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ϵ-Henig proper efficiency of set-valued optimization problems in real ordered linear spaces

Zhi-Ang Zhou · Xin-Min Yang · Jian-Wen Peng

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Abstract The aim of this paper is to investigate ϵ-Henig proper efficiency of set-valued optimization problems in linear spaces. Firstly, a new notion of ϵ-Henig properly efficient point is introduced in linear spaces. Secondly, scalarization theorems of set-valued optimization problems are established in the sense of ϵ-Henig proper efficiency. Finally, under the assumption of generalized cone subconvexlikeness, Lagrange multiplier theorems are obtained. Our results generalize some known results in the literature from topological spaces to linear spaces.

Keywords Set-valued map · Generalized cone subconvexlikeness · ϵ-Henig properly efficient solution · Scalarization · Lagrange multipliers

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1 Introduction

In multiobjective programming, it can happen that an optimization problem has no optimal solutions. To overcome the defect, some authors introduced the notion of

Z.-A. Zhou
Department of Applied Mathematics, Chongqing University of Technology, Chongqing 400054, China
e-mail: zhi_ang@163.com

X.-M. Yang · J.-W. Peng (✉)
School of Mathematics, Chongqing Normal University, Chongqing 400047, China
e-mail: jwpeng6@yahoo.com.cn

X.-M. Yang
e-mail: xmyang@cqnu.edu.cn

efficient solutions. Pareto [1] gave the idea of Pareto efficient solution to investigate some equilibrium problems in economics. Based on Pareto's idea, Koopmans [2] firstly introduced the notion of efficient solution of multiobjective optimization problems. Kuhn and Tucker [3] gave the notion of efficient solution of vector-valued optimization problems. Usually, the set containing efficient solutions is so big that it has poor properties. To refine the notion of efficient solution, some authors [4–9] introduced different kinds of properly efficient solution, such as Geoffrion properly efficient solution, Benson properly efficient solution, Henig properly efficient solution and super efficient solution. In different settings, Guerraggio et al. [10] gave some relations among properly efficient solutions. We know that Henig properly efficient solution not only has many desirable properties, but also has much weaker existence conditions than other properly efficient solutions.

With the development of set-valued analysis, properly efficient solutions have been generalized from vector-valued optimization problems to set-valued ones [11–14]. However, in the above mentioned references, properly efficient solutions were mainly studied in topological spaces. It is well known that linear spaces are much wider than topological spaces. To the best of our knowledge, there are only a few authors [15–17] to study properly efficient solutions of set-valued optimization problems in linear spaces. The aim of this paper is to investigate ϵ-Henig proper efficiency of set-valued optimization problems in linear spaces.

This paper is organized as follows. In Sect. 2, some preliminaries, including notations and lemmas, are given. In Sect. 3, we introduce a new notion of ϵ-Henig properly efficient solution of set-valued optimization problems in linear spaces and compare it with other ϵ- properly efficient solutions. In Sect. 4, we present scalarization theorems of set-valued optimization problems in the sense of ϵ-Henig proper efficiency. In Sect. 5, we obtain Lagrange multiplier theorems in linear spaces. Our results generalize some known results in the literature.

2 Preliminaries

Throughout this paper, we always suppose that X, Y and Z are three real ordered linear spaces and A is a nonempty set in X. Let 0 denote the zero element for every space. Let K be a nonempty subset in Y. The generated cone of K is defined as $\text{cone}(K) := \{\lambda k | k \in K, \lambda \geq 0\}$. K is called a convex cone iff

$$\lambda_1 k_1 + \lambda_2 k_2 \in K, \quad \forall \lambda_1, \lambda_2 \geq 0, \quad \forall k_1, k_2 \in K.$$

A cone K is said to be pointed iff $K \cap (-K) = \{0\}$. K is said to be nontrivial iff $K \neq \{0\}$ and $K \neq Y$.

The algebraic dual of Y and Z is denoted by Y^* and Z^*, respectively. Let C and D be two nontrivial, pointed and convex cones in Y and Z, respectively. The algebraic dual cone C^+ and strictly algebraic dual cone C^{+i} of C are, respectively, defined as

$$C^+ := \{y^* \in Y^* | \langle y, y^* \rangle \geqslant 0, \quad \forall y \in C\}, \quad C^{+i} := \{y^* \in Y^* | \langle y, y^* \rangle > 0, \quad \forall y \in C \setminus \{0\}\},$$

where $\langle y, y^* \rangle$ denotes the value of the linear functional y^* at the point y. The meaning of D^+ is similar to that of C^+.

Definition 2.1 [18] Let K be a nonempty subset in Y. The algebraic interior of K is the set

$$\mathrm{cor}(K) := \{k \in K | \forall k' \in Y, \exists \lambda' > 0, \quad \forall \lambda \in [0, \lambda'], k + \lambda k' \in K\}.$$

From now on, we suppose that $\mathrm{cor}(C) \times \mathrm{cor}(D) \neq \emptyset$.

Definition 2.2 [19] Let K be a nonempty subset in Y. K is called balanced iff, $\forall x \in K, \quad \forall \lambda \in [-1, 1], \lambda k \in K$. K is called absorbent iff $0 \in \mathrm{cor}(K)$.

Remark 2.1 It follows from Definitions 2.1 and 2.2 that, a nonempty subset K in Y is absorbent iff, $\forall y \in Y, \exists \lambda' > 0, \quad \forall \lambda \in [0, \lambda'], \lambda y \in K$.

