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ORIGINAL PAPER

Generalized vector quasi-equilibrium problems on Hadamard manifolds

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Abstract In this paper, a generalized vector quasi-equilibrium problem (GVQEP) is introduced and studied on Hadamard manifolds. An existence theorem of solutions for the GVQEP is established under some suitable conditions. Some applications to a generalized vector quasi-variational inequality, a generalized vector variational-like inequality and a vector optimization problem are also presented on Hadamard manifolds.

Keywords Hadamard manifold · Generalized vector quasi-equilibrium problem · Generalized vector quasi-variational inequality · Generalized vector variational-like inequality · Vector optimization problem

1 Introduction

The equilibrium problem was introduced and studied by Blum and Oettli [1] in 1994 as a generalization of optimization problems and variational inequalities. This problem contains many important problems as special cases, including Nash equilibrium, complementarity and fixed point problems. Recently, there has been an increasing interest in the study of vector equilibrium problems because it provides a unified way to research some nonlinear problems, for instance, vector variational inequalities, vector complementary problems, vector saddle point problems and vector optimization

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problems. In recent decades, many results concerned with the existence of solutions for the vector equilibrium problems and the vector quasi-equilibrium problems have been established (see, for example, [2–14] and the references therein).

On the other hand, recent interests of a number of researchers are focused on extending some concepts and techniques of nonlinear analysis in Euclidean spaces to Riemannian manifolds (see [15–18]). In general, a manifold is not a linear space. In this setting, the linear space is replaced by a Riemannian manifold and the line segment by a geodesic (see [16,19]). There are some advantages for a generalization of optimization methods from Euclidean spaces to Riemannian manifolds, because non-convex and non-smooth of constrained optimization problems can be seen as convex and smooth unconstrained optimization problems from the Riemannian geometry point of view (see [19–21]). Németh [22] and Colao et al. [23] introduced the variational inequalities and equilibrium problems on Hadamard manifolds, respectively. They studied the existence of the solutions for variational inequalities and equilibrium problems on Hadamard manifolds under some suitable conditions. Recently, Zhou and Huang [24] investigated the relationship between the vector variational inequality and the vector optimization problem on Hadamard manifolds, and proved the existence of solutions for the vector variational inequality on Hadamard manifolds. However, to the best of our knowledge, there is no paper to study the generalized vector quasi-equilibrium problem (GVQEP) on Hadamard manifolds. Therefore, it is an interesting problem to consider the GVQEP on Hadamard manifolds.

Motivated and inspired by the work mentioned above, in this paper, we introduce and study a GVQEP on Hadamard manifolds. We establish an existence theorem of solutions for the GVQEP under some suitable conditions. We also give some applications to a generalized vector quasi-variational inequality (GVQVI), a generalized vector variational-like inequality and a vector optimization problem on Hadamard manifolds.

2 Preliminaries

In this section we recall some fundamental definitions, properties and notations used throughout this paper. These can be founded in any introductory books on Riemannian geometry, for example, [25–27].

Let M be a simply connected m-dimensional manifold and $x \in M$. The tangent space of M at x is denoted by T_xM and the tangent bundle of M by $TM = \bigcup_{x\in M} T_xM$, which is naturally a manifold. We always assume that M is endowed with a Riemannian metric to become a Riemannian manifold. We denote by $\langle \cdot, \cdot \rangle_x$ the scalar product on T_xM with the associated norm $\| \cdot \|_x$, where the subscript x is sometimes omitted. Given a piecewise smooth curve $\gamma : [a, b] \to M$ joining x to y, that is, $\gamma(a) = x$ and $\gamma(b) = y$, we can define the length of γ by $l(\gamma) = \int_a^b \|\gamma'(t)\| dt$. Then for any $x, y \in M$, the Riemannian distance $d(x, y)$ which induces the original topology on M, is defined by minimizing this length over the set of all such curves joining x to y.

Let ∇ be the Levi-Civita connection associated with the Riemannian metric. If γ is a curve joining x to y in M, then for each $t \in [a, b]$, ∇ induces an isometry

$P_{\gamma,\gamma(t),x} : T_x M \to T_{\gamma(t)} M$, the so-called parallel transport along γ from x to $\gamma(t)$. When the reference to a curve joining x to y is not necessary, we use the notation $P_{y,x}$. We say that γ is a geodesic when $\nabla_{\gamma'}^{\gamma'} = 0$, in this case $\|\gamma'\| = 1$, γ is said to be normalized. A geodesic joining x to y in M is said to be minimal if its length equals $d(x, y)$.

A Riemannian manifold is complete if for any $x \in M$ all geodesic emanating from x are defined for all $-\infty < t < +\infty$. By the Hopf–Rinow Theorem, we know that, if M is complete, then any pair of points in M can be joined by a minimal geodesic. Moreover, (M, d) is a complete metric space and bounded closed subsets are compact.

Assuming that M is complete. The exponential map $\exp_x : T_x M \to M$ at x is defined by $\exp_x(v) = \gamma_v(1, x)$ for each $v \in T_x M$, where $\gamma(\cdot) = \gamma_v(\cdot, x)$ is the geodesic starting x with velocity v, that is, $\gamma(0) = x$ and $\gamma'(0) = v$. It is easy to see that $\exp_x(tv) = \gamma_v(t, x)$ for each real number t.

