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Subdifferentials of a perturbed minimal time function in normed spaces

Yongle Zhang · Yiran He · Yi Jiang

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Abstract In a general normed vector space, we study the perturbed minimal time function determined by a bounded closed convex set U and a proper lower semicontinuous function $f(\cdot)$. In particular, we show that the Fréchet subdifferential and proximal subdifferential of a perturbed minimal time function are representable by virtue of corresponding subdifferential of $f(\cdot)$ and level sets of the support function of U. Some known results is a special case of these results.

Keywords Perturbed minimal time function · Subdifferentials · Support function

1 Introduction

Let X be a normed vector space, U be a bounded closed convex subset of X, and $f: X \to \bar{R}$ be a proper lower semicontinuous function. We define the perturbed minimal time function $T^f: X \to R$ by

$$T^f(x) := \inf_{s \in X}\{T(x,s) + f(s)\}, \quad \text{for all } x \in X, \tag{1.1}$$

where $T(x,s) := \inf\{t \geq 0: s - x \in tU\}$. It is easy to see that, if $U \equiv B$, then $T(x-s) = \|x-s\|$, where B is the unit ball in X.

For $x \in X$, the perturbed minimal time problem is to find an element $z_0 \in X$ such that

$$T(x, z_0) + f(z_0) = T^f(x).$$

Y. Zhang (✉) · Y. He · Y. Jiang
Department of Mathematics, Sichuan Normal University,
Chengdu 610066, Sichuan, China
e-mail: zhang-yongle@hotmail.com

In particular, if $f = I_S$, where I_S denote the indicator function I_S of a closed set S (the definition will be given below), then the perturbed minimal time function T^f reduces to the minimal time function T_S in [14], which is defined by the following differential inclusion

$$\dot{x}(t) \in U, \quad x(0) = x. \tag{1.2}$$

In other words,

$$T_S(x) \equiv \begin{cases} \inf\{T > 0\colon \text{there exists a trajectory } x(\cdot) \text{ satisfying (1.2)} \\ \qquad\qquad \text{with } x(0) = x \text{ and } x(T) \in S\}, & x \notin S; \\ 0, & x \in S. \end{cases}$$

If $f = J + I_S$ and $U \equiv B$, then the perturbed minimal time function T^f and the perturbed minimal time problem reduce to the perturbed distance function d_S^J and the perturbed optimization problem $\min_J(x, S)$ defined in [23], respectively, that is,

$$T^f(x) = d_S^J(x) := \inf_{s \in S}\{\|s - x\| + J(s)\}, \quad \text{for all } x \in X,$$

and

$$\min_J(x, S) := \{z_0 \in S \,|\, \|x - z_0\| + J(z_0) = d_S^J(x)\}.$$

Baranger [1] proved that if S is a nonempty closed subset of a uniformly convex Banach space X, then the set of all $x \in X$ for the perturbed optimization problem $\min_J(x, S)$ has a solution is a dense G_δ-subset of X, which extends a result in [22] on the best approximation problem. For other results on perturbed optimization problems, see for example [3,8,9,15,16,18–21]. In particular, Cobzas [9] extended Baranger's result to the setting of reflexive Kadec Banach space; while Ni [18] relaxed the reflexivity assumption made in Cobzas' result. The existence results have been applied to optimal control problems governed by partial differential equations, see for example, [1–3,8, 12].

Assuming that the origin is an interior point of U, Colombo and Wolenski [10,11] studied the proximal and Fréchet subdifferentials of the function $T_S(x)$ in a Hilbert space. He and Ng [13] studied the Fréchet and proximal subdifferentials of T_S in a Banach space. When the origin is an interior point of U, the function T_S is globally Lipschitz, so the Clarke subdifferential of T_S is also discussed in [13]. Jiang and He [14] show the Frechét and proximal subdifferentials of the minimal time function T_S without requiring the origin be an interior point of U in normed space. In particular, if U is the (closed) unit ball in X, then $T_S(x)$ reduces to the usual distance $d_S(x)$, which is defined by

$$d_S(x) := \inf_{s \in S} \|s - x\| \quad \text{for all } x \in X.$$

The subdifferentials of d_S were studied in [4–7], and the subdifferentials of perturbed distance functions d_S^J were studied in [17,23].