Definition 2.3 [20] Let K be a nonempty subset in Y. The vector closure of K is the set

$$\mathrm{vcl}(K) := \{k \in Y | \exists k' \in Y, \quad \forall \lambda' > 0, \exists \lambda \in]0, \lambda'], k + \lambda k' \in K\}.$$

Let $F : A \rightrightarrows Y$ be a set-valued map on A. Write $\langle F(x), y^* \rangle := \{\langle y, y^* \rangle | y \in F(x)\}$, $F(A) := \bigcup_{x \in A} F(x)$ and $\langle F(A), y^* \rangle := \bigcup_{x \in A} \langle F(x), y^* \rangle$.

Definition 2.4 [21] A set-valued map $F : A \rightrightarrows Y$ is called generalized C-subconvexlike on A iff $\mathrm{cone}(F(A)) + \mathrm{cor}(C)$ is a convex set in Y.

Lemma 2.1 [17] *Let $F : A \rightrightarrows Y$ be a set-valued map on A. If $C \subseteq Y$ is a nontrivial, pointed and convex cone with* $\mathrm{cor}(C) \neq \emptyset$*, then the following statements are equivalent:*

(i) *F is generalized C-subconvexlike on A;*
(ii) $\mathrm{vcl}(\mathrm{cone}(F(A) + C))$ *is a convex set in Y.*

Lemma 2.2 [22] *Let K be a nonempty subset in Y. Then* $K^+ = (\mathrm{vcl}(K))^+$.

Lemma 2.3 [23] *Let C be a nontrivial, pointed and convex cone with* $\mathrm{cor}(C) \neq \emptyset$ *in Y. If $y \in$* $\mathrm{cor}(C)$ *and $y^* \in C^+ \backslash \{0\}$, then* $\langle y, y^* \rangle > 0$.

Lemma 2.4 [24] *Let V be a linear space, and let $M, N \subseteq V$ be two convex sets such that $M \neq \emptyset$,* $\mathrm{cor}(N) \neq \emptyset$ *and $M \cap$* $\mathrm{cor}(N) = \emptyset$*. Then, there exists a hyperplane separating M and* $\mathrm{cor}(N)$ *in V.*

3 Characterization of ϵ-Henig properly efficient point

Definition 3.1 [17] Let $K \subseteq Y$ and $\epsilon \in C$. $\overline{y} \in K$ is called an ϵ-weakly efficient point of K with respect to C (denoted by $\overline{y} \in \epsilon\text{-WE}(K, C)$) iff $(K - \overline{y} + \epsilon) \cap (-\mathrm{cor}(C)) = \emptyset$.

Definition 3.2 [17] Let $K \subseteq Y$ and $\epsilon \in C$. $\overline{y} \in K$ is called an ϵ-Benson properly efficient point of K with respect to C (denoted by $\overline{y} \in \epsilon\text{-BE}(K, C)$) iff $\text{vcl}(\text{cone}(K + C - \overline{y} + \epsilon)) \cap (-C) = \{0\}$.

Definition 3.3 [17] Let $K \subseteq Y$ and $\epsilon \in C$. $\overline{y} \in K$ is called an ϵ-global properly efficient point of K with respect to C (denoted by $\overline{y} \in \epsilon\text{-GPE}(K, C)$) iff there exists a nontrivial, pointed and convex cone C' with $C \backslash \{0\} \subseteq \text{cor}(C')$ such that $(K - \overline{y} + \epsilon) \cap (-C' \backslash \{0\}) = \emptyset$.

Remark 3.1 In [17], we have given the following inclusions:

$$\epsilon - \text{GPE}(K, C) \subseteq \epsilon - \text{BE}(K, C) \subseteq \epsilon - \text{WE}(K, C).$$

Definition 3.4 Let B be a nonempty convex subset in Y. B is a base of C iff $C = \text{cone}(B)$ and there exists a balanced, absorbent and convex set V such that $0 \notin B + V$ in Y.

Remark 3.2 When the linear space Y becomes a locally convex space, $0 \notin \text{cl}(B)$ is equivalent to the statement that there exists a circled and convex neighbour V such that $0 \notin B + V$ in Y, where $\text{cl}(B)$ stands for the topological closure of B. Thus, the definition of the base B of C in this paper coincides with the one in [6,8,9] when the linear space Y becomes a locally convex space.

Write $B^{st} := \{y^* \in Y^* | \text{there exists } t > 0 \text{ such that } \langle b, y^* \rangle \geq t, \quad \forall b \in B\}$. From now on, we suppose that B is a base of C. Let $V \subseteq Y$ is a balanced, absorbent and convex set with $0 \notin B + V$. Write $C_V(B) := \text{cone}(B + V)$.

Remark 3.3 It is easy to check that $C_V(B)$ is a nontrivial, pointed and convex cone in Y. Moreover, $0 \notin \text{cor}(C_V(B))$.

Now, we introduce a new notion of ϵ-Henig properly efficient point in linear spaces.

Definition 3.5 Let $K \subseteq Y$ and $\epsilon \in C$. $\overline{y} \in K$ is called an ϵ-Henig properly efficient point of K with respect to B (denoted by $\overline{y} \in \epsilon\text{-HE}(K, B)$) iff there exists a balanced, absorbent and convex set V with $0 \notin B + V$ such that $\text{cone}(K - \overline{y} + \epsilon) \cap (-C_V(B)) = \{0\}$.

The following proposition will give the relation between ϵ-Henig properly efficient point and ϵ-global properly efficient point in linear spaces.