Definition 2.1 A Hadamard manifold M is a complete simply connected Riemannian manifold of nonpositive sectional curvature.

Proposition 2.1 [27] *Let M be a Hadamard manifold and $x \in M$. Then $\exp_x : T_x M \to M$ is a diffeomorphism and for any two points $x, y \in M$, there exists a unique normalized geodesic $\gamma_{x,y} = \exp_x t \exp_x^{-1} y$ for all $t \in [0, 1]$ joining x to y.*

Definition 2.2 Let M be a Hadamard manifold. A subset $K \subseteq M$ is said to be convex if, for any points x and y in K, the geodesic joining x to y is contained in K; that is, if $\gamma : [a, b] \to M$ is a geodesic such that $x = \gamma(a)$ and $y = \gamma(b)$, then $\gamma_{x,y} = \exp_x t \exp_x^{-1} y \in K$ for all $t \in [0, 1]$.

Definition 2.3 Let M be a Hadamard manifold, $S : M \to 2^M$ be a set-valued mapping and $x_0 \in M$. Then S is said to be

(i) upper semicontinuous at x_0 if, for any open set $V \subseteq M$ satisfying $S(x_0) \subseteq V$, there exists an open neighborhood $U(x_0)$ of x_0 such that $S(x) \subseteq V$ for all $x \in U(x_0)$;
(ii) lower semicontinuous at x_0 if, for any open set $V \subseteq M$ satisfying $S(x_0) \bigcap V \neq \emptyset$, there exists an open neighborhood $U(x_0)$ of x_0 such that $S(x) \bigcap V \neq \emptyset$ for all $x \in U(x_0)$;
(iii) upper Kuratowski semicontinuous at x_0 if, for any sequences $\{x_k\}, \{y_k\} \subseteq M$ with each $y_k \in S(x_k)$, the relations $\lim_{k\to\infty} x_k = x_0$ and $\lim_{k\to\infty} y_k = y_0$ imply $y_0 \in S(x_0)$;
(iv) upper semicontinuous (resp. lower semicontinuous, upper Kuratowski semicontinuous) on M if S is upper semicontinuous (resp. lower semicontinuous, upper Kuratowski semicontinuous) at every point $x \in M$;
(v) continuous on M if S is upper semicontinuous and lower semicontinuous at every point $x \in M$.

Definition 2.4 [28] Let M be a Hadamard manifold, $A : M \to 2^{TM}$ be a set-valued vector field and $x_0 \in M$. Then A is said to be

(i) upper semicontinuous at x_0 if, for any open set $V \subseteq M$ satisfying $A(x_0) \subseteq V \subseteq T_{x_0} M$, there exists an open neighborhood $U(x_0)$ of x_0 such that $P_{x_0,x} A(x) \subseteq V$ for all $x \in U(x_0)$;

(ii) upper Kuratowski semicontinuous at x_0 if for any sequence $\{x_n\} \subseteq M$ and $\{u_n\} \subseteq TM$ with each $u_n \in A(x_n)$, the relations $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} u_n = u_0$ imply $u_0 \in A(x_0)$;

(iii) upper semicontinuous (resp. upper Kuratowski semicontinuous) on M if A is upper semicontinuous (resp. upper Kuratowski semicontinuous) at each point $x \in M$.

Remark 2.1 It is easy to check that any upper semicontinuous and closed valued vector field A is upper Kuratowski semicontinuous.

Definition 2.5 Let M be a Hadamard manifold and $K \subseteq M$ be a nonempty convex set, Y be a Hausdorff topological vector space and $W \subseteq Y$ be a convex cone. Then $f : K \to Y$ is said to be

(i) W-convex if, for any $x, y \in K$,

$$f(\exp_x t \exp_x^{-1} y) \in (1-t)f(x) + tf(y) - W, \quad \forall t \in [0,1];$$

(ii) natural W-quasi-convex if, for any $x, y \in K$, $t \in [0,1]$,

$$f(\exp_x t \exp_x^{-1} y) \in co(f(x), f(y)) - W,$$

where $co(E)$ is the convex hull of E;

(iii) W-quasi-convex if, for any $x, y \in K$, $t \in [0,1]$,

$$f(\exp_x t \exp_x^{-1} y) \in z - W, \quad \forall z \in \bigcup(f(x), f(y)),$$

where $\bigcup(f(x), f(y))$ is the upper boundary of $f(x)$ and $f(y)$, that is,

$$\bigcup(f(x), f(y)) = \{z \in Y : f(x) \in z - W \text{ and } f(y) \in z - W\}.$$

Remark 2.2 (i) The notions of W-convexity, natural W-quasi-convexity and W-quasi-convexity on Hadamard manifold are generalizations of these in Hausdorff topological vector space [29];

(ii) It is clear that W-convexity implies natural W-quasi-convexity and natural W-quasi-convexity implies W-quasi-convexity.

3 A generalized vector quasi-equilibrium problem

In this section, we introduce and study a GVQEP on Hadamard manifold and prove the existence theorem for it, this is a generalization of some results contained in [23,24].

