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SHORT COMMUNICATION

A Farkas-type theorem for interval linear inequalities

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Abstract We describe a Farkas-type condition for strong solvability of interval linear inequalities. The result is used to derive several descriptions of the set of strong solutions and to show that this set forms a convex polytope.

Keywords Linear inequalities · Interval data · Strong solvability · Farkas-type theorem

1 Introduction

The famous theorem (often called a lemma) proved by Julius Farkas 111 years ago [1] asserts that a system of linear equations

$$Ax = b \tag{1}$$

(with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$) has a nonnegative solution if and only if for each $p \in \mathbb{R}^m$, $A^T p \geq 0$ implies $b^T p \geq 0$. Several consequences for systems of other types than (1) can be drawn from this result. For instance, since a system of linear inequalities

$$Ax \leq b \tag{2}$$

has a solution if and only if the system of linear equations

$$Ax_1 - Ax_2 + x_3 = b$$

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has a nonnegative solution, from the Farkas theorem we obtain that (2) is solvable if and only if for each $p \geq 0$, $A^T p = 0$ implies $b^T p \geq 0$.

Rohn [4] and recently independently Karademir and Prokopyev [3] (see also [6]) formulated a Farkas-type theorem for interval linear equations. Given an $m \times n$ interval matrix $\boldsymbol{A} = [\underline{A}, \overline{A}] = [A_c - \Delta, A_c + \Delta] = \{ A \mid A_c - \Delta \leq A \leq A_c + \Delta \}$ (where $A_c = (\underline{A} + \overline{A})/2$ and $\Delta = (\overline{A} - \underline{A})/2$ are the midpoint matrix and the radius matrix of $\boldsymbol{A}$, respectively) and an interval m-vector $\boldsymbol{b} = [\underline{b}, \overline{b}] = [b_c - \delta, b_c + \delta] = \{ b \mid b_c - \delta \leq b \leq b_c + \delta \}$ (where again $b_c = (\underline{b} + \overline{b})/2$ and $\delta = (\overline{b} - \underline{b})/2$), they proved that the system

$$Ax = b$$

has a nonnegative solution for each $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$ if and only if for each p, $A_c^T p + \Delta^T |p| \geq 0$ implies $b_c^T p - \delta^T |p| \geq 0$. The presence of absolute values renders, however, this result practically useful for small values of m only: checking this Farkas-type condition is NP-hard [4].

In this paper we formulate a Farkas-type condition for interval linear inequalities. Given an interval matrix $\boldsymbol{A}$ and an interval vector $\boldsymbol{b}$ as above, we are interested in solvability of each system

$$Ax \leq b \tag{3}$$

with data satisfying

$$A \in \boldsymbol{A}, \ b \in \boldsymbol{b}. \tag{4}$$

This type of solvability is called *strong solvability* of a formally written system of interval linear inequalities $\boldsymbol{A}x \leq \boldsymbol{b}$, see Chapter 2 in [2] for a survey of results. In Theorem 1 we prove a Farkas-type condition for strong solvability which we then use to obtain another proof of the result by Rohn and Kreslová [7] saying that if $\boldsymbol{A}x \leq \boldsymbol{b}$ is strongly solvable, then all the systems (3), (4) have a common solution which is called a *strong* (it could also be termed "universal") *solution* of $\boldsymbol{A}x \leq \boldsymbol{b}$. As the main result of this paper we give in Theorem 4 four alternative descriptions of the set of strong solutions, and in the concluding Sect. 5 we show an interconnection between strong solvability of interval linear equations and that of interval linear inequalities.

2 The Farkas-type theorem

We have the following Farkas-type characterization.