Proposition 3.1 *Let $K \subseteq Y$ and $\epsilon \in C$. Then,* $\epsilon\text{-HE}(K, B) \subseteq \epsilon\text{-GPE}(K, C)$.

Proof Let $\overline{y} \in \epsilon\text{-HE}(K, B)$. Then, there exists a balanced, absorbent and convex set V with $0 \notin B + V$ such that $\text{cone}(K - \overline{y} + \epsilon) \cap (-C_V(B)) = \{0\}$. Clearly,

$$(K - \overline{y} + \epsilon) \cap (-C_V(B) \setminus \{0\}) = \emptyset. \tag{1}$$

Now, we prove that

$$C \setminus \{0\} \subseteq \text{cor}(C_V(B)). \tag{2}$$

Let $c \in C\backslash\{0\}$. Then, there exist $t > 0$ and $b \in B$ such that $c = tb$. Clearly,

$$c + tV = tb + tV = t(b + V) \subseteq \text{cone}(B + V). \tag{3}$$

Since V is absorbent, for any $y \in Y$, there exists $\lambda' > 0, \quad \forall\lambda \in [0, \lambda'], \lambda y \in V$. Let $\alpha = t\lambda'$. It follows from (3) and balance of V that

$$\begin{aligned} c + \alpha' y &= t\left(b + \frac{\alpha'}{t} y\right) \\ &= t\left(b + \frac{\alpha'}{t\lambda'}\lambda' y\right) \in t\left(b + \frac{\alpha'}{\alpha} V\right) \subseteq t(b + V) \subseteq C_V(B), \quad \forall\alpha' \in [0, \alpha]. \end{aligned} \tag{4}$$

It follows from (4) that $c \in \text{cor}(C_V(B))$. Therefore, (2) holds. By Remark 3.3, $C_V(B)$ is a nontrivial, pointed and convex cone in Y. According to (1) and (2), $\overline{y} \in \epsilon\text{-GPE}(K, C)$. Hence, $\epsilon\text{-HE}(K, B) \subseteq \epsilon\text{-GPE}(K, C)$. □

Remark 3.4 By Remark 3.1 and Proposition 3.1, we obtain the following inclusions:

$$\epsilon - \text{HE}(K, B) \subseteq \epsilon - \text{GPE}(K, C) \subseteq \epsilon - \text{BE}(K, C) \subseteq \epsilon - \text{WE}(K, C).$$

Proposition 3.2 *Let $K \subseteq Y$ and $\epsilon \in C$. Then, ϵ-HE$(K, B) \subseteq \epsilon$-HE$(K + C, B)$.*

Proof Let $\overline{y} \in \epsilon\text{-HE}(K, B)$. Then, there exists a balanced, absorbent and convex set V with $0 \notin B + V$ such that

$$\text{cone}(K - \overline{y} + \epsilon) \cap (-C_V(B)) = \{0\}. \tag{5}$$

Clearly, $\overline{y} \in K + C$. We assert that

$$\text{cone}(K + C - \overline{y} + \epsilon) \cap (-C_V(B)) = \{0\}. \tag{6}$$

Otherwise, there exist $\lambda_1 > 0, k \in K, c \in C, \lambda_2 > 0, b \in B, v \in V$ such that

$$\lambda_1(k + c - \overline{y} + \epsilon) = -\lambda_2(b + v) \neq 0. \tag{7}$$

Case 1: $c = 0$. It follows from (7) that

$$\lambda_1(k - \overline{y} + \epsilon) = -\lambda_2(b + v) \in -C_V(B). \tag{8}$$

Case 2: $c \neq 0$. Since $C\backslash\{0\} \subseteq \text{cor}(C_V(B))$, it follows from (7) that

$$\lambda_1(k - \overline{y} + \epsilon) = -\lambda_1 c - \lambda_2(b + v) \in -\text{cor}(C_V(B)) - C_V(B) \subseteq -C_V(B). \tag{9}$$

By (5), (8) and (9), we have $\lambda_1(k - \overline{y} + \epsilon) = 0$. Therefore,

$$\lambda_1 c + \lambda_2(b + v) = (\lambda_1 c + \lambda_2 b) + \lambda_2 v = 0. \tag{10}$$

Since B is a base of C, there exist $\lambda_3 \geq 0$ and $\overline{b} \in B$ such that $\lambda_1 c = \lambda_3 \overline{b}$. It follows from (10) that

$$\begin{aligned} 0 &= (\lambda_1 c + \lambda_2 b) + \lambda_2 v = (\lambda_3 \overline{b} + \lambda_2 b) + \lambda_2 v \\ &= (\lambda_2 + \lambda_3)\left[\left(\frac{\lambda_3}{\lambda_2 + \lambda_3}\overline{b} + \frac{\lambda_2}{\lambda_2 + \lambda_3} b\right) + \frac{\lambda_2}{\lambda_2 + \lambda_3} v\right]. \end{aligned} \tag{11}$$

According to convexity of B and balance of V, we obtain

$$\frac{\lambda_3}{\lambda_2 + \lambda_3}\overline{b} + \frac{\lambda_2}{\lambda_2 + \lambda_3} b \in B \tag{12}$$

and

$$\frac{\lambda_2}{\lambda_2 + \lambda_3} v \in V. \tag{13}$$

By (11), (12) and (13), $0 \in B + V$, which contradicts $0 \notin B + V$. Hence, (6) holds. Thus, we complete the proof. □

Remark 3.5 The following example shows that ϵ-HE$(K+C, B) \subseteq \epsilon$-HE$(K, B)$ does not hold.