From now on, let M be a Hadamard manifold and $K \subseteq M$ be a nonempty closed set, let $C \subseteq TM$ be a nonempty set and $A : K \to 2^C$ be a set-valued vector field, $S : K \to 2^K$, $F : K \times C \times K \to \mathbb{R}^p$ be two mappings. We consider the following GVQEP: find $\overline{x} \in K$ and $\overline{z} \in A(\overline{x})$ such that

$$\overline{x} \in S(\overline{x}) \text{ and } F(\overline{x}, \overline{z}, y) \notin -int\mathbb{R}^p_+, \quad \forall y \in S(\overline{x}).$$

Some special cases are as follows.

(I) Let $S(x) = K$ for every $x \in K$, $F(x, z, y) = \langle z, \exp_x^{-1} y\rangle$ for all $(x, z, y) \in K \times C \times K$ and A be a single-valued vector field. Then (GVQEP) reduces to the vector variational inequality on Hadamard manifold [24], which consists in finding $x^* \in K$ such that

$$\langle A(x^*), \exp_{x^*}^{-1} y\rangle \notin -int\mathbb{R}_+^p, \quad \forall y \in K.$$

(II) Let $S(x) = K$ and $A(x) = \overline{z}$ for every $x \in K$, $F(x, z, y) = h(x, y)$ for all $(x, z, y) \in K \times C \times K$ and $p = 1$. Then (GVQEP) reduces to the equilibrium problem on Hadamard manifold [23], which consists in finding $x^* \in K$ such that

$$h(x^*, y) \geq 0, \quad \forall y \in K.$$

Definition 3.1 [30] Let $Q \subseteq \mathbb{R}^p$ be a nonempty set and $\mathbb{R}_+^p \subseteq \mathbb{R}^p$ be a closed convex cone. Then $\overline{y} \in Q$ is said to be a weak minimizer of Q if, for any $y \in Q$, $y - \overline{y} \notin -int\mathbb{R}_+^p$. All weak minimizers of Q are denoted by $Wmin_{\mathbb{R}_+^p} Q$.

Lemma 3.1 [30] *Let $Q \subseteq \mathbb{R}^p$ be a nonempty compact set and $\mathbb{R}_+^p \subseteq \mathbb{R}^p$ be a closed convex cone. Then $Wmin_{\mathbb{R}_+^p} Q \neq \emptyset$.*

The following result provides a fixed point theorem in the setting of Hadamard manifold, it follows from Theorem 3.10 in [23].

Lemma 3.2 *Let M be a Hadamard manifold and $K \subseteq M$ be a compact convex set, $V : K \to 2^K$ be a upper Kuratowski semicontinuous mapping. Assume that for any $x \in K$, $V(x)$ is closed and convex. Then there exists a fixed point of V.*

Theorem 3.1 *Let M be a Hadamard manifold. Suppose that the following conditions hold:*

(i) *$K \subseteq M$ is nonempty compact convex;*
(ii) *$S : K \to 2^K$ is a continuous set-valued mapping such that $S(x)$ is nonempty closed convex for every $x \in K$;*
(iii) *$A : K \to 2^{TM}$ is an upper Kuratowski semicontinuous vector field such that $A(x)$ is compact convex for every $x \in K$, $C \subseteq TM$ is a compact convex set and $A(x) \subseteq C$ for every $x \in K$;*
(iv) *$F : K \times C \times K \to \mathbb{R}^p$ is a continuous mapping satisfying the following conditions:*
    (a) *$F(x, z, x) \in \mathbb{R}_+^p$ for every $(x, z) \in K \times C$;*
    (b) *the mapping $y \mapsto F(x, z, y)$ is $\mathbb{R}_+^p$-quasi-convex for every $(x, z) \in K \times C$.*

Then there exist $\overline{x} \in S(\overline{x})$ and $\overline{z} \in A(\overline{x})$ such that

$$F(\overline{x}, \overline{z}, y) \notin -int\mathbb{R}_+^p, \quad \forall y \in S(\overline{x}).$$

Proof We define a set-valued mapping $P : K \times C \to 2^K$ as follows:

$$P(x, z) = \{u \in S(x) : F(x, z, u) \in Wmin_{\mathbb{R}^p_+} F(x, z, S(x))\}.$$

Since K is compact, $S(x)$ is closed and $S(x) \subseteq K$ for every $x \in K$, we know that $S(x)$ is compact for every $x \in K$. It follows from the continuity of F that $F(x, z, S(x))$ is compact for every $(x, z) \in K \times C$. Moreover, Lemma 3.1 shows that $Wmin_{\mathbb{R}^p_+} F(x, z, S(x)) \neq \emptyset$ and so $P(x, z) \neq \emptyset$ for every $(x, z) \in K \times C$.

We shall show that

(I) $P(x, z)$ is convex for every $(x, z) \in K \times C$;
(II) $P(x, z)$ is compact for every $(x, z) \in K \times C$;
(III) P is upper Kuratowski semicontinuous on $K \times C$.

