Optim Lett (2014) 8:1031–1046
DOI 10.1007/s11590-013-0619-4

ORIGINAL PAPER

Weak convergence theorems of the modified relaxed projection algorithms for the split feasibility problem in Hilbert spaces

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Received: 2 January 2012 / Accepted: 1 February 2013 / Published online: 13 February 2013

Abstract In this paper, we introduce a modified relaxed projection algorithm and a modified variable-step relaxed projection algorithm for the split feasibility problem in infinite-dimensional Hilbert spaces. The weak convergence theorems under suitable conditions are proved. Finally, some numerical results are presented, which show the advantage of the proposed algorithms.

Keywords Split feasibility problem · Relaxed projection methods · CQ method · Projection method · Inverse strongly monotone

1 Introduction

Let C and Q be the nonempty closed convex subsets of the real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The split feasibility problem (SFP) is formulated as finding a point x^* with the property

$$x^* \in C \quad \text{and} \quad Ax^* \in Q, \tag{1}$$

Supported by the NSFC Tianyuan Youth Foundation of Mathematics of China (No. 11126136) and Fundamental Research Funds for the Central Universities (No. ZXH2011C002).

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where $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. The SFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. Many researchers studied the SFP and introduced various algorithms to solve it (see [2–12] and references therein).

For the split feasibility problem in the finite-dimensional real Hilbert spaces, Byrne [2,3] presented the so-called CQ algorithm for solving the SFP, that does not involve matrix inverses. Take an initial guess $x_0 \in \mathcal{H}_1$ arbitrarily, and define $\{x_n\}$ recursively as

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n, \tag{2}$$

where $0 < \gamma < 2/L$, and where P_C denotes the metric projection onto C and L is the spectral radius of the operator A^*A. Then the sequence $\{x_n\}$ generated by (2) converges strongly to a solution of SFP whenever $\mathcal{H}_1$ is finite-dimensional and whenever there exists a solution to SFP (1). In the CQ algorithm, Byrne assumed that the projections P_C and P_Q are easily calculated. However, in some cases it is impossible or needs too much work to exactly compute the metric projection. Yang [13] presented a relaxed CQ algorithm, in which P_C and P_Q are replaced by P_{C_n} and P_{Q_n}, which are the metric projections onto two halfspaces C_n and Q_n, respectively. Clearly, Yang's algorithm is easy to implement.

Recently, based on work of Yang [13] and Qu and Xiu [14], Wang et al. [15] presented two relaxed inexact projection methods and a variable-step relaxed inexact projection method. The second relaxed inexact projection methods (Algorithm 3.2 in [15]) is to find x_{n+1} satisfying

$$\|x_{n+1} - P_{C_n}(I - \gamma A^*(I - P_{Q_n})A)x_n\| \le \epsilon_n \|(I - P_{Q_n})Ax_n\|,\ n = 0, 1, \ldots. \tag{3}$$

So x_{n+1} is in the ball $B(s, r)$ with $s = P_{C_n}(x_n - \gamma A^*(I - P_{Q_n})Ax_n)$ and $r = \epsilon_n \|(I - P_{Q_n})Ax_n\|$. In the proofs of the convergence theorems, Wang et al. [15] used the fact that, from (3) and $\lim_{n\to\infty} \|(I - P_{Q_n})Ax_n\| = 0$, it follows that for sufficiently large n, x_{n+1} is the projection of x_n in C_n. However, it is obvious that x_{n+1} may not in C_n if $P_{C_n}(x_n - \gamma A^*(I - P_{Q_n})Ax_n) \in \partial C_n$, since C_n is a half-space. To overcome the gap, we modify the algorithms of [13] and [14] and proposed a relaxed projection algorithm as follows

$$x_{n+1} = \alpha_n P_{C_n} x_n + (1 - \alpha_n) P_{C_n}(I - \gamma A^*(I - P_{Q_n})A)x_n, \tag{4}$$

and a variable-step relaxed projection algorithm in infinite-dimensional real Hilbert space. From the following remark, it is easy to see that the algorithm (4) is the special case of (3) proposed by Wang et al. [15].

Remark 1.1 It follows, from (4),

$$\begin{aligned}\|x_{n+1}-P_{C_n}(I-\gamma A^*(I-P_{Q_n})A)x_n\| &= \alpha_n\|P_{C_n}x_n-P_{C_n}(x_n-\gamma A^*(I-P_{Q_n})Ax_n)\| \\ &\le \alpha_n\|\gamma A^*(I-P_{Q_n})Ax_n\| \\ &\le \alpha_n\gamma\sqrt{L}\|(I-P_{Q_n})Ax_n\|.\end{aligned}$$

Let $\epsilon_n = \alpha_n\gamma\sqrt{L}$ and then (3) is derived.

This paper is organized as follows. In Sect. 2, we review some concepts and existing results. In Sect. 3, we present a modified relaxed projection methods for the SFP and establish the convergence of the algorithm. A modified variable-step relaxed inexact projection algorithm is presented in Sect. 4. In Sect. 5, we report some computational results with the proposed algorithms.

2 Preliminaries

In this section, we review some definitions and lemmas which will be used in this paper.