Theorem 1 *A system $\boldsymbol{A}x \leq \boldsymbol{b}$ is strongly solvable if and only if for each $p \geq 0$, $\underline{A}^T p \leq 0 \leq \overline{A}^T p$ implies $\underline{b}^T p \geq 0$.*

Proof "If": Let for each $p \geq 0$, $\underline{A}^T p \leq 0 \leq \overline{A}^T p$ imply $\underline{b}^T p \geq 0$. Assume to the contrary that some system (3) with data satisfying (4) is not solvable. Then according to the above-quoted Farkas condition for (3) there exists a vector $p_0 \geq 0$ such that $A^T p_0 = 0$ and $b^T p_0 < 0$. But from $\underline{A} \leq A \leq \overline{A}$ and $p_0 \geq 0$ we obtain $\underline{A}^T \leq A^T \leq \overline{A}^T$ and $\underline{A}^T p_0 \leq A^T p_0 \leq \overline{A}^T p_0$, hence $\underline{A}^T p_0 \leq 0 \leq \overline{A}^T p_0$. Thus our assumption implies $\underline{b}^T p_0 \geq 0$, but we also have $\underline{b}^T p_0 \leq b^T p_0 < 0$, which is a contradiction.

"Only if": Let each system (3) with data satisfying (4) be solvable. Assume to the contrary that the Farkas-type condition does not hold, i.e., that there exists a vector $p_1 \geq 0$ such that

$$\underline{A}^T p_1 \leq 0 \leq \overline{A}^T p_1$$

and

$$\underline{b}^T p_1 < 0. \tag{5}$$

For each $i = 1, \ldots, n$ define a real function of one real variable by

$$f_i(t) = (t\underline{A}_{\bullet i} + (1-t)\overline{A}_{\bullet i})^T p_1.$$

Then $f_i(0) = (\overline{A}_{\bullet i})^T p_1 = (\overline{A}^T p_1)_i \geq 0$ and $f_i(1) = (\underline{A}_{\bullet i})^T p_1 = (\underline{A}^T p_1)_i \leq 0$, hence by continuity there exists a $t_i \in [0, 1]$ such that $f_i(t_i) = 0$. Now define A columnwise by

$$A_{\bullet i} = t_i \underline{A}_{\bullet i} + (1-t_i)\overline{A}_{\bullet i} \qquad (i = 1, \ldots, n).$$

Then $A_{\bullet i}$, as a convex combination of $\underline{A}_{\bullet i}$ and $\overline{A}_{\bullet i}$, belongs to $[\underline{A}_{\bullet i}, \overline{A}_{\bullet i}]$ for each i, hence $A \in \boldsymbol{A}$. Moreover, from the definition of t_i we have $(A^T p_1)_i = (A_{\bullet i})^T p_1 = f_i(t_i) = 0$ for each i, hence $A^T p_1 = 0$ which together with (5) implies that the system $Ax \leq \underline{b}$ has no solution in contradiction to our assumption. □

3 Strong solvability of interval linear inequalities

Let us now have a closer look at the Farkas-type condition of Theorem 1. If we write it in a slightly different form: for each p,

$$p \geq 0,\ \overline{A}^T p \geq 0,\ -\underline{A}^T p \geq 0 \text{ implies } \underline{b}^T p \geq 0,$$

we can see that it is just the Farkas condition for nonnegative solvability of the system

$$\overline{A}x_1 - \underline{A}x_2 \leq \underline{b}. \tag{6}$$

In this way we have found another proof of a theorem by Rohn and Kreslová [7] formulated, unlike the Farkas condition, in primal terms:

Theorem 2 *A system of interval linear inequalities $\boldsymbol{A}x \leq \boldsymbol{b}$ is strongly solvable if and only if the system (6) has a nonnegative solution.*

This result has a nontrivial consequence which is also contained in [7]: namely, the existence of strong solutions.

Theorem 3 *If a system of interval linear inequalities $\boldsymbol{A}x \leq \boldsymbol{b}$ is strongly solvable, then there exists an x_0 such that*

$$Ax_0 \leq b \tag{7}$$

holds for each $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$.