To show (I), let $u_1, u_2 \in P(x, z)$ and $t \in [0, 1]$. By the definition of $P(x, z)$, one has $u_1, u_2 \in S(x)$,

$$F(x, z, y) - F(x, z, u_1) \in \mathbb{R}^p \backslash (-int\mathbb{R}^p_+) = \mathbb{R}^p_+, \quad \forall y \in S(x),$$

and

$$F(x, z, y) - F(x, z, u_2) \in \mathbb{R}^p \backslash (-int\mathbb{R}^p_+) = \mathbb{R}^p_+, \quad \forall y \in S(x).$$

Since $y \mapsto F(x, z, y)$ is $\mathbb{R}^p_+$-quasi-convex for every $(x, z) \in K \times C$, we have

$$F(x, z, y) - F(x, z, \exp_{u_1} t \exp^{-1}_{u_1} u_2) \in \mathbb{R}^p_+, \quad \forall y \in S(x)$$

and so

$$F(x, z, \exp_{u_1} t \exp^{-1}_{u_1} u_2) \in Wmin_{\mathbb{R}^p_+} F(x, z, S(x)), \quad \forall (x, z) \in K \times C. \tag{1}$$

Again since $S(x)$ is convex for every $x \in K$, one has $\exp_{u_1} t \exp^{-1}_{u_1} u_2 \in S(x)$. This together with (1) implies that $\exp_{u_1} t \exp^{-1}_{u_1} u_2 \in P(x, z)$ and so $P(x, z)$ is convex for every $(x, z) \in K \times C$. Then (I) holds.

Next, we prove (II). Since K is compact and $P(x, z) \subseteq K$. We need to prove that $P(x, z)$ is closed for every $(x, z) \in K \times C$. Let $\{u_n\} \subseteq P(x, z)$ and $u_n \to \overline{u}$. We show that $\overline{u} \in P(x, z)$. In fact, since $\{u_n\} \subset P(x, z)$, one has $u_n \in S(x)$ and

$$F(x, z, y) - F(x, z, u_n) \notin -int\mathbb{R}^p_+, \quad \forall y \in S(x).$$

That is,

$$F(x, z, y) - F(x, z, u_n) \in \mathbb{R}^p \backslash (-int\mathbb{R}^p_+) = \mathbb{R}^p_+, \quad \forall y \in S(x).$$

Since $\mathbb{R}^p_+$ is closed, F is continuous and $u_n \to \overline{u}$, we have

$$F(x, z, y) - F(x, z, \overline{u}) \notin -int\mathbb{R}^p_+, \quad \forall y \in S(x)$$

and so

$$F(x, z, \overline{u}) \in Wmin_{\mathbb{R}^p_+} F(x, z, S(x)). \tag{2}$$

Since $S(x)$ is closed valued for every $x \in K$, we know that $\overline{u} \in S(x)$. This together with (2) implies that $\overline{u} \in P(x, z)$. Thus, $P(x, z)$ is compact for every $(x, z) \in K \times C$. Then (II) holds.

Next we prove that P is upper Kuratowski semicontinuous on $K \times C$. Let $\{(x_k, z_k)\} \subseteq K \times C$ and $(x_k, z_k) \to (x, z)$ with $u_k \in P(x_k, z_k)$ and $u_k \to u \in K$. Since $u_k \in P(x_k, z_k)$, one has $u_k \in S(x_k)$. Since S is continuous and closed valued, we know that $u \in S(x)$ and for every $y \in S(x)$, there exists $y_k \in S(x_k)$ such that $y_k \to y$. On the other hand, since $u_k \in P(x_k, z_k)$, one has

$$F(x_k, z_k, y_k) - F(x_k, z_k, u_k) \notin -int\mathbb{R}^p_+.$$

The continuity of F shows that

$$\lim_{n\to\infty} F(x_k, z_k, u_k) = F(x, z, u).$$

This together with

$$\lim_{n\to\infty} F(x_k, z_k, y_k) = F(x, z, y)$$

implies that

$$\lim_{n\to\infty} F(x_k, z_k, y_k) - \lim_{n\to\infty} F(x_k, z_k, u_k) = \lim_{n\to\infty} \{F(x_k, z_k, y_k) - F(x_k, z_k, u_k)\} \in \mathbb{R}^p \setminus (-int\mathbb{R}^p_+).$$

Thus, for any $y \in S(x)$, we have

$$F(x, z, y) - F(x, z, u) \notin -int\mathbb{R}^p_+$$

and so

$$F(x, z, u) \in Wmin_{\mathbb{R}^p_+} F(x, z, S(x)).$$

Therefore, $u \in P(x, z)$. This shows that P is upper Kuratowski semicontinuous on $K \times C$. Thus, (III) holds.

Now we define a set-valued mapping $G : K \times C \to 2^{K\times C}$ as follows:

$$G(x, z) = P(x, z) \times A(x).$$

It is obvious that G is upper Kuratowski semicontinuous on $K \times C$ and $G(x, z)$ is compact convex for every $(x, z) \in K \times C$. By Lemma 3.2, there exists $(\overline{x}, \overline{z}) \in K \times C$

such that

$$(\overline{x}, \overline{z}) \in G(\overline{x}, \overline{z}) = P(\overline{x}, \overline{z}) \times A(\overline{x}).$$

That is, $\overline{x} \in P(\overline{x}, \overline{z})$ and $\overline{z} \in A(\overline{x})$. Thus, there exists $\overline{z} \in A(\overline{x})$ such that

$$\overline{x} \in S(\overline{x}) \text{ and } F(\overline{x}, \overline{z}, y) - F(\overline{x}, \overline{z}, \overline{x}) \notin -int\mathbb{R}^p_+, \quad \forall y \in S(\overline{x}).$$