Definition 2.1 Let F be a mapping from a set $\Omega \subset \mathcal{H}$ into $\mathcal{H}$. Then

- F is said to be monotone on Ω, if

$$\langle F(x) - F(y), x - y\rangle \ge 0, \quad \forall x, y \in \Omega;$$

- F is said to be α-inverse strongly monotone (α-ism), with $\alpha > 0$, if

$$\langle F(x) - F(y), x - y\rangle \ge \alpha\|F(x) - F(y)\|^2, \quad \forall x, y \in \Omega;$$

- F is said to be Lipschitz continuous on Ω with constant $\lambda > 0$, if

$$\|F(x) - F(y)\| \le \lambda\|x - y\|, \quad \forall x, y \in \Omega.$$

The projection is an important tool for our work in this paper. Let Ω be a closed convex subset of real Hilbert space $\mathcal{H}$. Recall that the (nearest point or metric) projection from H onto Ω, denoted P_Ω, is defined in such a way that, for each $x \in \mathcal{H}$, $P_\Omega x$ is the unique point in Ω such that

$$\|x - P_\Omega x\| = \min\{\|x - z\| : z \in \Omega\}$$

The following two lemmas are useful characterizations of projections.

Lemma 2.1 *Given $x \in \mathcal{H}$ and $z \in \Omega$. Then $z = P_\Omega x$ if and only if*

$$\langle x - z, y - z\rangle \le 0, \quad \textit{for all} \quad y \in \Omega.$$

Lemma 2.2 *For any $x, y \in \mathcal{H}$ and $z \in \Omega$, it holds*

(i) $\|P_\Omega(x) - P_\Omega(y)\|^2 \le \langle P_\Omega(x) - P_\Omega(y), x - y\rangle$;
(ii) $\|P_\Omega(x) - z\|^2 \le \|x - z\|^2 - \|P_\Omega(x) - x\|^2$.

Remark 2.1 From Lemma 2.2 (i), we know that P_Ω is a monotone, 1-ism and nonexpansive (i.e., $\|P_\Omega(x) - P_\Omega(y)\| \le \|x - y\|$) operator. Moreover, it is easily verified that the operator $I - P_\Omega$ is also 1-ism, where I denotes the identity operator, i.e., for all $x, y \in \mathcal{H}$:

$$\langle (I - P_\Omega)x - (I - P_\Omega)y, x - y\rangle \ge \|(I - P_\Omega)x - (I - P_\Omega)y\|^2. \tag{5}$$

Let F denote a mapping on $\mathcal{H}$. For any $x \in \mathcal{H}$ and $\alpha > 0$, define:

$$x(\alpha) = P_\Omega(x - \alpha F(x)), \quad e(x, \alpha) = x - x(\alpha).$$

From the nondecreasing property of $\|e(x, \alpha)\|$ on $\alpha > 0$ by Toint [16] (see Lemma 2(1)) and the nonincreasing property of $\|e(x, \alpha)\|/\alpha$ on $\alpha > 0$ by Gafni and Bertsekas [17] (see Lemma 1a), we immediately conclude a useful lemma.

Lemma 2.3 *Let F be a mapping on $\mathcal{H}$. For any $x \in \mathcal{H}$ and $\alpha > 0$, we have:*

$$\min\{1, \alpha\}\|e(x, 1)\| \le \|e(x, \alpha)\| \le \max\{1, \alpha\}\|e(x, 1)\|.$$

For the SFP, we firstly assume that the following conditions are satisfied:

(1) The solution set of the SFP denoted by $\Gamma := \{x \in C : Ax \in Q\}$ is nonempty.
(2) The set C is given by

$$C = \{x \in \mathcal{H}_1 : c(x) \le 0\},$$

where $c : \mathcal{H}_1 \to \mathbb{R}$ is a convex function, and C is nonempty.
The set Q is given by

$$Q = \{y \in \mathcal{H}_2 : q(y) \le 0\},$$

where $q : \mathcal{H}_2 \to \mathbb{R}$ is a convex function, and Q is nonempty.
(3) c and q are subdifferentiable on C and Q. (Note that the convex function is subdifferentiable everywhere in $\mathbb{R}^N$.) For any $x \in \mathcal{H}_1$, at least one subgradient $\xi \in \partial c(x)$ can be calculated, where $\partial c(x)$ is defined as follows:

$$\partial c(x) = \{z \in \mathcal{H}_1 : c(u) \ge c(x) + \langle u - x, z\rangle, \quad \text{for all } u \in \mathcal{H}_1\}.$$

For any $y \in \mathcal{H}_2$, at least one subgradient $\eta \in \partial q(y)$ can be calculated, where

$$\partial q(y) = \{w \in \mathcal{H}_2 : q(v) \ge q(y) + \langle v - y, w\rangle, \quad \text{for all } v \in \mathcal{H}_2\}.$$

(4) c and q are bounded on bounded sets. (Note that this condition is automatically satisfied if $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite dimensional.)

From Banach-Steinhaus Theorem (see Theorem 2.5 in [18]), it is easy to get the following result.

Remark 2.2 Assumption (4) guarantees that if $\{x_n\}$ is a bounded sequence in $\mathcal{H}_1$ (resp. $\mathcal{H}_2$) and $\{x_n^*\}$ is a sequence in $\mathcal{H}_1$ (resp. $\mathcal{H}_2$) such that $x_n^* \in \partial c(x_n)$ (resp. $x_n^* \in \partial q(x_n)$) for each n, then $\{x_n^*\}$ is bounded.