In order to reduce the symmetry of the norm, we replace the distance function in [23] by $T(\cdot,\cdot)$, which does not have the symmetry, and explore the Fréchet subdifferentials and the Proximal subdifferentials of its perturbed functions $T^f(\cdot)$, the perturbed functions $T^f(\cdot)$ are encountered in constraint optimization, via applying various perturbation, penalization, and approximation techniques. Our main results extend the corresponding ones in [14] from the minimal time function to perturbed minimal time function, and extend the corresponding ones in [23] from the general perturbed distance functions to general perturbed minimal time functions.

2 Preliminaries

Let X be a normed vector space with norm denoted by $\|\cdot\|$. Let X^* denote the topological dual of X. We use $B(x;r)$ to denote the open ball centered at x with radius $r > 0$ and $\langle\cdot,\cdot\rangle$ to denote the pairing between X^* and X. Let $g\colon X \to \mathbb{R}$ be a lower semicontinuous function and $x \in X$. g is said to be center Lipschitz on S at x with Lipschitz constant L, if

$$|g(y) - g(x)| \le L\|y - x\|, \quad \forall y \in S.$$

Let us recall the following well-known classes of subdifferentials for g at x.

- The *proximal subdifferential* of g at x is the set

$$\partial^P g(x) := \left\{\xi \in X^*\colon \liminf_{\|v\|\to 0} \frac{g(x+v) - g(x) - \langle \xi, v\rangle}{\|v\|^2} > -\infty\right\}.$$

  In other words, $\xi \in \partial^P g(x)$ if and only if there exist $\sigma > 0$ and $\delta > 0$ such that

$$g(x+v) - g(x) \ge \langle \xi, v\rangle - \sigma\|v\|^2, \quad \text{for all } v \in B(0,\delta).$$

- The *Frechét subdifferential* of g at x is the set

$$\partial^F g(x) := \left\{\xi \in X^*\colon \liminf_{\|v\|\to 0} \frac{g(x+v) - g(x) - \langle \xi, v\rangle}{\|v\|} \ge 0\right\}.$$

  That is, $\xi \in \partial^F g(x)$ if and only if for any $\sigma > 0$, there exists $\delta > 0$ such that

$$g(x+v) - g(x) \ge \langle \xi, v\rangle - \sigma\|v\|, \quad \text{for all } v \in B(0,\delta).$$

Recall that f satisfies the center Lipschitz condition on X at x, if there exists $L > 0$ such that

$$|f(y) - f(x)| \le L\|y - x\|, \quad \text{for each } y \in X.$$

The support function of a set $K \subset X$ is defined by

$$\Im_K(\xi) := \sup_{x\in K}\langle \xi, x\rangle.$$

The indicator function I_S of S is defined by

$$I_S(x) \equiv \begin{cases} 0, & x \in S; \\ +\infty, & x \notin S. \end{cases}$$

In view of [14, Proposition 2.2], we have the following result.

Proposition 2.1 *$T(x, s) = 0$ if and only if $x = s$.*

We use S_0 to denote the set of all points $x \in X$ such that x is a solution of the perturbed optimization problem, i.e.,

$$S_0 = \{x \in X | T^f(x) = f(x)\}.$$

Remark 2.1 It is obviously that, if $f = I_S$, then S_0 equals S in [14]; if $f = I_S + J$ and U is the unit ball in X, then S_0 equals S_0 in [23].

3 Fréchet subdifferential of a minimal time function

Theorem 3.1 *Let $x \in S_0$. The following assertions hold.*

1. $\partial^F T^f(x) \subset \partial^F f(x) \cap \{\xi \in X^*: \Im_U(-\xi) \le 1\}$.
2. *If $f(\cdot)$ is center Lipschitz on X at x with Lipschitz constant $0 \le L < 1/M$, where $M := \sup_{u\in U}\|u\|$, then we have*

$$\partial^F T^f(x) = \partial^F f(x) \cap \{\xi \in X^*: \Im_U(-\xi) \le 1\}.$$

Proof (1) Let $\xi \in \partial^F T^f(x)$. Then for any $\sigma > 0$, there exists $\delta > 0$ such that

$$T^f(y) - T^f(x) - \langle \xi, y - x\rangle \ge -\sigma\|y - x\| \tag{3.1}$$

for all $y \in B(x; \delta)$.