Next we show that $F(\overline{x}, \overline{z}, y) \notin -int\mathbb{R}^p_+$ for all $y \in S(\overline{x})$. If there exists some $y^* \in S(x^*)$ such that $F(\overline{x}, \overline{z}, y^*) \in -int\mathbb{R}^p_+$, then it follows from $F(\overline{x}, \overline{z}, \overline{x}) \in \mathbb{R}^p_+$ that

$$F(\overline{x}, \overline{z}, y^*) - F(\overline{x}, \overline{z}, \overline{x}) \in -int\mathbb{R}^p_+ - \mathbb{R}^p_+ = -int\mathbb{R}^p_+,$$

which is a contradiction. Thus, there exist $\overline{x} \in K$ and $\overline{z} \in A(\overline{x})$ such that

$$\overline{x} \in S(\overline{x}) \text{ and } F(\overline{x}, \overline{z}, y) \notin -int\mathbb{R}^p_+, \quad \forall y \in S(\overline{x}).$$

This completes the proof. □

By Remark 2.2 and Theorem 3.1, we have the following result.

Corollary 3.1 *Let M be a Hadamard manifold. Suppose that the following conditions hold:*

(i) *$K \subseteq M$ is nonempty compact convex;*
(ii) *$S : K \to 2^K$ is a continuous set-valued mapping and $S(x)$ is nonempty closed convex for every $x \in K$;*
(iii) *$A : K \to 2^{TM}$ is a upper Kuratowski semicontinuous vector field and $A(x)$ is compact convex for every $x \in K$, $C \subseteq TM$ is a compact convex set and $A(x) \subseteq C$ for every $x \in K$;*
(iv) *$F : K \times C \times K \to \mathbb{R}^p$ is a continuous mapping satisfying the following conditions:*
   (a) *$F(x, z, x) \in \mathbb{R}^p_+$ for every $(x, z) \in K \times C$;*
   (b) *the mapping $y \mapsto F(x, z, y)$ is natural $\mathbb{R}^p_+$-quasi-convex for every $(x, z) \in K \times C$.*

Then there exist $\overline{x} \in S(\overline{x})$ and $\overline{z} \in A(\overline{x})$ such that

$$F(\overline{x}, \overline{z}, y) \notin -int\mathbb{R}^p_+, \quad \forall y \in S(\overline{x}).$$

Remark 3.1 (i) Let $S(x) = K$ for every $x \in K$, $F(x, z, y) = \langle z, \exp_x^{-1} y\rangle$ for all $(x, z, y) \in K \times C \times K$ and A be a single-valued vector field. Then Theorem 3.1 and Corollary 3.1 can be regarded as a generalization of Theorem 3.2 in [24].

(ii) Let $S(x) = K$ and $A(x) = \overline{z}$ for every $x \in K$, $F(x, z, y) = h(x, y)$ for all $(x, z, y) \in K \times C \times K$ and $p = 1$. Then Theorem 3.1 and Corollary 3.1 can be regarded as a generalization of Theorem 3.2 in [23].

Next, we present an example to illustrate that all conditions in Theorem 3.1 can be satisfied and so we can solve the vector quasi-equilibrium problem in the setting of Hadamard manifold by employing Theorem 3.1. However, the classical existence theorems for the vector quasi-equilibrium problems in the setting of Euclidean spaces are not valid.

Example 3.1 Let $M = (\mathbb{R}_{++}, \langle , \rangle)$ be the Riemannian manifold, where $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ and $\langle , \rangle$ is the Riemannian metric $\langle , \rangle = g(x)uv$ with $g : \mathbb{R}_{++} \to (0, +\infty)$. Since the map $\phi : \mathbb{R} \to M$ given by $\phi(x) = e^x$ is an isometry, the sectional curvature of M is $K \equiv 0$. Moreover, the tangent plane at $x \in M$ denoted by $T_x M$, equals $\mathbb{R}$. The Riemannian distance $d : M \times M \to \mathbb{R}_+$ is given by

$$d(x, y) = |\phi^{-1}(x) - \phi^{-1}(y)| = |\ln(x/y)|,$$

(see, for example [31]). Therefore, $(\mathbb{R}_{++}, \langle , \rangle)$ is a Hadamard manifold. The geodesic curve $\gamma : \mathbb{R} \to M$ starting form $x(\gamma(0) = x)$ will have the equation

$$\gamma(t) = xe^{(v/x)t},$$

where $v = \gamma'(0) \in T_x M$ is the tangent unit vector of γ in the starting point.