Lemma 2.4 *[19] Let $\{a_n\}$ be a sequence of nonnegative number such that*

$$a_{n+1} \leq (1 + \lambda_n) a_n,$$

where $\{\lambda_n\}$ satisfies $\sum_{n=1}^{\infty} \lambda_n < +\infty$. Then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.5 *[20] Let K be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $\{x_n\}$ be a bounded sequence which satisfies the following properties:*

- *every weak limit point of $\{x_n\}$ lies in K;*
- *$\lim_{n\to\infty} \|x_n - x\|$ exists for every $x \in K$.*

Then $\{x_n\}$ converges weakly to a point in K.

3 A modified relaxed projection algorithm

In this section we propose a modified relaxed algorithm and prove the weak convergence of the proposed algorithm.

Algorithm 3.1 *Let x_0 be arbitrary. Let $\{\alpha_n\} \subset (0, \infty)$, and*

$$x_{n+1} = \alpha_n P_{C_n} x_n + (1 - \alpha_n) P_{C_n} (I - \gamma A^*(I - P_{Q_n})A) x_n, \tag{6}$$

where $\gamma \in (0, M)$, $M := \min\{2/\|A\|^2, \sqrt{2}/\|A\|\}$, and $\{C_n\}$ and $\{Q_n\}$ are the sequences of closed convex sets constructed as follows:

$$C_n = \{x \in \mathcal{H}_1 : c(x_n) + \langle \xi_n, x - x_n \rangle \leq 0\},$$

where $\xi_n \in \partial c(x_n)$, and

$$Q_n = \{y \in \mathcal{H}_2 : q(Ax_n) + \langle \eta_n, y - Ax_n \rangle \leq 0\},$$

where $\eta_n \in \partial q(Ax_n)$.

Remark 3.1 By the definition of subdifferentials, it is clear that $C \subset C_n$ and $Q \subset Q_n$. Also note that C_n and Q_n are half-space, thus, the projections P_{C_n} and P_{Q_n} have closed-form expression.

Theorem 3.1 *Let $\{x_n\}$ be the sequence generated by the Algorithm 3.1 Let $\{\alpha_n\} \subset (0, \infty)$ satisfy*

$$\sum_{n=0}^{\infty} \alpha_n^2 < +\infty.$$

Then $\{x_n\}$ *converges weakly to a solution of SFP(1).*

Proof Set $\sigma = \gamma^2\|A\|^2 - 2\gamma$, then $\sigma < 0$. Let $\delta_n = \sqrt{-\frac{2}{\sigma}\alpha_n}$. From (6), we have

$$\begin{aligned}
&\|x_{n+1} - P_{C_n}(I - \gamma A^*(I - P_{Q_n})A)x_n\| \\
&\quad = \alpha_n\|P_{C_n}x_n - P_{C_n}(x_n - \gamma A^*(I - P_{Q_n})Ax_n)\| \\
&\quad \le \alpha_n\|\gamma A^*(I - P_{Q_n})Ax_n\| \\
&\quad \le \alpha_n\gamma\|A\|\|(I - P_{Q_n})Ax_n\|. \qquad (7)
\end{aligned}$$

Take arbitrarily $x^* \in \Gamma$, then we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|x_{n+1} - P_C(I - \gamma A^*(I - P_{Q_n})A)x_n\|^2 \\
&+ \|P_C(I - \gamma A^*(I - P_{Q_n})A)x_n - x^*\|^2 \\
&+ 2\langle x_{n+1} - P_C(I - \gamma A^*(I - P_{Q_n})A)x_n, P_C(I - \gamma A^*(I - P_{Q_n})A)x_n - x^*\rangle. \qquad (8)
\end{aligned}$$

From Cauchy-Schwarz inequality and arithmetic and geometric means inequality, it is follows

$$\begin{aligned}
&2\langle x_{n+1} - P_C(I - \gamma A^*(I - P_{Q_n})A)x_n, P_C(I - \gamma A^*(I - P_{Q_n})A)x_n - x^*\rangle \\
&\quad \le \frac{1}{\delta_n^2}\|x_{n+1} - P_C(I - \gamma A^*(I - P_{Q_n})A)x_n\|^2 \\
&\quad +\delta_n^2\|P_C(I - \gamma A^*(I - P_{Q_n})A)x_n - x^*\|^2. \qquad (9)
\end{aligned}$$

Substituting (9) into (8), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le (1 + \delta_n^2)\|P_{C_n}(I - \gamma A^*(I - P_{Q_n})A)x_n - x^*\|^2 \\
&+ \left(1 + \frac{1}{\delta_n^2}\right)\|x_{n+1} - P_{C_n}(I - \gamma A^*(I - P_{Q_n})A)x_n\|^2. \qquad (10)
\end{aligned}$$

Since $x^* \in C_n$, we have

$$\begin{aligned}
&\|P_{C_n}(I - \gamma A^*(I - P_{Q_n})A)x_n - x^*\|^2 \le \|(x_n - x^*) - \gamma A^*(I - P_{Q_n})Ax_n\|^2 \\
&\quad = \|x_n - x^*\|^2 + \gamma^2\|A^*(I - P_{Q_n})Ax_n\|^2 - 2\gamma\langle x_n - x^*, A^*(I - P_{Q_n})Ax_n\rangle \\
&\quad \le \|x_n - x^*\|^2 + \gamma^2\|A\|^2\|(I - P_{Q_n})Ax_n\|^2 - 2\gamma\langle Ax_n - Ax^*, Ax_n - P_{Q_n}Ax_n\rangle. \qquad (11)
\end{aligned}$$