Example 3.1 Let $Y = \mathbb{R}^2$, $K = \{(y_1, y_2) | 2 \leq y_1 \leq 4, -0.2 \leq y_2 \leq 0.8\} \subseteq Y$, $B = \{(y_1, y_2) | y_1 + y_2 = 1, y_1 \geq 0, y_2 \geq 0\} \subseteq Y$, $C = \{(y_1, y_2) | y_1 \geq 0, y_2 \geq 0\} \subseteq Y$ and $\epsilon = (0.3, 0.5)$, $\overline{y} = (2.1, 0.9)$. Clearly, $\overline{y} \in \epsilon$-HE$(K + C, B)$. However, $\overline{y} \notin \epsilon$-HE$(K, B)$.

Remark 3.6 It is easy to check that ϵ-HE$(K, B) = K \cap \epsilon$-HE$(K + C, B)$.

4 Scalarization

In this section, we will establish scalarization theorems of an unconstrained set-valued optimization problem in the sense of ϵ-Henig proper efficiency. Let $F : A \rightrightarrows Y$ be a set-valued maps on A. Now, we consider the following unconstrained set-valued optimization problem:

$$\text{(VP1) Min } F(x) \text{ subject to } x \in A.$$

Definition 4.1 Let $\epsilon \in C$. $\overline{x} \in A$ is called an ϵ-Henig properly efficient solution of (VP1) iff there exists $\overline{y} \in F(\overline{x})$ such that $\overline{y} \in \epsilon$-HE$(F(A), B)$. The pair $(\overline{x}, \overline{y})$ is called an ϵ-Henig properly efficient element of (VP1).

The following example shows that the ϵ-Henig properly efficient element of a set-valued optimization problem in real ordered linear spaces may exist, but it does not exist in topological spaces.

Example 4.1 In (VP1), let $X = Y = \mathbb{R}^2$, $A = [0, 1]\times[0, 1] \subseteq X$, $B = \{(y_1, y_2)|y_1 + y_2 = 1, y_1 \geq 0, y_2 \geq 0\} \subseteq Y$, $C = \{(y_1, y_2)|y_1 \geq 0, y_2 \geq 0\} \subseteq Y$ and $\epsilon = (0.1, 0.1)$. The set-valued map $F : A \rightrightarrows Y$ is defined as follows:

$$F(x_1, x_2) = [1, 1 + x_1] \times [1, 1 + x_2], (x_1, x_2) \in A.$$

Let $\overline{x} = (1, 1)$ and $\overline{y} = (1, 1)$. For the real ordered linear space Y, it is easy to check that $(\overline{x}, \overline{y})$ is an ϵ-Henig properly efficient element of (VP1). On the other hand, let $\Gamma = \{Y, \emptyset\} \cup \{V_\alpha|\alpha > \frac{\sqrt{2}}{2}\}$, where $V_\alpha = \{(y_1, y_2)|y_1^2 + y_2^2 < \alpha, (y_1, y_2) \in Y\}$. Clearly, Y is equipped with the topology Γ. For the topological space (Y, Γ), it is easy to check that the set of ϵ-Henig properly efficient elements of (VP1) is an empty set.

The scalar minimization problem of (VP1) is defined as follows:

$$(\text{VP1})_\varphi \ \text{Min}\ \langle F(x), \varphi\rangle \ \text{subject to}\ x \in A,$$

where $\varphi \in Y^* \setminus \{0\}$.

Definition 4.2 [25] Let $\epsilon \in C$. $\overline{x} \in A$ is called an ϵ-optimal solution of $(\text{VP1})_\varphi$ iff there exists $\overline{y} \in F(\overline{x})$ such that

$$\langle \overline{y}, \varphi\rangle \leq \langle y, \varphi\rangle + \langle \epsilon, \varphi\rangle, \quad \forall x \in A, \quad \forall y \in F(x).$$

The pair $(\overline{x}, \overline{y})$ is called an ϵ-optimal element of $(\text{VP1})_\varphi$.

The following two theorems involve the relations between ϵ-Henig properly efficient element of (VP1) and ϵ-optimal element of $(\text{VP1})_\varphi$.

Theorem 4.1 *Let $\epsilon \in C$. Suppose that the following conditions hold:*

(i) $(\overline{x}, \overline{y})$ *is an ϵ-Henig properly efficient element of (VP1);*
(ii) $F - \overline{y} + \epsilon$ *is generalized C-subconvexlike on A.*

Then, there exists $\varphi \in B^{st}$ such that $(\overline{x}, \overline{y})$ is an ϵ-optimal element of $(\text{VP1})_\varphi$.