The above equation implies that

$$\exp_x tv = xe^{(v/x)t}.$$

To get the expression of the inverse exponential map, for any $x, y \in M$, we write

$$y = \exp_x \left(d(x, y) \frac{\exp_x^{-1} y}{d(x, y)} \right) = xe^{\left(\frac{\exp_x^{-1} y}{xd(x,y)}\right) d(x,y)} = xe^{\frac{\exp_x^{-1} y}{x}}.$$

It follows that

$$\exp_x^{-1} y = x \ln(y/x).$$

Let $K = \{x | x = e^t, t \in [0, 1]\}$ and $p = 2$. For any $x \in K$, let $S(x) = \{y | y = e^t, t \in [\ln x, 1]\}$ and $A(x) = [1, e]$. For any $x, y \in K$ and $z \in T_x M$, define $F(x, z, y) = (f_1(x, z, y), f_2(x, z, y))$, where

$$f_1(x, z, y) = \langle z, \exp_x^{-1} y \rangle + \ln y - \ln x, \quad f_2(x, z, y) = (zx - 3)(\ln y - \ln x).$$

Note that K is non-convex in $\mathbb{R}$, $S(x)$ is non-convex for all $x \in K$, f_1 and f_2 are both non-convex in the third variable under the usual sense, i.e., in the case M is endowed with the Euclidean metric. Thus, the classical existence theorems for the vector quasi-equilibrium problems in the setting of Euclidean spaces are no longer valid. However, by using the expression of geodesics, K is convex, $S(x)$ is convex for all $x \in K$, f_1 and f_2 are convex in the third variable. It is easy to verify that all conditions in Theorem 3.1 hold and so Theorem 3.1 implies the existence of solutions for (GVQEP) and the set $\{x|x = e^t, t \in [\ln(3/e), 1]\}$ is the solution set of (GVQEP) on M.

4 Applications

This section is devoted to some applications of the existence theorem for GVQEPs on Hadamard manifolds. The first one is to GVQVIs, followed by the application to generalized vector variational-like inequalities. The last one is to vector optimization problems.

4.1 Generalized vector quasi-variational inequalities

Let M be a Hadamard manifold, $K \subseteq M$ be a nonempty closed set, $V_i : K \to 2^{TM} (i = 1, 2, \ldots, p)$ be a set-valued vector field and $S : K \to 2^K$ be a set-valued mapping. We denote by $V = V_1 \times V_2 \times \cdots \times V_p$ and consider the following GVQVI: find $\overline{x} \in K$ and $\overline{z} \in V(\overline{x})$ such that

$$\overline{x} \in S(\overline{x}) \text{ and } \langle \overline{z}, \exp_{\overline{x}}^{-1} y \rangle \notin -int\mathbb{R}_+^p, \quad \forall y \in S(\overline{x}).$$

Lemma 4.1 [28] *Let M be a Hadamard manifold, $x_0 \in M$ and $\{x_n\} \subseteq M$ with $x_n \to x_0$. Then the following assertions hold.*

(i) *For any $y \in M$, we have*

$$\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} y \text{ and } \exp_y^{-1} x_n \to \exp_y^{-1} x_0.$$

(ii) *Given $u_n, v_n \in T_{x_n}M$ and $u_0, v_0 \in T_{x_0}M$, if $u_n \to u_0$ and $v_n \to v_0$, then $\langle u_n, v_n \rangle \to \langle u_0, v_0 \rangle$.*

Lemma 4.2 [23,32] *Let M be a Hadamard manifold, $K \subseteq M$ be a nonempty set, $x \in K$ and $u \in T_xM$. Define a mapping $g : M \to \mathbb{R}$ by*

$$g(y) = \langle u, \exp_x^{-1} y \rangle.$$

Then both g are affine, in other words, g and $-g$ are convex functions.

Theorem 4.1 *Let M be a Hadamard manifold. Suppose that the following conditions hold:*

(i) *$K \subseteq M$ is nonempty compact convex;*
(ii) *$S : K \to 2^K$ is a continuous set-valued mapping and $S(x)$ is nonempty closed convex for every $x \in K$;*
(iii) *For each $i = 1, 2, \ldots, p$, $V_i : K \to 2^{TM}$ is an upper Kuratowski semicontinuous vector field and $V_i(x)$ is compact convex for every $x \in K$, $C_i \subseteq TM$ is a compact convex set and $V_i(x) \subseteq C_i$ for every $x \in K$.*

Then there exist $\overline{x} \in S(\overline{x})$ *and* $\overline{z} \in V(\overline{x})$ *such that*

$$\langle \overline{z}, \exp_{\overline{x}}^{-1} y \rangle \notin -int\mathbb{R}_+^p, \quad \forall y \in S(\overline{x}).$$

Proof We denote by $C = C_1 \times C_2 \times \cdots \times C_p$ and define $\overline{F} : K \times C \times K \to \mathbb{R}^p$ as

$$\overline{F}(x, z, y) = \langle z, \exp_x^{-1} y \rangle.$$

Obviously the solutions of (GVQVI) are the solutions of $\overline{F}$. It is straightforward to see that $\overline{F}$ satisfies hypotheses (i), (ii) and (iii) in Theorem 3.1. Now we only need to prove (iv) of Theorem 3.1. It follows from Lemma 4.1 that the mapping $\overline{F}$ is continuous on $K \times C \times K$. Since $\langle z, \exp_x^{-1} x \rangle = 0$ for every $(x, z) \in K \times C$, condition (a) in Theorem 3.1 holds. By Lemma 4.2, we know that $y \mapsto \overline{F}(x, z, y)$ is $\mathbb{R}_+^p$-convex. This yields that $y \mapsto \overline{F}(x, z, y)$ is $\mathbb{R}_+^p$-quasi-convex by Remark 2.2. As a consequence of Theorem 3.1, there exists $\overline{x} \in K$ and $\overline{z} \in V(\overline{x})$ such that