It is easily seen that

$$\begin{aligned}
&\langle Ax_n - Ax^*, Ax_n - P_{Q_n}Ax_n\rangle \\
&\quad = \|(I - P_{Q_n})Ax_n\|^2 + \langle P_{Q_n}Ax_n - Ax^*, Ax_n - P_{Q_n}Ax_n\rangle. \qquad (12)
\end{aligned}$$

Since $Ax^* \in Q \subset Q_n$, by Lemma 2.1, we get

$$\langle P_{Q_n} Ax_n - Ax^*, Ax_n - P_{Q_n} Ax_n \rangle \geq 0. \tag{13}$$

Combining (11)–(13), we have

$$\begin{aligned} &\|P_{C_n}(I - \gamma A^*(I - P_{Q_n})A)x_n - x^*\|^2 \\ &\quad \leq \|x_n - x^*\|^2 + (\gamma^2\|A\|^2 - 2\gamma)\|(I - P_{Q_n})Ax_n\|^2 \\ &\quad = \|x_n - x^*\|^2 + \sigma\|(I - P_{Q_n})Ax_n\|^2 \end{aligned} \tag{14}$$

Applying (7), (10) and (14), we obtain

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \leq (1 + \delta_n^2)\|x_n - x^*\|^2 \\ &\quad + \left[\sigma(1 + \delta_n^2) + \gamma^2\|A\|^2\alpha_n^2\left(1 + \frac{1}{\delta_n^2}\right)\right]\|(I - P_{Q_n})Ax_n\|^2 \\ &\quad \leq \left(1 + \left(-\frac{2}{\sigma}\right)\alpha_n^2\right)\|x_n - x^*\|^2 \\ &\quad + \left(\gamma^2\|A\|^2 - 2\right)\left(\alpha_n^2 - \frac{1}{2}\sigma\right)\|(I - P_{Q_n})Ax_n\|^2, \end{aligned} \tag{15}$$

which with $\sigma < 0$ and $\gamma \leq \sqrt{2}/\|A\|$ implies

$$\|x_{n+1} - x^*\|^2 \leq \left(1 + \left(-\frac{2}{\sigma}\right)\alpha_n^2\right)\|x_n - x^*\|^2. \tag{16}$$

So, using Lemma 2.4, we have $\lim_{n\to\infty}\|x_n - x^*\|$ exists, and hence $\{x_n\}$ is bounded. From (15), it follows that

$$\begin{aligned} &\sigma\left(\frac{\gamma^2\|A\|^2}{2} - 1\right)\|(I - P_{Q_n})Ax_n\|^2 \leq \left(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\right) \\ &\quad + \left(-\frac{2}{\sigma}\right)\alpha_n^2\|x_n - x^*\|^2. \end{aligned}$$

Consequently we get by $\alpha_n \to 0$,

$$\lim_{n\to\infty}\|Ax_n - P_{Q_n}Ax_n\| = 0. \tag{17}$$

Set

$$u_n = A^*(I - P_{Q_n})Ax_n \to 0. \tag{18}$$

We next demonstrate that

$$\|x_{n+1} - x_n\| \to 0.$$

To see this, we note the identity

$$\|x_{n+1} - x_n\|^2 = \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\langle x_{n+1} - x_n, x_{n+1} - x^*\rangle. \quad (19)$$

On the other hand, using $x_{n+1} = \alpha_n P_{C_n} x_n + (1 - \alpha_n) P_{C_n}(x_n - \gamma u_n)$, we have

$$\begin{aligned}&\langle x_{n+1} - x_n, x_{n+1} - x^*\rangle = \alpha_n \langle P_{C_n} x_n - x_n, x_{n+1} - x^*\rangle \\ &\quad + \alpha_n (1 - \alpha_n)\langle P_{C_n}(x_n - \gamma u_n) - x_n, P_{C_n} x_n - x^*\rangle \\ &\quad + (1 - \alpha_n)^2 \langle x_n - P_{C_n}(x_n - \gamma u_n), x^* - P_{C_n}(x_n - \gamma u_n)\rangle. \end{aligned} \quad (20)$$

Since $x^* \in C \subset C_n$, we have by Lemma 2.1

$$\langle (x_n - \gamma u_n) - P_{C_n}(x_n - \gamma u_n), x^* - P_{C_n}(x_n - \gamma u_n)\rangle \leq 0.$$

Therefore,

$$\begin{aligned}&\langle x_n - P_{C_n}(x_n - \gamma u_n), x^* - P_{C_n}(x_n - \gamma u_n)\rangle \\ &\quad = \langle (x_n - \gamma u_n) - P_{C_n}(x_n - \gamma u_n), x^* - P_{C_n}(x_n - \gamma u_n)\rangle \\ &\qquad + \langle \gamma u_n, x^* - P_{C_n}(x_n - \gamma u_n)\rangle \\ &\quad \leq \langle \gamma u_n, x^* - P_{C_n}(x_n - \gamma u_n)\rangle \\ &\quad \leq \gamma \|u_n\| \|x^* - P_{C_n}(x_n - \gamma u_n)\|. \end{aligned} \quad (21)$$

Since $\{x_n\}$ is bounded, by $\alpha_n \to 0$ and (18), (20), (21), we get

$$\begin{aligned}&\langle x_{n+1} - x_n, x_{n+1} - x^*\rangle \leq \alpha_n \langle P_{C_n} x_n - x_n, x_{n+1} - x^*\rangle \\ &\quad + \alpha_n (1 - \alpha_n)\langle P_{C_n}(x_n - \gamma u_n) - x_n, \\ &P_{C_n} x_n - x^*\rangle + (1 - \alpha_n)^2 \gamma \|u_n\| \|x^* - P_C(x_n - \gamma u_n)\| \\ &\quad \to 0, \end{aligned}$$

which by (19) and existence of $\lim_{n\to\infty} \|x_n - x^*\|$ yields

$$\|x_{n+1} - x_n\| \to 0. \quad (22)$$

Since $\{x_n\}$ is bounded, which implies that $\{\xi_n\}$ is bounded, we see that the set of weak limit points of $\{x_n\}$, $\omega_w(x_n)$, is nonempty. We now show □

Claim $\omega_w(x_n) \subset \Gamma$.