Proof Since $(\overline{x}, \overline{y})$ is an ϵ-Henig properly efficient element of (VP1), it follows from Proposition 3.2 that there exists a balanced, absorbent and convex set $V \subseteq Y$ with $0 \notin B + V$ such that

$$\text{cone}(F(A) + C - \overline{y} + \epsilon) \cap (-C_V(B)) = \{0\}. \tag{14}$$

It follows from (14) and Remark 3.3 that

$$\text{cone}(F(A) + C - \overline{y} + \epsilon) \cap \text{cor}(-C_V(B)) = \emptyset. \tag{15}$$

According to (15) and the proof of Proposition 2.1 [17], we have

$$\mathrm{vcl}(\mathrm{cone}(F(A) + C - \overline{y} + \epsilon)) \cap \mathrm{cor}(-C_V(B)) = \emptyset.$$

Since C is nontrivial, it follows from (2) that $\mathrm{cor}(-C_V(B)) \neq \emptyset$. By Condition (ii) and Lemma 2.1, $\mathrm{vcl}(\mathrm{cone}(F(A)+C-\overline{y}+\epsilon))$ is a convex set in Y. Thus, all conditions of Lemma 2.4 are satisfied. Therefore, there exists $\varphi \in Y^* \setminus \{0\}$ such that

$$\langle y_1, \varphi\rangle \geq \langle y_2, \varphi\rangle, \quad \forall y_1 \in \mathrm{vcl}(\mathrm{cone}(F(A) + C - \overline{y} + \epsilon)), \quad \forall y_2 \in -C_V(B).$$

Clearly,

$$\langle y_1, \varphi\rangle \geq \langle y_2, \varphi\rangle, \quad \forall y_1 \in \mathrm{cone}(F(A) + C - \overline{y} + \epsilon), \quad \forall y_2 \in -C_V(B). \tag{16}$$

Since $0 \in C_V(B)$ and $F(A)+C-\overline{y}+\epsilon \subseteq \mathrm{cone}(F(A)+C-\overline{y}+\epsilon)$, it follows from (16) that

$$\langle y_1, \varphi\rangle \geq 0, \quad \forall y_1 \in F(A) + C - \overline{y} + \epsilon. \tag{17}$$

Because $0 \in C$, it follows from (17) that

$$\langle \overline{y}, \varphi\rangle \leq \langle y, \varphi\rangle + \langle \epsilon, \varphi\rangle, \quad \forall x \in A, \quad \forall y \in F(x). \tag{18}$$

Since $0 \in \mathrm{cone}(F(A) + C - \overline{y} + \epsilon)$, it follows from (16) that

$$\langle y_2, \varphi\rangle \geq 0, \quad \forall y_2 \in C_V(B).$$

Clearly,

$$\langle b + v, \varphi\rangle \geq 0, \quad \forall b \in B, \quad \forall v \in V. \tag{19}$$

It follows from (19) and balance of V that

$$\langle b, \varphi\rangle \geq \langle v, \varphi\rangle, \quad \forall b \in B, \quad \forall v \in V. \tag{20}$$

Since V is balanced and absorbent, there exists $\overline{v} \in V$ such that $\langle \overline{v}, \varphi\rangle > 0$. Write $t := \langle \overline{v}, \varphi\rangle$. According to (20), we have

$$\langle b, \varphi\rangle \geq t, \quad \forall b \in B.$$

Therefore, $\varphi \in B^{st}$. It follows from (18) that $(\overline{x}, \overline{y})$ is an ϵ-optimal element of $(\mathrm{VP1})_\varphi$. □

Remark 4.1 The conclusion of Theorem 4.1 in [17] is $\varphi \in C^+ \setminus \{0\}$ and meanwhile the conclusion of Theorem 4.1 in this paper is $\varphi \in B^{st}$. It follows from Remark 3.2 that Theorem 4.1 reduces to the necessity of Theorem 3.3.1 in [26] when $\epsilon = 0$ and the linear space Y becomes a locally convex space. Following the line of Theorem 1 in [27], it is easy to check that the notion of ϵ-strictly efficient point [14] is equivalent to the notion of ϵ-Henig properly efficient point [28] in locally convex spaces. In [29], Xu and Song have shown that, in locally convex spaces, near-subconvexlikeness of the set-valued map F introduced by Yang et al.[30] is equivalent to ic-cone-convexlikeness of the set-valued map F introduced by Sach [31] when the topological interior $\text{int}(C) \neq \emptyset$. Therefore, it follows from Lemma 2.1 and the above statements that Theorem 4.1 is a generalization of Theorem 3.1 in [14].

Theorem 4.2 *Let $\epsilon \in C$. If there exists $\varphi \in C^{+i}$ such that $(\overline{x}, \overline{y})$ is an ϵ-optimal element of (VP1)$_\varphi$, then $(\overline{x}, \overline{y})$ is an ϵ-Henig properly efficient element of (VP1).*

Proof Since $(\overline{x}, \overline{y})$ is an ϵ-optimal element of (VP1)$_\varphi$, we have

$$\langle y, \varphi\rangle + \langle \epsilon, \varphi\rangle \geq \langle \overline{y}, \varphi\rangle, \quad \forall x \in A, \quad \forall y \in F(x). \tag{21}$$

Let α be a positive constant and $B := \{b \in C | \langle b, \varphi\rangle = \alpha\}$. Clearly, $\text{cone}(B) = C$. Let $V := \{y \in Y | |\langle y, \varphi\rangle| < \overline{\alpha}\}$, where $\overline{\alpha}$ is a positive constant with $\overline{\alpha} < \alpha$. Clearly, V is a balanced, absorbent and convex set in Y.

We assert that $0 \notin B + V$. Otherwise, there exist $\overline{b} \in B$ and $\overline{v} \in V$ such that $\overline{b} + \overline{v} = 0$. Therefore, $\langle \overline{b} + \overline{v}, \varphi\rangle = 0$. On the other hand, $\langle \overline{b} + \overline{v}, \varphi\rangle = \alpha + \langle \overline{v}, \varphi\rangle > \alpha - \overline{\alpha} > 0$. This is a contradiction. Hence, $0 \notin B + V$.