$$\overline{x} \in S(\overline{x}) \text{ and} \langle \overline{z}, \exp_{\overline{x}}^{-1} y \rangle \notin -int\mathbb{R}_+^p, \quad \forall y \in S(\overline{x})$$

and so $\overline{x}$ is a solution of (GVQVI). This completes the proof. □

Example 4.1 Let M, K, p, S and A be the same as in Example 3.1. For any $x, y \in K$ and $z \in T_x M$, define

$$F(x, z, y) = (f_1(x, z, y), f_2(x, z, y)),$$

where

$$f_1(x, z, y) = \langle z, \exp_x^{-1} y \rangle, \quad f_2(x, z, y) = \langle z - 1, \exp_x^{-1} y \rangle.$$

Then it is easy to verify that all conditions in Theorem 4.1 hold. Therefore, Theorem 4.1 implies that there exists a solution set for (GVQVI).

4.2 Generalized vector variational-like inequalities

Let M be a Hadamard manifold, $K \subseteq M$ be a nonempty closed set, $V_i : K \to 2^{TM}$ $(i = 1, 2, \ldots, p)$ be a set-valued vector field and $\eta : K \times K \to TM$ be a continuous vector field. Then the generalized vector variational-like inequality (GVVLI) is formulated as follows: find $\overline{x} \in K$ and $\overline{z} \in V(\overline{x})$ such that

$$\langle \overline{z}, \eta(y, \overline{x}) \rangle \notin -int\mathbb{R}_+^p, \quad \forall y \in S(\overline{x}).$$

Theorem 4.2 *Let M be a Hadamard manifold. Suppose that the following conditions hold:*

(i) *$K \subseteq M$ is nonempty compact convex;*

(ii) *$S : K \to 2^K$ is a continuous set-valued mapping such that $S(x)$ is nonempty closed convex for every $x \in K$;*
(iii) *For each $i = 1, 2, \dots, p$, $V_i : K \to 2^{TM}$ is an upper Kuratowski semicontinuous vector field and $V_i(x)$ is compact convex for every $x \in K$, $C_i \subseteq TM$ is a compact convex set and $V_i(x) \subseteq C_i$ for every $x \in K$;*
(iv) *$\eta : K \times K \to TM$ is a continuous vector field satisfying the following conditions:*
  (a) *$\langle z, \eta(x, x)\rangle \in \mathbb{R}^p_+$ for every $(x, z) \in K \times C$;*
  (b) *the mapping $y \mapsto \langle z, \eta(y, x)\rangle$ is $\mathbb{R}^p_+$-quasi-convex for every $(x, z) \in K \times C$.*

Then there exist $\overline{x} \in K$ and $\overline{z} \in V(\overline{x})$ such that

$$\langle \overline{z}, \eta(y, \overline{x})\rangle \notin -int\mathbb{R}^p_+, \quad \forall y \in K.$$

Proof We define $C = C_1 \times C_2 \times \cdots \times C_p$ and $\widehat{F} : K \times C \times K \to \mathbb{R}^p$ as follows:

$$\widehat{F}(x, z, y) = \langle z, \eta(y, x)\rangle.$$

Obviously, the solutions of (GVVLI) are the solutions of $\widehat{F}$. It is easy to check that $\widehat{F}$ satisfies all the hypotheses in Theorem 3.1. As a consequence of Theorem 3.1, there exist $\overline{x} \in K$ and $\overline{z} \in V(\overline{x})$ such that

$$\langle \overline{z}, \eta(y, \overline{x})\rangle \notin -int\mathbb{R}^p_+, \quad \forall y \in K.$$

This completes the proof. □

4.3 Vector optimization problems

In this section, some relationships between a generalized variational-like inequality and a vector optimization problem are established, which can be considered as a generalization of results presented in [24].

Definition 4.1 Let M be a Hadamard manifold and $f : M \to \mathbb{R} \bigcup \{+\infty\}$ be a proper function. We said that f is locally Lipschitz on M if, for each $x \in dom f$, there exists $\epsilon_x > 0$ such that

$$|f(z) - f(y)| \le L_x d(z, y), \quad \forall z, y \in B(x, \epsilon_x),$$

where L_x is some positive number (called the Lipschitz constant of f in neighborhood of x) and

$$B(x, \epsilon_x) = \{y \in M; d(x, y) < \epsilon_x\}.$$

Definition 4.2 [33] Let M be a Hadamard manifold, $K \subseteq M$ be a convex set and $f : M \to \mathbb{R} \bigcup \{+\infty\}$ be a proper locally Lipschitz function. Given $x \in K$, the generalized directional derivative of f at x in the direction $v \in T_x M$, denoted by $f^\circ(x, v)$, is defined by

$$f^\circ(x, v) = \limsup_{t \downarrow 0 y \to x} \frac{f(\exp_y t(D\exp_x)_{\exp_x^{-1} y} v) - f(x)}{t},$$

where $(D\exp_x)_{\exp_x^{-1} y}$ denotes the differential of $\exp_x$ at $\exp_x^{-1} y$.

It is worth to pointed out that an equivalent definition has appeared in [34].