We will prove

$$f(y) - f(x) - \langle \xi, y - x\rangle \ge -\sigma\|y - x\|, \quad \text{for all} \quad y \in B(x; \delta). \tag{3.2}$$

Thus $\xi \in \partial^F f(x)$.

By (3.1) and definition of S_0, (3.2) is trivial if $y \in B(x; \delta) \cap S_0$, we may assume that $y \in B(x; \delta)\backslash S_0$, by the definition of T^f, we have $T^f(y) \le f(y)$, and as $x \in S_0$, we have from (3.1) that

$$f(y) - f(x) - \langle \xi, y - x\rangle \ge T^f(y) - T^f(x) - \langle \xi, y - x\rangle \ge -\sigma\|y - x\|$$

Hence, $f(y) - f(x) - \langle \xi, y - x\rangle \ge -\sigma\|y - x\|$, for all $y \in B(x; \delta)$.

Fix any $v \in U$. Let $t_\lambda := T^f(x - \lambda v)$, where $\lambda > 0$. Since $x - (x - \lambda v) \in \lambda U$, $T(x - \lambda v, x) \leq \lambda$, $t_\lambda \leq \lambda + f(x) < \infty$. It follows from (3.1) that for sufficiently small $\lambda > 0$,

$$\lambda + f(x) \geq t_\lambda \geq f(x) + \lambda\langle -\xi, v\rangle - \lambda\sigma\|v\|,$$

which implies that $\langle -\xi, v\rangle \leq 1 + \sigma\|v\|$. Since $\sigma > 0$ and $v \in U$ are arbitrary, $\Im_U(-\xi) \leq 1$.

(2) It is sufficient to prove

$$\partial^F f(x) \cap \{\xi \in X^*\colon \Im_U(-\xi) \leq 1\} \subset \partial^F T^f(x).$$

Let $\xi \in \partial^F f(x)$ be such that $\Im_U(-\xi) \leq 1$.

For any $\sigma > 0$, take $\sigma_0 \in \left(0, \frac{(1-LM)\sigma}{(1+M\|\xi\|)}\right)$. By the definition of Fréchet normal cone, there exists $\delta > 0$ such that

$$f(y) - f(x) - \langle \xi, y - x\rangle \geq -\sigma_0\|y - x\|, \text{ for all } y \in B(x;\delta). \tag{3.3}$$

Then

$$T^f(y) - T^f(x) - \langle \xi, y - x\rangle \geq -\sigma_0\|y - x\|, \text{ for all } y \in S_0 \cap B(x;\delta). \tag{3.4}$$

Let $\delta_0 := \frac{(1-LM)\delta}{3(1+M\|\xi\|)} < \delta$. Then

$$T^f(y) - T^f(x) - \langle \xi, y - x\rangle \geq -\sigma_0\|y - x\|, \text{ for all } y \in S_0 \cap B(x;\delta_0). \tag{3.5}$$

Now we prove that (3.5) holds for all $y \in B(x;\delta_0)\backslash S_0$. Therefore, $\xi \in \partial^F T^f(x)$.

If not, then there is $y_0 \notin S_0$ such that

$$\|y_0 - x\| < \delta_0 \quad \text{and} \quad T^f(y_0) < T^f(x) + \langle \xi, y_0 - x\rangle - \sigma\|y_0 - x\|. \tag{3.6}$$

The latter implies that

$$T^f(y_0) \leq f(x) + \|\xi\|\|y_0 - x\|. \tag{3.7}$$

Let $t := T^f(y_0)$. By the definition of T^f, for any $\varepsilon \in \left(0, \frac{(1-LM)\delta}{3M}\right)$, there are $t_1 \in (0, t + \varepsilon)$, and $s \in X$ such that $t_1 = T(y_0, s) + f(s) < t + \varepsilon$. By the definition of T, for any $\varepsilon' \in \left(0, \frac{(1-LM)\delta}{3M}\right)$, there are $t_2 \in (t_1 - f(s), t_1 - f(s) + \varepsilon')$, $u \in U$, such that $s - y_0 = t_2 u$. Thus (3.7) and f is center Lipshitz on X at x yield that

$$
\begin{aligned}
\|s-x\| &\le \|y_0-x\| + (t_1 - f(s) + \varepsilon')\|u\| \le \|y_0-x\| + (t+\varepsilon - f(s) + \varepsilon')M \\
&\le \|y_0-x\| + (f(x) + \|\xi\|\|y_0-x\| + \varepsilon - f(s) + \varepsilon')M \\
&\le (1+M\|\xi\|)\|y_0-x\| + (f(x)-f(s))M + (\varepsilon+\varepsilon')M \\
&\le (1+M\|\xi\|)\|y_0-x\| + LM\|s-x\| + (\varepsilon+\varepsilon')M
\end{aligned}
$$