Indeed, assume $\hat{x} \in \omega_w(x_n)$ and $\{x_{n_j}\}$ is a subsequence of $\{x_n\}$ which converges weakly to $\hat{x}$. Since $x_{n_j+1} \in C_{n_j}$, we obtain

$$c(x_{n_j}) + \langle \xi_{n_j}, x_{n_j+1} - x_{n_j}\rangle \leq 0.$$

Thus,

$$c(x_{n_j}) \leq -\langle \xi_{n_j}, x_{n_j+1} - x_{n_j} \rangle \leq \xi \|x_{n_j+1} - x_{n_j}\|,$$

where ξ satisfies $\|\xi_{n_j}\| \leq \xi$ for all n. By virtue of the lower semicontinuity of c, we get by (22)

$$c(\hat{x}) \leq \liminf_{j\to\infty} c(x_{n_j}) \leq 0,$$

Therefore, $\hat{x} \in C$.

Next we show that $A\hat{x} \in Q$. To see this, by (17), set $y_n = Ax_n - P_{Q_n}Ax_n \to 0$ and let η be such that $\|\eta_n\| \leq \eta$. Then, since $Ax_{n_j} - y_{n_j} = P_{Q_{n_j}}Ax_{n_j} \in Q_{n_j}$, we get

$$q(Ax_{n_j}) + \langle \eta_{n_j}, (Ax_{n_j} - y_{n_j}) - Ax_{n_j} \rangle \leq 0.$$

Hence,

$$q(Ax_{n_j}) \leq \langle \eta_{n_j}, y_{n_j} \rangle \leq \eta \|y_{n_j}\| \to 0.$$

By the weak lower semicontinuity of q and the fact that $Ax_{n_j} \to A\hat{x}$ weakly, we arrive at the conclusion

$$q(A\hat{x}) \leq \liminf_{j\to\infty} q(Ax_{n_j}) \leq 0.$$

Namely, $A\hat{x} \in Q$.

Therefore, $\hat{x} \in \Gamma$. Now we can apply Lemma 2.5 to $K := \Gamma$ to get that the full sequence $\{x_n\}$ converges weakly to a point in Γ.

4 A modified variable-step relaxed projection algorithm

In the Algorithm 3.1, the stepsizes γ are all fixed. In this section, based on the algorithm of Qu and Xiu [14], we present a modified variable-step projection method which needs not compute the spectral radius of the operator A^*A, and the objective function can sufficiently decrease at each iteration.

For every n, using Q_n we define the function $F_n : \mathcal{H}_1 \to \mathcal{H}_1$ by

$$F_n(x) = A^*(I - P_{Q_n})Ax.$$

The variable-step relaxed projection algorithm is defined as follows:

Algorithm 4.1 *Given constants $\gamma > 0$, $l \in (0, 1)$, $\mu \in (0, \sqrt{2}/2)$. Let x_0 be arbitrary. Let $\{\alpha_n\} \subset (0, \infty)$, and let*

$$y_n = P_{C_n}(x_n - \beta_n F_n(x_n)),$$

where $\beta_n = \gamma l^{m_n}$ *and* m_n *is the smallest nonnegative integer* m *such that*

$$\|F_n(x_n) - F_n(y_n)\| \le \mu \frac{\|x_n - y_n\|}{\beta_n}. \tag{23}$$

Let

$$x_{n+1} = \alpha_n y_n + (1 - \alpha_n) P_{C_n}(x_n - \beta_n F_n(y_n)), \tag{24}$$

where $\{C_n\}$ *and* $\{Q_n\}$ *are the sequences of closed convex sets constructed as follows:*

$$C_n = \{x \in \mathcal{H}_1 : c(x_n) + \langle \xi_n, x - x_n \rangle \le 0\},$$

where $\xi_n \in \partial c(x_n)$*, and*

$$Q_n = \{y \in \mathcal{H}_2 : q(Ax_n) + \langle \eta_n, y - Ax_n \rangle \le 0\},$$

where $\eta_n \in \partial q(Ax_n)$*.*

Although C_n, Q_n and F_n depend on n, we have the following two nice lemmas (see [14]).