From the above statements, we obtain that B is a base of C. We assert that

$$\text{cone}(F(A) - \overline{y} + \epsilon) \cap (-C_V(B)) = \{0\}. \tag{22}$$

Otherwise, there exist $r_1 > 0, r_2 > 0, x_1 \in A, y_1 \in F(x_1), b_1 \in B$ and $v_1 \in V$ such that

$$r_1(y_1 - \overline{y} + \epsilon) = -r_2(b_1 + v_1) \neq 0. \tag{23}$$

It follows from (23) that $\langle y_1 - \overline{y} + \epsilon, \varphi\rangle = -\frac{r_2}{r_1}\langle b_1 + v_1, \varphi\rangle < 0$, i.e., $\langle y_1, \varphi\rangle + \langle \epsilon, \varphi\rangle < \langle \overline{y}, \varphi\rangle$. which contradicts (21). Therefore, (22) holds. Thus, $(\overline{x}, \overline{y})$ is an ϵ-Henig properly efficient element of (VP1). □

Remark 4.2 By Remark 3.4, the notion of ϵ-Henig proper efficiency of set-valued optimization problems is stronger ϵ-globally proper efficiency of set-valued optimization problems. However, comparing Theorem 4.2 with Theorem 4.2 in [17], we obtain a stronger conclusion under the same assumptions. Therefore, Theorem 4.2 is a generalization of Theorem 4.2 in [17]. Since the condition $\varphi \in C^{+i}$ is weaker than the condition $\varphi \in B^{st}$ and the locally convex space Y is replaced by a linear space, Theorem 4.2 generalizes Theorem 3.2 in [14]. Furthermore, if $\epsilon = 0$ and the linear space is replaced by the locally convex space Y, then Theorem 4.2 reduces to the sufficiency of Theorem 3.3.1 in [26].

5 Lagrange multipliers

In this section, we will derive Lagrange Multiplier rules in the sense of ϵ-Henig proper efficiency in linear spaces. Let $F : A \rightrightarrows Y$ and $G : A \rightrightarrows Z$ be two set-valued maps from A to Y and Z, respectively. we consider the following set-valued optimization problem:

$$\text{(VP2) Min } F(x) \text{ subject to } G(x) \cap (-D) \neq \emptyset.$$

The feasible set of (VP2) is defined by $S := \{x \in A | G(x) \cap (-D) \neq \emptyset\}$.

Definition 5.1 Let $\epsilon \in C$. $\overline{x} \in S$ is called an ϵ-Henig properly efficient solution of (VP2) iff there exists $\overline{y} \in F(\overline{x})$ such that $\overline{y} \in \epsilon\text{-HE}(F(S), B)$. The pair $(\overline{x}, \overline{y})$ is called an ϵ-Henig properly efficient element of (VP2).

Let $L(Z, Y)$ be the set of all linear operators from Z to Y. Write $L^+(Z, Y) := \{T \in L(Z, Y) | T(D) \subseteq C\}$. The Lagrangian set-valued map of (VP) is defined by

$$L(x, T) := F(x) + T(G(x)), \quad \forall (x, T) \in A \times L^+(Z, Y).$$

Consider the following unconstrained set-valued optimization problem:

$$(\text{UVP})_T \text{ Min } L(x, T) \quad \text{subject to } (x, T) \in A \times L^+(Z, Y).$$

Let $I(x) = F(x) \times G(x), \quad \forall x \in A$. By Definition 2.4, the set-valued map $I : A \rightrightarrows Y \times Z$ is generalized $C \times D$-subconvexlike on A iff $\text{cone}(I(A)) + \text{cor}(C \times D)$ is a convex set in $Y \times Z$.

Theorem 5.1 *Let $\epsilon \in C$, $\overline{x} \in S$ and $0 \in G(\overline{x})$. Suppose that the following conditions hold:*

(i) *$(\overline{x}, \overline{y})$ is an ϵ-Henig properly efficient element of (VP2);*
(ii) *$\overline{I}(x)$ is generalized $C \times D$-subconvexlike on A, where $\overline{I}(x) = (F(x) - \overline{y} + \epsilon) \times G(x)$;*
(iii) $\text{vcl}(\text{cone}(G(A) + D)) = Z$.

Then, there exists $\overline{T} \in L^+(Z, Y)$ such that $-\overline{T}(G(\overline{x}) \cap (-D)) \subseteq (\text{cor}(C) \cup \{0\}) \setminus (\epsilon + C_V(B) \setminus \{0\})$ and $(\overline{x}, \overline{y})$ is an ϵ-Henig properly efficient element of $(\text{UVP})_{\overline{T}}$.

Proof Since $(\overline{x}, \overline{y})$ is an ϵ-Henig properly efficient element of (VP2), it follows from Proposition 3.2 that there exists a balanced, absorbent and convex set $V \subseteq Y$ with $0 \notin B + V$ such that

$$\text{cone}(F(S) + C - \overline{y} + \epsilon) \cap (-C_V(B)) = \{0\}. \tag{24}$$

It follows from (24) that

$$(F(S) + C - \overline{y} + \epsilon) \cap \text{cor}(-C_V(B)) = \emptyset, \tag{25}$$

which implies that $(\overline{I}(S) + C \times D) \cap \text{cor}(-(C_V(B) \times D)) = \emptyset$. We assert that

$$(\overline{I}(A) + C \times D) \cap \text{cor}(-(C_V(B) \times D)) = \emptyset. \tag{26}$$