Then, we have

$$
\|s-x\| \le \frac{1+M\|\xi\|}{1-LM}\|y_0-x\| + \frac{M}{1-LM}(\varepsilon+\varepsilon') < \delta. \tag{3.8}
$$

This verifies that $s \in B(x;\delta)$. Applying (3.3), (3.8) and $\Im_U(-\xi) \le 1$, we have

$$
\begin{aligned}
&T^f(y_0) - T^f(x) - \langle \xi, y_0 - x\rangle = t - f(x) - \langle \xi, y_0 - s\rangle - \langle \xi, s - x\rangle \\
&\quad \ge t - f(x) - \langle \xi, y_0 - s\rangle + (f(x) - f(s) - \sigma_0\|s-x\|) \\
&\quad = t - (t_1 - f(s) + \varepsilon')\langle -\xi, u\rangle - f(s) - \sigma_0\|s-x\| \\
&\quad \ge t - (t_1 - f(s) + \varepsilon') - f(s) - \sigma_0\|s-x\| \\
&\quad \ge -\varepsilon - \varepsilon' - \sigma_0\|s-x\| \\
&\quad \ge -\varepsilon - \varepsilon' - \sigma_0\left(\frac{1+M\|\xi\|}{1-LM}\|y_0-x\| + \frac{M}{1-LM}(\varepsilon+\varepsilon')\right) \\
&\quad \ge -\left(1 + \frac{M}{1-LM}\sigma_0\right)(\varepsilon+\varepsilon') - \frac{1+M\|\xi\|}{1-LM}\sigma_0\|y_0-x\| \\
&\quad \ge -\left(1 + \frac{M}{1-LM}\sigma_0\right)(\varepsilon+\varepsilon') - \sigma\|y_0-x\|.
\end{aligned}
$$

Letting $\varepsilon' \to 0+$ and $\varepsilon \to 0+$, it yields that

$$
T^f(y_0) - T^f(x) - \langle \xi, y_0 - x\rangle \ge -\sigma\|y_0-x\|,
$$

which contradicts to (3.6). □

In particular, letting $f = I_S$, we get the following corollary, which was proved in [14].

Corollary 3.1 *Assume that $f = I_S$, where S is a closed convex subset of X, if $x \in S$, then*

$$
\partial^F T^f(x) = \partial^F T(x) = N_S^F(x) \cap \{\xi \in X^*: \Im_U(-\xi) \le 1\}.
$$

In particular, letting $f = I_S + J$ and $U \equiv B$, we get the following corollary, which was proved in [23].

Corollary 3.2 *Assume that $f = I_S + J$ and $U \equiv B$, where B is the unit ball in X and S is a closed convex subset of X, let $x \in S_0$. The following assertions hold.*

1. $\partial^F T^f(x) = \partial^F d_S^J(x) \subset \partial^F(J + I_S)(x) \cap B^*$.
2. *If $J(\cdot)$ is center Lipschitz on S at x with Lipschitz constant $0 \le L < 1$, then we have*

$$\partial^F T^f(x) = \partial^F d_S^J(x) = \partial^F (J + I_S)(x) \cap B^*.$$

4 Proximal subdifferential of a minimal time function

Theorem 4.1 *Let $x \in S_0$. The following assertions hold.*

1. $\partial^P T^f(x) \subset \partial^P f(x) \cap \{\xi \in X^*: \Im_U(-\xi) \leq 1\}$.
2. *If $f(\cdot)$ is center Lipschitz on X at x with Lipschitz constant $0 \leq L < 1/M$, where $M := \sup_{u\in U} \|u\|$, then we have*

$$\partial^P T^f(x) = \partial^P f(x) \cap \{\xi \in X^*: \Im_U(-\xi) \leq 1\}.$$

Proof (1) Let $\xi \in \partial^P T^f(x)$. Then there exist $\sigma, \delta > 0$ such that

$$T^f(y) - T^f(x) - \langle \xi, y - x\rangle \geq -\sigma \|y - x\|^2 \tag{4.1}$$

for all $y \in B(x; \delta)$.