Definition 4.3 [33–35] Let M be a Hadamard manifold, $K \subseteq M$ be a convex set and $f : M \to \mathbb{R} \bigcup \{+\infty\}$ be a proper locally Lipschitz function. The generalized subdifferential of f at $x \in K$, denoted by $\partial^\circ f(x)$, is defined by

$$\partial^\circ f(x) = \{w \in T_x M : f^\circ(x, v) \geq \langle w, v\rangle, \quad \forall v \in T_x M\}.$$

Lemma 4.3 [35] *Let M be a Hadamard manifold, $K \subseteq M$ be a convex set and $f : M \to \mathbb{R} \bigcup \{+\infty\}$ be a proper locally Lipschitz function. Then for every $x \in K$,*

(i) *$\partial^\circ f(x)$ is nonempty compact convex;*
(ii) *$\partial^\circ f$ is upper Kuratowski semicontinuous on K.*

Remark 4.1 Let M be a Hadamard manifold, $K \subseteq M$ be a convex set, $f = (f_1, f_2, \ldots, f_p) : M \to \mathbb{R}^p$ be a vector valued function and $f_i (i = 1, 2, \ldots, p)$ be a proper locally Lipschitz function. Let $\partial^\circ f(x) = \partial^\circ f_1(x) \times \partial^\circ f_2(x) \times \cdots \times \partial^\circ f_p(x)$ for all $x \in K$. Then it is easy to check that $x \mapsto \partial^\circ f(x)$ is upper Kuratowski semicontinuous on K and $\partial^\circ f(x)$ is compact convex valued for every $x \in K$.

By Remark 4.1 and Theorem 3.1, we obtain the following result.

Theorem 4.3 *Let M be a Hadamard manifold, $K \subseteq M$ be a compact convex set, $f_i : M \to \mathbb{R}^p (i = 1, 2, \ldots, p)$ be a proper locally Lipschitz function, $\eta : K \times K \to TM$ be a continuous vector field satisfying the following conditions:*

(i) *$\langle z, \eta(x, x)\rangle \in \mathbb{R}^p_+$ for every $(x, z) \in K \times C$;*
(ii) *the mapping $y \mapsto \langle z, \eta(y, x)\rangle$ is $\mathbb{R}^p_+$-quasi-convex for every $(x, z) \in K \times C$.*

Then there exist $\overline{x} \in K$ and $\overline{z} \in V(\overline{x})$ such that

$$\langle \overline{z}, \eta(y, \overline{x})\rangle \notin -int\mathbb{R}^p_+, \quad \forall y \in K.$$

Next, we consider the following vector optimization problem:

$$\text{(VOP)} : \begin{cases} Wmin_{\mathbb{R}^p_+} f(x) \\ x \in K, \end{cases}$$

where $K \subseteq M$ is a nonempty set and $f : M \to \mathbb{R}^p$ is a vector valued function. It is easy to check that $x^* \in K$ is a efficient weak solution of (VOP) iff

$$f(y) - f(x^*) \notin -int\mathbb{R}^p_+, \quad \forall y \in K.$$

Definition 4.4 [36] Let M be a Hadamard manifold, $K \subseteq M$ be a convex set and $h : M \to \mathbb{R} \bigcup \{+\infty\}$ be a proper locally Lipschitz function. Then h is said to be invex with respect to η, if there exists $\eta : M \times M \to TM$ such that

$$h(y) - h(x) \geq h^{\circ}(x, \eta(y, x)), \quad \forall y, x \in M.$$

Theorem 4.4 *Let M be a Hadamard manifold, $K \subseteq M$ be a nonempty set and $f_i : M \to \mathbb{R}^p (i = 1, 2, \ldots, p)$ be invex with respect to η. If $\overline{x}$ is a solution of the following generalized variational-like inequality: find $\overline{x} \in K$ and $\overline{z} \in \partial^{\circ} f(\overline{x})$ such that*

$$\langle \overline{z}, \eta(y, \overline{x}) \rangle \notin -int\mathbb{R}^p_+, \quad \forall y \in K.$$

Then $\overline{x}$ is a weak efficient solution of (VOP).

Proof Suppose that $\overline{x}$ is not a weak efficient solution of (VOP). Then there exists $x^* \in K$ such that

$$f(x^*) - f(\overline{x}) \in -int\mathbb{R}^p_+.$$

This implies that, for each $i = 1, 2, \ldots, p$,

$$f_i(x^*) - f_i(\overline{x}) < 0.$$

Since $f_i (i = 1, 2, \ldots, p)$ is invex with respect to η, one has

$$f_i(x^*) - f_i(\overline{x}) \geq f_i^{\circ}(\overline{x}, \eta(x^*, \overline{x})).$$

By the definition of $\partial^{\circ} f$, we know that, for every $z \in \partial^{\circ} f(\overline{x})$,

$$\langle z, \eta(x^*, \overline{x}) \rangle \in -int\mathbb{R}^p_+,$$

which is a contradiction. This completes the proof. □

Remark 4.2 If f is convex and differentiable on M, $\eta(y, x) = \exp_x^{-1} y$ for all $(y, x) \in M \times M$, then Theorems 4.3 and 4.4 can be regarded as a generalization of Theorem 3.1 in [24].

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