Proof According to Theorem 2, strong solvability of $\boldsymbol{A}x \leq \boldsymbol{b}$ implies nonnegative solvability of (6). If $x_1 \geq 0$ and $x_2 \geq 0$ solve (6), then for each $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$ we have $A(x_1 - x_2) \leq \overline{A}x_1 - \underline{A}x_2 \leq \underline{b} \leq b$, so that $x_0 = x_1 - x_2$ possesses the required property. □

In order to be able to convert solutions of (6) into complementary ones, we need the following auxiliary result. Minimum of two vectors is understood entrywise.

Proposition 1 *If x_1, x_2 is a nonnegative solution of (6), then for each d with*

$$0 \leq d \leq \min\{x_1, x_2\},$$

$x_1' = x_1 - d$, $x_2' = x_2 - d$ is also a nonnegative solution of (6).

Proof Since $d \leq \min\{x_1, x_2\} \leq x_1$, we have $x_1' \geq 0$, and similarly $x_2' \geq 0$. Next, $\overline{A}x_1' - \underline{A}x_2' = \overline{A}x_1 - \underline{A}x_2 + (\underline{A} - \overline{A})d \leq \overline{A}x_1 - \underline{A}x_2 \leq \underline{b}$, so that x_1', x_2' solve (6). □

A vector x_0 satisfying (7) for each $A \in \boldsymbol{A}, b \in \boldsymbol{b}$ is called a *strong solution* of $\boldsymbol{A}x \leq \boldsymbol{b}$. Denote by $\boldsymbol{X}_S(\boldsymbol{A}, \boldsymbol{b})$ the set of strong solutions. For $x = (x_i)_{i=1}^n$ let

$$x^+ = (\max\{x_i, 0\})_{i=1}^n,$$
$$x^- = (\max\{-x_i, 0\})_{i=1}^n,$$

and

$$|x| = (|x_i|)_{i=1}^n,$$

then $x = x^+ - x^-$, $|x| = x^+ + x^-$, $(x^+)^T x^- = 0$, $x^+ \geq 0$ and $x^- \geq 0$. The following theorem brings several alternative descriptions of the set of strong solutions.

Theorem 4 *We have*

$$\boldsymbol{X}_S(\boldsymbol{A}, \boldsymbol{b}) = \{\, x_1 - x_2 \mid \overline{A}x_1 - \underline{A}x_2 \leq \underline{b},\ x_1 \geq 0,\ x_2 \geq 0 \,\} \tag{8}$$
$$= \{\, x \mid \overline{A}x^+ - \underline{A}x^- \leq \underline{b} \,\} \tag{9}$$
$$= \{\, x \mid A_c x + \Delta|x| \leq \underline{b} \,\} \tag{10}$$
$$= \{\, x \mid A_c x + \Delta t \leq \underline{b},\ -t \leq x \leq t \,\}. \tag{11}$$

Proof The equality (8) was proved in [7, Prop. 1]. Denote by X_1, X_2, X_3, X_4 the right-hand side sets in (8)–(11), respectively. We shall prove that $X_1 \subseteq X_2 \subseteq X_3 \subseteq X_4 \subseteq X_1$.

"$X_1 \subseteq X_2$": Let $x_1 \geq 0, x_2 \geq 0$ satisfy $\overline{A}x_1 - \underline{A}x_2 \leq \underline{b}$. Put $d = \min\{x_1, x_2\}$ (entrywise) and $x = x_1 - x_2$. Then $d \geq 0, x^+ = x_1 - d, x^- = x_2 - d$, and $\overline{A}x^+ - \underline{A}x^- \leq \underline{b}$ by Proposition 1 which proves that $X_1 \subseteq X_2$.

"$X_2 \subseteq X_3$": From $x = x^+ - x^-$, $|x| = x^+ + x^-$ we have $x^+ = (|x| + x)/2$, $x^- = (|x| - x)/2$. Substituting these quantities into $\overline{A}x^+ - \underline{A}x^- \leq \underline{b}$ leads to $A_c x + \Delta|x| \leq \underline{b}$.

"$X_3 \subseteq X_4$": If $A_c x + \Delta|x| \leq \underline{b}$, then for $t = |x|$ we have $A_c x + \Delta t \leq \underline{b}$ and $-t \leq x \leq t$.