We wil prove

$$f(y) - f(x) - \langle \xi, y - x\rangle \geq -\sigma \|y - x\|^2, \text{ for all } \ y \in B(x; \delta). \tag{4.2}$$

Then $\xi \in \partial^P f(x)$.

By (4.1) and the definition of S_0, (4.2) is trivial if $y \in B(x; \delta) \cap S_0$, we may assume that $y \in B(x; \delta) \setminus S_0$. By the definition of T^f, we have $T^f(y) \leq f(y)$, and as $x \in S_0$, we have from (4.1) that

$$f(y) - f(x) - \langle \xi, y - x\rangle \geq T^f(y) - T^f(x) - \langle \xi, y - x\rangle \geq -\sigma \|y - x\|^2.$$

Hence, $f(y) - f(x) - \langle \xi, y - x\rangle \geq -\sigma \|y - x\|^2$, for all $y \in B(x; \delta)$.

Fix any $v \in U$. Let $t_\lambda := T^F(x - \lambda v)$, where $\lambda > 0$. Since $x - (x - \lambda v) \in \lambda U$, $T(x - \lambda v, x) \leq \lambda$, $t_\lambda \leq \lambda + f(x) < \infty$. It follows from (4.1) that for sufficiently small $\lambda > 0$,

$$\lambda + f(x) \geq t_\lambda \geq f(x) + \lambda\langle -\xi, v\rangle - \lambda^2\sigma \|v\|^2,$$

which implies that $\langle -\xi, v\rangle \leq 1$. Therefore, $\Im_U(-\xi) \leq 1$.

(2) It is sufficient to prove

$$\partial^P f(x) \cap \{\xi \in X^*: \Im_U(-\xi) \leq 1\} \subset \partial^P T^f(x).$$

Let $\xi \in \partial^P f(x)$ be such that $\Im_U(-\xi) \leq 1$. Then there exist $\sigma_1, \delta > 0$ such that

$$f(y) - f(x) - \langle \xi, y - x\rangle \geq -\sigma_1 \|y - x\|^2, \text{ for all } y \in B(x; \delta). \tag{4.3}$$

Take $\sigma := 2\left(\frac{1+M\|\xi\|}{1-LM}\right)^2 \sigma_1 > \sigma_1$. Thus (4.3) implies that

$$f(y) - f(x) - \langle \xi, y - x\rangle \geq -\sigma \|y - x\|^2, \text{ for all } y \in B(x; \delta). \tag{4.4}$$

and

$$T^f(y) - T^f(x) - \langle \xi, y - x \rangle \geq -\sigma \|y - x\|^2, \text{ for all } y \in S_0 \cap B(x; \delta). \tag{4.5}$$

Let $\delta_0 := \frac{(1-LM)\delta}{3(1+M\|\xi\|)} < \delta$. Then

$$T^f(y) - T^f(x) - \langle \xi, y - x \rangle \geq -\sigma \|y - x\|^2, \text{ for all } y \in S_0 \cap B(x; \delta_0). \tag{4.6}$$

Now we prove that (4.6) holds for all $y \in B(x; \delta_0) \backslash S_0$. Therefore, $\xi \in \partial^P T^f(x)$.
If not, then there is $y_0 \notin S_0$ such that

$$\|y_0 - x\| < \delta_0 \quad \text{and} \quad T^f(y_0) < T^f(x) + \langle \xi, y_0 - x \rangle - \sigma \|y_0 - x\|^2. \tag{4.7}$$