Lemma 4.1 *For all* $n = 0, 1, 2, \ldots,$ F_n *is Lipschitz continuous on* $\mathcal{H}_1$ *with constant* L *and* $1/L$*-ism on* $\mathcal{H}_1$*, where* L *is the spectral radius of the operator* A^*A*. Therefore, Armijo-like search rule (23) is well defined.*

Lemma 4.2 *For all* $n = 0, 1, \ldots,$

$$\frac{\mu l}{L} < \beta_n \le \gamma.$$

Theorem 4.1 *Let* $\{x_n\}$ *be the sequence generated by the Algorithm 4.1 Let* $\{\alpha_n\} \subset (0, \infty)$ *satisfy*

$$\sum_{n=0}^{\infty} \alpha_n^2 < +\infty.$$

Then $\{x_n\}$ *converges weakly to a solution of SFP(1).*

Proof From (23) and (24), it is easily seen that

$$\begin{aligned}
\|x_{n+1} - P_{C_n}(x_n - \beta_n F_n(y_n))\| &= \alpha_n \|y_n - P_{C_n}(x_n - \beta_n F_n(y_n))\| \\
&= \alpha_n \|P_{C_n}(x_n - \beta_n F_n(x_n)) - P_{C_n}(x_n - \beta_n F_n(y_n))\| \\
&\le \alpha_n \beta_n \|F_n(x_n) - F_n(y_n)\| \\
&\le \alpha_n \mu \|x_n - y_n\|.
\end{aligned} \tag{25}$$

Let x^* be a solution of the SFP (1), using the similar procedure in the proof of Theorem 3.1, we obtain

$$\|x_{n+1}-x^*\|^2 \le (1+\alpha_n^2)\|P_{C_n}(x_n-\beta_n F_n(y_n))-x^*\|^2 + \left(1+\frac{1}{\alpha_n^2}\right)\|x_{n+1}-P_{C_n}(x_n-\beta_n F_n(y_n))\|^2. \tag{26}$$

Following the line of the proof of (7) in [14], we have

$$\|P_{C_n}(x_n-\beta_n F_n(y_n))-x^*\|^2 \le \|x_n-x^*\|^2-(1-\mu^2)\|x_n-y_n\|^2. \tag{27}$$

Combining (25)–(27), we get

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le (1+\alpha_n^2)\|x_n-x^*\|^2 \\ &\quad+\left[\alpha_n^2\mu^2\left(1+\frac{1}{\alpha_n^2}\right)-(1+\alpha_n^2)(1-\mu^2)\right]\|x_n-y_n\|^2 \\ &\le \left(1+\alpha_n^2\right)\|x_n-x^*\|^2+(1+\alpha_n^2)(2\mu^2-1)\|x_n-y_n\|^2,\end{aligned} \tag{28}$$

which with $\mu\in(0,\sqrt{2}/2)$ yields

$$\|x_{n+1}-x^*\|^2 \le \left(1+\alpha_n^2\right)\|x_n-x^*\|^2.$$

So, using Lemma 2.4, we get that $\lim_{n\to\infty}\|x_n-x^*\|$ exists, and hence $\{x_n\}$ is bounded. From (28), it follows that

$$(1-2\mu^2)\|x_n-y_n\|^2 \le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_n^2\|x_n-x^*\|^2. \tag{29}$$

Consequently we get

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{30}$$

We next demonstrate that

$$\|x_{n+1}-x_n\|\to 0. \tag{31}$$

Using (23), we obtain

$$\begin{aligned}\|x_{n+1}-x_n\|^2 &\le 2(\|x_{n+1}-y_n\|^2+\|y_n-x_n\|^2) \\ &= 2(1-\alpha_n)^2\|P_{C_n}(x_n-\beta_n F_n(y_n))-y_n\|^2+2\|y_n-x_n\|^2 \\ &= 2(1-\alpha_n)^2\|P_{C_n}(x_n-\beta_n F_n(y_n))-P_{C_n}(x_n-\beta_n F_n(x_n))\|^2+2\|y_n-x_n\|^2 \\ &\le 2(1-\alpha_n)^2\beta_n^2\|F_n(x_n)-F_n(y_n)\|^2+2\|y_n-x_n\|^2 \\ &\le 2[\mu^2(1-\alpha_n)^2+1]\|x_n-y_n\|^2,\end{aligned}$$

which by (30) yields (31).

Since $\{x_n\}$ is bounded, we see that the set of weak limit points of $\{x_n\}$, $\omega_w(x_n)$, is nonempty and $\{Ax_n\}$ is bounded which implies that $\{\eta_n\}$ is bounded. Let η be such that $\|\eta_n\| \le \eta$. We now show □

Claim $\omega_w(x_n) \subset \Gamma$.

Assume $\hat{x} \in \omega_w(x_n)$ and $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ which converges weakly to $\hat{x}$. Using the similar procedure in the proof of Theorem 3.1, by (31), we get $\hat{x} \in C$.

Next we show that $A(\hat{x}) \in Q$. Define

$$e_n(x, \beta) = x - P_{C_n}(x - \beta F_n(x)), \quad n = 0, 1, 2, \ldots.$$

Then from Lemmas 2.3 and 4.2, and Eq. (30), we have:

$$\min\left\{1, \frac{\mu l}{L}\right\} \|e_{n_i}(x_{n_i}, 1)\| \le \|e_{n_i}(x_{n_i}, \beta_{n_i})\| = \|x_{n_i} - y_{n_i}\| \to 0,$$

which implies

$$\lim_{i\to\infty} \|e_{n_i}(x_{n_i}, 1)\| = 0. \tag{32}$$

Using Lemma 2.1 and $x^* \in C \subset C_{n_i}$, we have for all $i = 1, 2, \ldots,$

$$\langle x_{n_i} - F_{n_i}(x_{n_i}) - P_{C_{n_i}}(x_{n_i} - F_{n_i}(x_{n_i})), P_{C_{n_i}}(x_{n_i} - F_{n_i}(x_{n_i})) - x^*\rangle \ge 0,$$

which implies

$$\langle e_{n_i}(x_{n_i}, 1) - F_{n_i}(x_{n_i}), x_{n_i} - x^* - e_{n_i}(x_{n_i}, 1)\rangle \ge 0,$$

which implies

$$\begin{aligned}\langle x_{n_i} - x^*, e_{n_i}(x_{n_i}, 1)\rangle \ge\ & \|e_{n_i}(x_{n_i}, 1)\|^2 - \langle F_{n_i}(x_{n_i}), e_{n_i}(x_{n_i}, 1)\rangle \\ & + \langle F_{n_i}(x_{n_i}), x_{n_i} - x^*\rangle \end{aligned} \tag{33}$$