Otherwise, there exists $x' \in A$ such that $(\overline{I}(x') + C \times D) \cap \text{cor}(-(C_V(B) \times D)) \neq \emptyset$. Hence,

$$(F(x') + C - \overline{y} + \epsilon) \cap \text{cor}(-C_V(B)) \neq \emptyset \tag{27}$$

and

$$(G(x') + D) \cap \text{cor}(-D) \neq \emptyset. \tag{28}$$

It follows from (28) that $(G(x') + D) \cap (-D) \neq \emptyset$. Since D is a convex cone in Z, it is easy to check that $G(x') \cap (-D) \neq \emptyset$. Therefore, $x' \in S$. According to (27) and $x' \in S$, we obtain $(F(S) + C - \overline{y} + \epsilon) \cap \text{cor}(-C_V(B)) \neq \emptyset$, which contradicts (25). So, (26) holds. By (26) and the proof of Theorem 4.1, we can prove that $\text{vcl}(\text{cone}(\overline{I}(A) + C \times D)) \cap \text{cor}(-(C_V(B) \times D)) = \emptyset$. It follows from condition (ii) and Lemma 2.1 that $\text{vcl}(\text{cone}(\overline{I}(A) + C \times D))$ is a convex set in $Y \times Z$. Thus, all conditions of Lemma 2.4 are satisfied. Therefore, there exists $(y^*, z^*) \in (Y^* \times Z^*) \setminus \{(0, 0)\}$, for any $r \geq 0$, $x \in A$, $(y, z) \in F(x) \times G(x)$, $(c, d) \in C_V(B) \times D$,

$$r\langle y - \overline{y} + \epsilon, y^*\rangle + r\langle z, z^*\rangle + \langle c, y^*\rangle + \langle d, z^*\rangle \geq 0. \tag{29}$$

Letting $r = 0$ in (29), we have

$$\langle c, y^*\rangle + \langle d, z^*\rangle \geq 0, \quad \forall (c, d) \in C_V(B) \times D. \tag{30}$$

Fixing c and d in (29), we obtain

$$\langle y - \overline{y} + \epsilon, y^*\rangle + \langle z, z^*\rangle + \frac{1}{r}(\langle c, y^*\rangle + \langle d, z^*\rangle) \geq 0, \quad \forall r > 0, x \in A, (y, z) \in F(x) \times G(x). \tag{31}$$

Letting $r \to +\infty$ in (31), we have

$$\langle y - \overline{y} + \epsilon, y^*\rangle + \langle z, z^*\rangle \geq 0, \quad \forall x \in A, (y, z) \in F(x) \times G(x). \tag{32}$$

Since both $C_V(B)$ and D are cones, it follows from (30) that $y^* \in (C_V(B))^+$ and $z^* \in D^+$. We assert that $y^* \neq 0$. Otherwise, $z^* \neq 0$. It follows from (32) that

$$\langle z, z^*\rangle \geq 0, \quad \forall x \in A, z \in G(x). \tag{33}$$

Since $z^* \in D^+$, it follows from (33) that $z^* \in (\text{cone}(G(A) + D))^+$. According to Lemma 2.2, $z^* \in (\text{vcl}(\text{cone}(G(A) + D)))^+$. By Condition (iii), $z^* \in Z^+$. Therefore, $z^* = 0$, which contradicts $z^* \neq 0$. Hence, $y^* \neq 0$. Thus, $y^* \in (C_V(B))^+ \setminus \{0\}$.

Because $C \setminus \{0\} \subseteq \mathrm{cor}(C_V(B))$, it follows from Lemma 2.3 that $y^* \in C^{+i}$. Therefore, there exists $c_1 \in \mathrm{cor}(C)$ such that $\langle c_1, y^* \rangle = 1$. The map $\overline{T} : Z \to Y$ is defined as follows

$$\overline{T}(z) := \langle z, z^* \rangle c_1, \quad \forall z \in Z. \tag{34}$$

Since $0 \in G(\overline{x})$, it follows from (34) that $\overline{T} \in L^+(Z, Y)$ and $\overline{y} \in F(\overline{x}) \subseteq F(\overline{x}) + \overline{T}(G(\overline{x}))$. Letting $x = \overline{x}$ and $y = \overline{y}$ in (32), we obtain

$$-\langle \epsilon, y^* \rangle \leq \langle \overline{z}, z^* \rangle \leq 0, \quad \forall \overline{z} \in G(\overline{x}) \cap (-D). \tag{35}$$

Using (34) and (35), we obtain

$$-\overline{T}(\overline{z}) = -\langle \overline{z}, z^* \rangle c_1 \in \mathrm{cor}(C) \cup \{0\}, \quad \forall \overline{z} \in G(\overline{x}) \cap (-D). \tag{36}$$

We assert that

$$-\overline{T}(\overline{z}) \notin \epsilon + \mathrm{cor}(C_V(B)), \quad \forall \overline{z} \in G(\overline{x}) \cap (-D). \tag{37}$$