The latter implies that

$$T^f(y_0) \leq J(x) + \|\xi\| \|y_0 - x\|. \tag{4.8}$$

Let $t := T^f(y_0)$. By the definition of T^f, for any $\varepsilon \in \left(0, \frac{(1-LM)\delta}{3M}\right)$, there are $t_1 \in (t, t + \varepsilon)$, and $s \in X$ such that $t_1 = T(y_0, s) + f(s) < t + \varepsilon$, by the definition of T, for any $\varepsilon' \in \left(0, \frac{(1-LM)\delta}{3M}\right)$, there are $t_2 \in (t_1 - f(s), t_1 - f(s) + \varepsilon')$ $u \in U$, such that $s - y_0 = t_2 u$. Thus (4.8) and f is center Lipshitz on X at x yield that

$$\begin{aligned}
\|s - x\| &\leq \|y_0 - x\| + (t_1 - f(s) + \varepsilon')\|u\| \leq \|y_0 - x\| + (t + \varepsilon - f(s) + \varepsilon')M \\
&\leq \|y_0 - x\| + (f(x) + \|\xi\| \|y_0 - x\| + \varepsilon - f(s) + \varepsilon')M \\
&\leq (1 + M\|\xi\|)\|y_0 - x\| + (f(x) - f(s))M + (\varepsilon + \varepsilon')M \\
&\leq (1 + M\|\xi\|)\|y_0 - x\| + LM\|s - x\| + (\varepsilon + \varepsilon')M
\end{aligned}$$

Then, we have

$$\|s - x\| \leq \frac{1 + M\|\xi\|}{1 - LM}\|y_0 - x\| + \frac{M}{1 - LM}(\varepsilon + \varepsilon') < \delta. \tag{4.9}$$

This verifies that $s \in B(x; \delta)$. Applying (4.4), (4.9) and $\Im_U(-\xi) \leq 1$, we have

$$\begin{aligned}
&T^f(y_0) - T^f(x) - \langle \xi, y_0 - x \rangle = t - f(x) - \langle \xi, y_0 - s \rangle - \langle \xi, s - x \rangle \\
&\geq t - f(x) - \langle \xi, y_0 - s \rangle + (f(x) - f(s) - \sigma_1 \|s - x\|^2) \\
&= t - (t_1 - f(s) + \varepsilon')\langle -\xi, u \rangle - f(s) - \sigma_1 \|s - x\|^2 \\
&\geq t - (t_1 - f(s) + \varepsilon') - f(s) - \sigma_1 \|s - x\|^2 \\
&\geq -\varepsilon - \varepsilon' - \sigma_1 \|s - x\|^2 \\
&\geq -\varepsilon - \varepsilon' - 2\sigma_1 \left(\frac{1 + M\|\xi\|}{1 - LM}\right)^2 \|y_0 - x\|^2 - 2\sigma_0 \left(\frac{M}{1 - LM}\right)^2 (\varepsilon + \varepsilon')^2 \\
&\geq -\varepsilon - \varepsilon' - 2\sigma_1 \left(\frac{M}{1 - LM}\right)^2 (\varepsilon + \varepsilon')^2 - \sigma \|y_0 - x\|^2.
\end{aligned}$$

Letting $\varepsilon' \to 0+$ and $\varepsilon \to 0+$, it yields that

$$T^f(y_0) - T^f(x) - \langle \xi, y_0 - x \rangle \geq -\sigma \|y_0 - x\|^2,$$

which contradicts to (4.7). □

In particular, letting $f = I_S$, we get the following corollary, which is proved in [14].

Corollary 4.1 *Assume that* $f = I_S$, *where* S *is a closed convex subset of* X, *if* $x \in S$, *then*

$$\partial^P T^f(x) = \partial^P T(x) = N_S^P(x) \cap \{\xi \in X^* : \Im_U(-\xi) \leq 1\}.$$

In particular, letting $f = I_S + J$ and $U \equiv B$, we get the following corollary, which was proved in [23].

Corollary 4.2 *Assume that* $f = I_S + J$ *and* $U \equiv B$, *where* B *is the unit ball in* X *and* S *is a closed convex subset of* X, *let* $x \in S_0$. *The following assertions hold.*

1. $\partial^P T^f(x) = \partial^P d_S^J(x) \subset \partial^P(J + I_S)(x) \cap B^*$.
2. *If* $J(\cdot)$ *is center Lipschitz on* S *at* x *with Lipschitz constant* $0 \leq L < 1$, *then we have*

$$\partial^P T^f(x) = \partial^P d_S^J(x) = \partial^P(J + I_S)(x) \cap B^*.$$

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