"$X_4 \subseteq X_1$": If $A_c x + \Delta t \leq \underline{b}$ and $-t \leq x \leq t$, then $|x| \leq t$ and nonnegativity of Δ implies $A_c x + \Delta|x| \leq A_c x + \Delta t \leq \underline{b}$. Substituting $x = x^+ - x^-$ and $|x| = x^+ + x^-$, we obtain $\overline{A}x^+ - \underline{A}x^- \leq \underline{b}$, hence $x_1 = x^+$, $x_2 = x^-$ satisfy $x_1 - x_2 = x$, $x_1 \geq 0$, $x_2 \geq 0$ and $\overline{A}x_1 - \underline{A}x_2 \leq \underline{b}$. □

In the proof we have relied on the equality (8) proved in [7]. But we can also pursue another path starting from the following general-purpose theorem.

Theorem 5 *Let* $\boldsymbol{A} = [A_c - \Delta, A_c + \Delta]$ *be an* $m \times n$ *interval matrix,* $\boldsymbol{b} = [b_c - \delta, b_c + \delta]$ *an interval* m*-vector, and let* $x \in \mathbb{R}^n$. *Then we have*

$$\{ Ax - b \mid A \in \boldsymbol{A},\ b \in \boldsymbol{b} \} = [A_c x - b_c - (\Delta|x| + \delta),\ A_c x - b_c + (\Delta|x| + \delta)]. \tag{12}$$

Proof According to [2, Prop. 2.27], there holds

$$\{ Ax \mid A \in \boldsymbol{A} \} = [A_c x - \Delta|x|,\ A_c x + \Delta|x|].$$

Then it is sufficient to apply this result to the augmented interval matrix $\boldsymbol{A}' = (\boldsymbol{A} \mid -\boldsymbol{b})$ and to the augmented vector $x' = (x^T \mid 1)^T$ to obtain (12). □

Now, if x is a strong solution of $\boldsymbol{A}x \leq \boldsymbol{b}$, then $Ax - b \leq 0$ for each $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$ which is equivalent to nonpositivity of the upper bound of the right-hand side interval in (12) which is the case if and only if $A_c x + \Delta|x| \leq \underline{b}$. This proves (10), and the other three descriptions can be proved in the same way as in the main proof.

As a direct consequence of (11) we obtain the next result. As the terminology varies, we mention explicitly that a polytope can be bounded as well as unbounded.

Corollary 1 *The set* $\boldsymbol{X}_S(\boldsymbol{A}, \boldsymbol{b})$ *is a convex polytope.*

Proof The above set X_4, described by a set of linear inequalities (11), is a convex polytope in $\mathbb{R}^{2n}$, hence $\boldsymbol{X}_S(\boldsymbol{A}, \boldsymbol{b})$, as an x-projection of it, is a convex polytope as well. □

4 Computation of a strong solution

The inequalities in (9), (10) provide for direct checks of whether a given x is a strong solution, or not, whereas the description in (8) makes it possible to find a strong solution by solving the linear program

$$\min\{ e^T x_1 + e^T x_2 \mid \overline{A}x_1 - \underline{A}x_2 \le \underline{b},\ x_1 \ge 0,\ x_2 \ge 0 \}, \tag{13}$$

where $e = (1, 1, \dots, 1)^T \in \mathbb{R}^n$. Indeed, solve (13); if an optimal solution x_1^*, x_2^* is found, then $\boldsymbol{A}x \le \boldsymbol{b}$ is strongly solvable and $x = x_1^* - x_2^*$ is a strong solution of it; if (13) is infeasible, then $\boldsymbol{A}x \le \boldsymbol{b}$ is not strongly solvable (the problem (13) cannot be unbounded since its objective is nonnegative).

Finally we show that the very form of (6) enforces complementarity of any optimal solution.

Theorem 6 *Each optimal solution* x_1^*, x_2^* *of* (13) *satisfies* $(x_1^*)^T x_2^* = 0$.

