $i \in M$, and since $\alpha_i(\widetilde{x}, \bar{x}) > 0$ for all $i \in M$, it implies that

$$\left\langle \bar{x}_i^*, \eta(\widetilde{x}, \bar{x}) \right\rangle < 0, \quad \forall \bar{x}_i^* \in \partial^\diamond f_i(\bar{x}), \forall i \in M,$$

a contradiction to the fact that $\bar{x}$ solves MP-SWVVLI (f, S) and hence the result.

Now, we consider the weak formulation of the vector variational-like inequalities of Minty type in terms of the Michel–Penot subdifferentials, denoted by MP-MWVVLI (f, S), as follows:

(MP-MWVVLI) To find $\bar{x} \in S$ such that, for all $x \in S$, and for all $x_i^* \in \partial^\diamond f_i(x)$, $i \in M := \{1, \ldots, m\}$, one has, $(\left\langle x_1^*, \eta(x, \bar{x}) \right\rangle, \ldots, \left\langle x_m^*, \eta(x, \bar{x}) \right\rangle) \notin -int\mathbb{R}_+^m$.

Remark 5 Obviously, every solution of the MP-MVVLI (f, S) is also a solution of the MP-MWVVLI (f, S), but the converse is not true in general.

The following result is a direct consequence of Proposition 5.

Corollary 3 *Let S be a nonempty subset of X and let $f := (f_1, \ldots, f_m) : X \to \mathbb{R}^m$ be locally Lipschitz and M-P V-invex on S with respect to η and α_i, $i \in M$ such that η is skew and α_i, $i \in M$ is symmetric. If $\bar{x}$ is an efficient solution of the VOP (f, S), then $\bar{x}$ solves the MP-MWVVLI (f, S) with respect to η.*

The following corollary is a direct consequence of Propositions 4 and 5.

Corollary 4 *Let S be a nonempty subset of X and let $f := (f_1, \ldots, f_m) : X \to \mathbb{R}^m$ be locally Lipschitz and M-P V-invex on S with respect to η and α_i, $i \in M$ such that η is skew and α_i, $i \in M$ is symmetric. If $\bar{x}$ solves the MP-SVVLI (f, S) with respect to η, then $\bar{x}$ solves the MP-MWVVLI (f, S) with respect to η.*

The following result gives the relationship between the MP-SWVVLI (f, S) and MP-MWVVLI (f, S).

Proposition 8 *Let S be a nonempty subset of X and let f be M-P V-invex with respect to η and α_i, $i \in M$ on S such that η is skew. If $\bar{x} \in S$ solves the MP-SWVVLI (f, S) with respect to η, then it also solves the MP-MWVVLI (f, S) with respect to η.*

Proof Suppose that $\bar{x}$ solves the MP-SWVVLI (f, S) with respect to η. Then, for every $i \in M$, there exists $\bar{x}_i^* \in \partial^\diamond f_i(\bar{x})$ such that, for all $x \in S$, one has

$$(\left\langle \bar{x}_1^*, \eta(x, \bar{x}) \right\rangle, \ldots, \left\langle \bar{x}_m^*, \eta(x, \bar{x}) \right\rangle) \notin -int\mathbb{R}_+^m. \tag{18}$$

Since f is M-P V-invex with respect to η and $\alpha_i, i \in M$ on S, by Proposition 1, $\partial^\diamond f$ is V-invariant monotone with respect to η and $\alpha_i, i \in M$ on S and hence, by skewness of η, for all $i \in M$, $x \in S$ and $x_i^* \in \partial^\diamond f_i(x)$, one has

$$\frac{\alpha_i(x,\bar{x})}{\alpha_i(\bar{x},x)}\left\langle \bar{x}_i^*, \eta(x,\bar{x})\right\rangle \leq \left\langle x_i^*, \eta(x,\bar{x})\right\rangle. \tag{19}$$

From (18) and (19), for all $x \in S$, $i \in M$ and $x_i^* \in \partial^\diamond f_i(x)$, one has

$$\left(\left\langle x_1^*, \eta(x,\bar{x})\right\rangle, \ldots, \left\langle x_m^*, \eta(x,\bar{x})\right\rangle\right) \notin -int\mathbb{R}_+^m.$$

Thus, $\bar{x}$ solves the MP-MWVVLI (f, S) and hence the result. □

The following result gives the condition under which a weak efficient solution of the NVOP (f, S) also solves the MP-MWVVLI (f, S).

Proposition 9 *Let S be a nonempty subset of X and let f be locally Lipschitz and M-P V-invex on S with respect to η and $\alpha_i, i \in M$ such that η is skew and $\alpha_i, i \in M$ is symmetric. If $\bar{x}$ is a weak efficient solution of the VOP (f, S), then $\bar{x}$ solves the MP-MWVVLI (f, S) with respect to η.*

Proof Suppose to the contrary that $\bar{x}$ does not solve the MP-MWVVLI (f, S) with respect to η. Then, there exists $\tilde{x} \in S$ and $\tilde{x}_i^* \in \partial^\diamond f_i(\tilde{x}), i \in M$ such that, $\left\langle \tilde{x}_i^*, \eta(\tilde{x},\bar{x})\right\rangle < 0, \forall i \in M$. Since, η is skew, $\alpha_i, i \in M$ is symmetric and $\alpha_i(\tilde{x},\bar{x}) > 0, i \in M$, it follows that $\alpha_i(\bar{x},\tilde{x})\left\langle \tilde{x}_i^*, \eta(\bar{x},\tilde{x})\right\rangle > 0, \forall i \in M$. By the M-P V-invexity of f on S, it implies that $f_i(\tilde{x}) - f_i(\bar{x}) < 0, \forall i \in M$, a contradiction to the fact that $\bar{x}$ is a weak efficient solution of the VOP (f, S) and hence the result.

6 Conclusions

In this paper, we have formulated Stampacchia and Minty type vector variational-like inequalities in terms of the Michel–Penot subdifferentials which is the smallest among all the convex valued subdifferentials. We have established the relationships among the solutions of the Stampacchia and Minty type vector variational-like inequalities and the efficient solutions of the vector optimization problems involving locally Lipschitz Michel–Penot V-invex functions and could thus overcome the restriction of the requirement of the same kernel function for all the involved objective functions. We have also considered the corresponding weak versions of the Stampacchia and Minty vector variational-like inequalities and also established relationships between their solutions and the weak efficient solutions of the nonsmooth vector optimization problem under the assumption of Michel–Penot V-invexity. The results of this paper are more general and sharper than the corresponding results present in literature (see, e.g. [3,24]) due to the use of V-invexity and Michel–Penot subdifferentials. Further, the results of this paper may be extended using some more general locally Lipschitz V-r-invexity assumptions, which was introduced in [5] and further studied in [31] and [32].

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