From (5), it follows

$$\begin{aligned}\langle F_{n_i}(x_{n_i}) - F_{n_i}(x^*), x_{n_i} - x^*\rangle &= \langle A^*(I - P_{Q_{n_i}})Ax_{n_i} - A^*(I - P_{Q_{n_i}})Ax^*, x_{n_i} - x^*\rangle \\ &= \langle (I - P_{Q_{n_i}})Ax_{n_i} - (I - P_{Q_{n_i}})Ax^*, Ax_{n_i} - Ax^*\rangle \\ &\ge \|(I - P_{Q_{n_i}})Ax_{n_i} - (I - P_{Q_{n_i}})Ax^*\|^2 \\ &= \|(I - P_{Q_{n_i}})Ax_{n_i}\|^2 \end{aligned} \tag{34}$$

Combining (33) and (34), we have

$$\begin{aligned}\langle x_{n_i} - x^*, e_{n_i}(x_{n_i}, 1)\rangle \ge\ & \|e_{n_i}(x_{n_i}, 1)\|^2 - \langle F_{n_i}(x_{n_i}), e_{n_i}(x_{n_i}, 1)\rangle \\ & + \|(I - P_{Q_{n_i}})Ax_{n_i}\|^2. \end{aligned} \tag{35}$$

Since

$$\|F_{n_i}(x_{n_i})\| = \|F_{n_i}(x_{n_i}) - F_{n_i}(x^*)\| \leq L\|x_{n_i} - x^*\|, \quad \forall i = 1, 2, \ldots,$$

and $\{x_{n_i}\}$ is bounded, the sequence $\{F_{n_i}(x_{n_i})\}$ is also bounded. Therefore, from (32) and (35), we get

$$\lim_{n_i \to \infty} \|P_{Q_{n_i}}(Ax_{n_i}) - Ax_{n_i}\| = 0.$$

By $P_{Q_{n_i}}(Ax_{n_i}) \in Q_{n_i}$, we have

$$q(Ax_{n_i}) + \langle \eta_{n_i}, P_{Q_{n_i}}(Ax_{n_i}) - Ax_{n_i} \rangle \leq 0.$$

Hence,

$$q(Ax_{n_i}) \leq -\langle \eta_{n_i}, P_{Q_{n_i}}(Ax_{n_i}) - Ax_{n_i} \rangle \leq \eta \|P_{Q_{n_i}}(Ax_{n_i}) - Ax_{n_i}\| \to 0.$$

By the weak lower semicontinuity of q and the fact that $Ax_{n_j} \to A\hat{x}$ weakly, we arrive at the conclusion

$$q(A\hat{x}) \leq \liminf_{j \to \infty} q(Ax_{n_j}) \leq 0.$$

Namely, $A\hat{x} \in Q$.

Therefore, $\hat{x} \in \Gamma$. Now we can apply Lemma2.5 to $K := \Gamma$ to get that the full sequence $\{x_n\}$ converges weakly to a point in Γ.

5 Numerical results

In this section, we will present the results of numerical tests for a problem (see [15]). We consider the following problem:

(P) Let $C = \{x \in R^{10} | c(x) \leq 0\}$ where $c(x) = -x_1 + x_2^2 + \cdots + x_{10}^2$ and $Q = \{y \in R^{20} | q(y) \leq 0\}$ where $q(x) = y_1 + y_2^2 + \cdots + y_{20}^2 - 1$. Note that C is the set above the function $x_1 = x_2^2 + \cdots + x_{10}^2$ and Q is the set below the function $y_1 = -y_2^2 - \cdots - y_{20}^2 + 1$. $A \in R^{20\times 10}$ is a random matrix where every element of A is in (0, 1) satisfying $A(C) \cap Q \neq \emptyset$. Let x_0 be a random vector in R^{10} where every element of x_0 is in (0, 1).

The terminal condition is that $\|x_n - x^*\| \leq \epsilon$, where x^* is a solution of (P). Let $\alpha_n = 1/n^2, n = 0, 1, \ldots,$ in the Algorithms 3.1 and 4.1 In the tables, ϵ denotes the tolerance of the solution point of the problem, 'Iter.' and 'InIt' denote the terminating iterative numbers and the number of total iterations of finding suitable β_n in (23), respectively. $C(x)$ and $Q(Ax)$ denote the value of $c(x)$ and $q(Ax)$ at the terminal point, respectively.

We compare the algorithms 3.1 and 4.1 with the relaxed CQ algorithm in [13] and the variable-step CQ algorithm in [14], respectively.