Otherwise, there exists $\hat{z} \in G(\overline{x}) \cap (-D)$ such that $-\overline{T}(\hat{z}) - \epsilon \in \mathrm{cor}(C_V(B))$. Clearly, $\langle -\overline{T}(\hat{z}) - \epsilon, y^* \rangle > 0$, i.e., $-\langle \epsilon, y^* \rangle > \langle \hat{z}, z^* \rangle$, which contradicts (35). Therefore, (37) holds. By (36) and (37), we obtain $-\overline{T}(G(\overline{x}) \cap (-D)) \subseteq (\mathrm{cor}(C) \cup \{0\}) \setminus (\epsilon + \mathrm{cor}(C_V(B)))$. Since $0 \in G(\overline{x})$, it follows from (32) and (34) that

$$\begin{aligned}
\langle \overline{y} + \overline{T}(0), y^* \rangle &= \langle \overline{y}, y^* \rangle \leq \langle y + \epsilon, y^* \rangle + \langle z, z^* \rangle = \langle y + \epsilon, y^* \rangle + \langle z, z^* \rangle \langle c_1, y^* \rangle \\
&= \langle y + \epsilon, y^* \rangle + \langle \langle z, z^* \rangle c_1, y^* \rangle \\
&= \langle y + \overline{T}(z) + \epsilon, y^* \rangle, \quad \forall x \in A, y \in F(x), z \in G(x).
\end{aligned}$$

Therefore, $(\overline{x}, \overline{y})$ is an ϵ-optimal element of the following scalar minimization problem

$$\mathrm{Min}\ \langle L(x, \overline{T}), y^* \rangle \quad \text{subject to } x \in A.$$

According to Theorem 4.2, $(\overline{x}, \overline{y})$ is an ϵ-Henig properly efficient element of $(\mathrm{UVP})_{\overline{T}}$. □

Remark 5.1 When linear spaces become locally convex spaces, Theorem 5.1 reduces to Theorem 4.1 in [14]. Moreover, the conclusion $-\overline{T}(G(\overline{x}) \cap (-D)) \subseteq (\mathrm{cor}(C) \cup \{0\}) \setminus (\epsilon + C_V(B) \setminus \{0\})$ is stronger than the conclusion $-\overline{T}(G(\overline{x}) \cap (-D)) \subseteq C \setminus (\epsilon + C \setminus \{0\})$ in [14]. When linear spaces become locally convex spaces and $\epsilon = 0$, Theorem 5.1 reduces to Theorem 3.1 in [32]. Moreover, the condition $\mathrm{vcl}(\mathrm{cone}(G(A) + D)) = Z$ is weaker than the condition $G(A) \cap (-\mathrm{cor}(D)) \neq \emptyset$ in [32].

Theorem 5.2 *Let $\epsilon \in C$, $\overline{x} \in S$ and $\overline{y} \in F(\overline{x})$. If there exists $\overline{T} \in L^+(Z, Y)$ such that $(\overline{x}, \overline{y})$ is an ϵ-Henig properly efficient element of $(UVP)_{\overline{T}}$, then $(\overline{x}, \overline{y})$ is an ϵ-Henig properly efficient element of (VP2).*

Proof Since $(\overline{x}, \overline{y})$ is an ϵ-Henig properly efficient element of $(\text{UVP})_{\overline{T}}$, it follows from Proposition 3.3 that there exists a balanced, absorbent and convex set $V \subseteq Y$ with $0 \notin B + V$ such that

$$\text{cone}(F(A) + \overline{T}(G(A)) - \overline{y} + \epsilon + C) \cap (-C_V(B)) = \{0\}. \tag{38}$$

Let $x \in S$. Taking $z \in G(x) \cap (-D)$, we obtain $-\overline{T}(z) \in C$, which implies $0 \in \overline{T}(G(x)) + C$. Therefore, for any $x \in S$, $F(x) - \overline{y} + \epsilon \subseteq F(x) + \overline{T}(G(x)) - \overline{y} + \epsilon + C$. Clearly,

$$\text{cone}(F(S) - \overline{y} + \epsilon) \subseteq \text{cone}(F(A) + \overline{T}(G(A)) - \overline{y} + \epsilon + C). \tag{39}$$

It follows from (38) and (39) that $\text{cone}(F(S) - \overline{y} + \epsilon) \cap (-C_V(B)) = \{0\}$. Hence, $(\overline{x}, \overline{y})$ is an ϵ-Henig properly efficient element of (VP2). □

Remark 5.2 Since ic-cone-convexlikeness of $\overline{I}$ of Theorem 4.2 in [14] is removed and locally convex spaces are replaced by linear spaces, Theorem 5.2 generalizes Theorem 4.2 in [14]. When linear spaces become locally convex spaces and $\epsilon = 0$, Theorem 5.2 reduces to Theorem 3.2 in [32]. Moreover, Theorem 5.2 removes the condition $0 \in \overline{T}(G(\overline{x}) \cap (-D))$ of Theorem 3.2 in [32].

6 Conclusions

In this work, we extend ϵ-Henig properly efficient solution from topological spaces to linear spaces. Some characterizations of ϵ-Henig properly efficient solution are given in linear spaces. Some comparisons are made between ϵ-Henig properly efficient solution and other properly efficient solutions. Using a separation theorem in linear spaces, we obtain a necessary condition of ϵ-Henig properly efficient solution of set-valued optimization problems under the assumption of generalized cone subconvexlikeness. Without any convexity, we obtain a sufficient condition of ϵ-Henig properly efficient solution of set-valued optimization problems. Under suitable conditions, we derive Lagrange multiplier rules in the sense of ϵ-Henig proper efficiency. It is worth noting that the way to define the base B of C in linear spaces is different from the one in topological spaces. We do not know that whether the condition $0 \notin B + V$ in the definition of base B is equivalent to the condition $0 \notin \text{vcl}(B)$, which is one of conditions to define a base of C in topological spaces. Following our line, whether super efficiency can be discussed in linear spaces is an interesting topic.

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