Proof Assume that it is not so, so that some optimal solution x_1^*, x_2^* of (13) satisfies $(x_1^*)^T x_2^* > 0$. Then $(x_1^*)_i (x_2^*)_i > 0$ for some i. Set $d = \min\{x_1^*, x_2^*\}$, then $d \ge 0$ and $d_i > 0$, and by Proposition 1, $x_1^* - d, x_2^* - d$ is a feasible solution of (13) whose objective value $e^T(x_1^* - d) + e^T(x_2^* - d) = e^T x_1^* + e^T x_2^* - 2e^T d < e^T x_1^* + e^T x_2^*$ is less than the optimal value, a contradiction. □

5 Strong solvability of interval linear equations and inequalities

For each $y \in \{-1, 1\}^m$ (a $\pm$ 1-vector) denote by $T_y = \operatorname{diag}(y)$ the $m \times m$ diagonal matrix with diagonal vector y. Given an $m \times n$ interval matrix $\boldsymbol{A}$, for each $y \in \{-1, 1\}^m$ let

$$\boldsymbol{A}_y = \{ T_y A \mid A \in \boldsymbol{A} \}. \tag{14}$$

It can be easily seen that if $\boldsymbol{A} = [A_c - \Delta, A_c + \Delta]$, then

$$\boldsymbol{A}_y = [T_y A_c - \Delta,\ T_y A_c + \Delta],$$

i.e., $\boldsymbol{A}_y$ is again an interval matrix. The same notation also applies to interval vectors that are special cases of interval matrices.

Given an $m \times n$ interval matrix $\boldsymbol{A}$ and an interval m-vector $\boldsymbol{b}$, the system of interval linear equations $\boldsymbol{A}x = \boldsymbol{b}$ is called strongly solvable [2] if each system

$$Ax = b$$

with data satisfying

$$A \in \boldsymbol{A},\ b \in \boldsymbol{b}$$

is solvable.

The following theorem establishes an interconnection between strong solvability of interval linear equations and inequalities. It shows that with each strongly solvable system of interval linear equations $\boldsymbol{A}x = \boldsymbol{b}$ with an $m \times n$ interval matrix $\boldsymbol{A}$ we may associate 2^m strongly solvable systems of interval linear inequalities.

Theorem 7 *A system $\boldsymbol{A}x = \boldsymbol{b}$ is strongly solvable if and only if $\boldsymbol{A}_y x \leq \boldsymbol{b}_y$ is strongly solvable for each $y \in \{-1, 1\}^m$.*

Proof "Only if": Let $y \in \{-1, 1\}^m$, $A' \in \boldsymbol{A}_y$ and $b' \in \boldsymbol{b}_y$. In view of (14), $A' = T_y A$ and $b' = T_y b$ for some $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$. Since $\boldsymbol{A}x = \boldsymbol{b}$ is strongly solvable, $Ax = b$ and thus also $T_y Ax = T_y b$ and $T_y Ax \leq T_y b$ are solvable, so that $\boldsymbol{A}_y x \leq \boldsymbol{b}_y$ is strongly solvable.

"If": Conversely, let each $\boldsymbol{A}_y x \leq \boldsymbol{b}_y$, $y \in \{-1, 1\}^m$, have a strong solution x_y, so that

$$T_y A x_y \leq T_y b,$$

which we can write as

$$T_y (b - A x_y) \geq 0, \tag{15}$$

holds for each $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$. Take some fixed $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$. Then (15) shows that the residual set

$$\{ b - Ax \mid x \in \mathbb{R}^n \}$$

intersects all orthants, hence Theorem 3 in [5] implies that $Ax = b$ has a solution. Since $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$ were arbitrary, the system $\boldsymbol{A}x = \boldsymbol{b}$ is strongly solvable, which was to be proved. □

This theorem offers some answer to the question why checking strong solvability on interval linear inequalities is essentially easier than checking that of interval linear equations: in the former case only one linear program (13) needs to be solved whereas 2^m of them are required in the latter case.

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