Table 1 MRCQ and RCQ applied for (P)

ϵ	MRCQ			RCQ		
	Iter.	$C(x)$	$Q(Ax)$	Iter.	$C(x)$	$Q(Ax)$
0.01	28	3.71E−01	2.10E−02	34	4.52E−02	2.204−02
0.01	5	2.41E−04	−0.114535	5	1.12E−04	−0.486864
0.01	11	−0.439145	1.65E−02	7	2.30E−06	1.36E−02
0.01	48	−2.48E−01	2.00E−02	38	−2.06E−02	2.04E−02
0.001	4	−6.62E−02	−0.282846	5	5.05E−07	−0.579155
0.001	5	5.83E−05	−0.193581	5	1.32E−05	−0.542675
0.001	14	−0.232628	1.43E−03	10	2.05E−08	1.12E−03
0.001	49	−0.125805	2.43E−03	18	1.34E−08	2.34E−03
0.0001	4	−0.223021	−0.171751	5	2.37E−06	−0.344674
0.0001	5	9.28E−05	−0.256516	5	9.44E−06	−0.664877
0.0001	11	−8.69E−03	1.26E−04	7	3.59E−010	−8.06E−7
0.0001	44	−9.44E−02	2.27E−04	36	−4.55E−05	2.14E−04

Table 2 MRCQ applied to (P) with different stepsizes

Time of test	γ	ϵ	Iter.	$C(x)$	$Q(Ax)$
1	0.5M	0.0001	198	−0.295319	2.25E−04
1	0.7M	0.0001	142	−0.295280	2.24E−04
1	0.99M	0.0001	101	−0.295296	2.25E−04
2	0.5M	0.0001	11	−2.88E−02	1.80E−04
2	0.7M	0.0001	6	−2.37E−02	−3.85E−04
2	0.99M	0.0001	4	−1.64E−02	−0.370022
3	0.5M	0.0001	120	−0.462144	2.21E−04
3	0.7M	0.0001	94	−0.460538	2.34E−04
3	0.99M	0.0001	80	−0.457319	2.17E−04
4	0.5M	0.0001	39	−0.267348	2.31E−04
4	0.7M	0.0001	28	−0.265630	2.22E−04
4	0.99M	0.0001	20	−0.262080	2.22E−04

Firstly, we compare the modified relaxed projection algorithm 3.1 (**MRCQ**) with relaxed CQ algorithm (**RCQ**) in [13] applied to (P). Let $\gamma = 0.99M$ where $M = \min\{2/\|A\|^2, \sqrt{2}/\|A\|\}$. In Table 1 we display the iteration history when using three different values of ϵ and for each ϵ four different start vectors (chosen at random). We see from Table 1 that none of the two tested algorithms come out as the best. We next use three different values of the stepsize γ. For each test the same start vector was used. The results, Table 2, show that 'Iter.' decreases slowly as the stepsize approaches M.

Secondly, we compare the modified variable-step relaxed projection 4.1 (**MVRCQ**) with variable-step CQ algorithm (**VRCQ**) in [14] applied to (P). Let $\gamma = 0.5, l =$

Table 3 MVRCQ and VRCQ applied for (P)

ϵ	MVRCQ				VRCQ			
	Iter.	InIt.	$C(x)$	$Q(Ax)$	Iter.	InIt.	C(x)	Q(Ax)
0.01	617	5057	−0.116069	2.22E−02	618	5066	−0.118461	2.24E−02
0.01	552	4407	−0.144963	2.09E−02	555	4432	−0.150549	2.08E−02
0.01	812	6805	−0.585519	2.53E−02	814	6821	−0.591187	2.50E−02
0.01	677	5581	−0.174444	2.36E−02	679	5599	−0.177482	2.34E−02
0.001	553	4506	−0.300212	1.99E−03	554	4515	−0.302575	2.00E−03
0.001	1036	8723	−0.301105	2.11E−03	1038	8740	−0.304699	2.12E−03
0.001	637	4866	−4.47E−02	2.04E−03	643	4920	−4.95E−02	2.03E−03
0.001	1000	8322	−0.172833	2.18E−03	1002	8340	−0.175848	2.19E−03
0.0001	999	7984	−0.239796	2.17E−04	1001	8000	−0.242994	2.17E−04
0.0001	1187	10133	−0.111612	2.05E−04	1188	10141	−0.114309	2.06E−04
0.0001	1030	8281	−0.425143	2.25E−04	1031	8287	−0.427735	2.27E−04
0.0001	928	7344	−0.191356	1.97E−04	931	7369	−0.194958	1.97E−04

0.4, $\mu = 0.7$. From Table 3, we first note that 'Iter.' and 'InIt.' are slightly smaller for (**MVRCQ**) than that of (**VRCQ**); secondly we see that the iteration is not sensitive to the tolerance. It is also observed that the numbers of steps needed in Table 3 are larger than that of Table 2. This phenomenon might be caused by the choice of β_n in (23) which maybe be much smaller than M. This issue is left for future studies.

6 Concluding remarks

In this paper, a modified relaxed projection algorithm and a modified variable-step relaxed projection algorithm with Armijo-like searches for solving the split feasibility problem have been presented. For the two algorithms, the orthogonal projections can be calculated directly. The algorithm 4.1 does not need to compute or estimate the norm of the matrix and the stepsize can be computed using a number of steps of the power method. Moreover, the objective function can decrease significantly at each iteration in these algorithms. In the feasible case of the SFP, the corresponding convergence properties have been established. We perform some numerical experiments, which have confirmed the theoretical results obtained.

Acknowledgments The authors express their thanks to the reviewers, whose constructive suggestions led to improvements in the presentation of the results.

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