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ORIGINAL PAPER

A column generation-based heuristic algorithm for an inventory routing problem with perishable goods

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Abstract An inventory routing problem is a variation of the vehicle routing problem in which inventory and routing decisions are determined simultaneously over a given time horizon. The objective is to minimize the sum of transportation and inventory costs. In this paper, we study a specific inventory routing problem in which goods are perishable (PIRP). We develop a mathematical model for PIRP and exploit its structure to develop a column generation-based solution approach. Cutting planes are added to improve the formulation. We present computational experiments to demonstrate that our methodology is effective, and that the integration of routing and inventory can yield significant cost savings.

Keywords Inventory routing · Perishable goods · Column generation · Vehicle routing · Integer programming

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1 Introduction

In many distribution systems, suppliers must determine when and how much to deliver to their customers over a finite planning horizon. Representative examples are the deliveries of goods to customers in Vendor Managed Inventory (VMI) systems [12] and the deliveries of air products [5]. In such examples, replenishment quantities of goods, delivery consolidations and days to visit customers must be determined concurrently. Industrial gas companies pioneered the use of the IRP model in their delivery problems and have benefited greatly from the integration of inventory and routing decisions [7]. Such practical applications have motivated the development of the inventory routing problem (IRP) along with its variants.

IRPs are a generalization of the classical vehicle routing problem (VRP) [26]. While VRPs deal only with routing decisions, IRPs combine routing and inventory decisions in a single model. In a classical VRP, customers place orders on a given day and the delivery company assigns the orders to routes for its fleet, whereas in a classical IRP, there are no customer orders—instead the delivery company decides the delivery quantity and timing for each customer over a planning horizon [7].

IRP is known to be a difficult combinatorial optimization problem. The cost savings from IRP models comes at the expense of an increase in the complexity of the solution approach. The main challenge resides in the inventory balance constraints, which link routing decisions and quantities of goods shipped to customers over a given planning horizon.

In this research, we consider a deterministic multi-period IRP for perishable goods (PIRP) in which customer demands for each time period are given. The objective of PIRP is to determine: (1) when to deliver to each customer; (2) how much to deliver to each customer in each time period; and (3) how to route vehicles such that the sum of transportation and inventory costs is minimized while still meeting customer demands and perishability constraints. In our model, perishable goods are assumed to have a fixed shelf-life, and will be discarded at the end of their shelf-life. Consequently, quantities of goods delivered to customers at any given time are limited not only by the holding capacity at the customer's site but also by the shelf-life of the goods. A detailed description of the PIRP model used in our research is given in Sect. 3.

This work was motivated by operational problems in food distribution in the Academic Model for the Prevention and Treatment of HIV (AMPATH) program, a partnership between the medical school of Indian University and Moi University in Kenya. The AMPATH nutrition program currently provides food support for more than 30,000 HIV-infected patients, and the main function of the food distribution system is to transport fresh foods from production farms and dry foods from warehouses to distribution sites [22].

Our research explores modeling issues and solution approaches for IRP models with perishable goods. The main contributions of this research are:

1. We formulate the path flow formulation for the multi-period IRP with perishability constraints. In order to solve the LP relaxation of the formulation by the column generation approach, we introduce a method to derive the associated pricing problem.

2. We show that the lower bound from the linear relaxation of PIRP is not as strong as that obtained in the column generation literature for vehicle routing problems.
3. We derive three valid inequalities to strengthen the PIRP formulation, and apply these valid inequalities in our column generation approach. Computational experiments show that good solutions to PIRP are obtained in reasonable time. As a result of solving the LP relaxation by column generation, we obtain a strong lower bound on the optimal solution to PIRP.

The remainder of this paper is organized as follows. Section 3 formulates PIRP for perishable good deliveries. In Sect. 4, a reduced dual model is presented, and a column generation based algorithm is designed from this reduced model. Numerical experiments are provided in Sect. 5, and concluding remarks follow in Sect. 6.

2 Literature review

Bell et al. [5] were among the first to apply IRP to a gas distribution problem; their paper won the first Franz Edelman Award. They developed real-time computerized software to determine routing schedules based on a multi-period, combined inventory control/vehicle scheduling model. The software is capable of handling problems with up to 800,000 variables and 200,000 constraints, and uses a lagrangian relaxation algorithm to solve these large scale problems to proven near optimality as explained in Fisher [15]. Their software has been utilized by several companies and has reportedly achieved cost savings of between 6 and 10 % of operational costs. Gaur and Fisher [16] have developed a multi-period IRP model for Albert Heijn, a leading supermarket chain in the Netherlands, and have claimed a 4 % savings in distribution costs in the first year and a potential savings of 12–20 % in the following years. Beyond its application to inventory routing for land vehicles, IRP has been used to schedule large volume vessel shipments in marine transportation, where it is known as inventory marine routing [8,9,24].

As IRP is a difficult problem, solution approaches in the literature are typically heuristic [7]. Dror and Ball [13] were among the first to consider a heuristic approach to solving the multi-period IRP. They managed to reduce the annual distribution problem to a single-period problem. Using the probability of stock-outs at the customers, they developed a set of rules to assign vehicle routes and replenishment quantities. Bertazzi et al. [6] studied a multi-period, multi-product IRP with deterministic demand and an order-up-to-level policy. They proposed a two-stage heuristic algorithm to solve this problem based on constructive customer allocation and customer exchange heuristics. Archetti et al. [2] introduced an exact algorithm for IRP. They proposed an arc flow formulation for the problem and developed a branch and cut algorithm to solve it. However, their algorithm is limited to the case of a single vehicle and order-up-to-maximum inventory level policies. A detailed literature review of the solution approaches for IRP can be found in [1].

Column generation approaches have successfully been used to solve many large scale hard integer programming problems such as VRP with time windows [11]. However, to the best of our knowledge, research on column generation approaches for IRPs is very limited. Gronhaug et al. [19] implemented a column generation approach for the marine inventory routing problem. However, since their formulation and constraints are customized for the marine transportation, their approach is not applicable to the multi-period IRP for perishable goods.

The literature on the delivery of perishable goods with the consideration of inventory costs is scant. Federgruen et al. [14] were among the first to integrate transportation and inventory models for perishable goods. The model they proposed is formulated as a complex, nonlinear, mixed integer program with the objective of minimizing the sum of transportation, shortage and out-of-date costs. To solve this model, they introduced a heuristic algorithm based on interchanges. However, as they only considered a single period problem, this study does not apply to situations in which a delivery covers multiple-period demands.

In another line of research, some studies have proposed extensions of the economic order quantity (EOQ) policy for inventory models with perishable goods. Giri and Chaudhuri [18] presented an extended EOQ inventory model for a perishable product in which the demand rate is a function of the on-hand inventory and the holding cost is a non-linear stock dependent function for an infinite planning horizon. Panda et al. [23] presented a single-item order level inventory model for a seasonal product over a finite planning horizon in which the demand rate was modeled as a ramp-type time dependent function. Tarantilis and Kiranoudis [25] modeled a fresh milk distribution system in Greece as a heterogeneous fixed fleet vehicle routing problem and proposed a threshold-based algorithm to solve it. Hsu et al. [20] has extended the conventional vehicle routing problem with time windows to incorporate the randomness of food spoilage during the delivery process. However, neither EOQ models nor VRP approaches support the combination of inventory and transportation components.

3 An inventory routing problem with perishable goods

We consider a distribution problem involving a depot, a set of customers and a homogeneous fleet of capacitated vehicles. Perishable goods are transported from the depot to customers in such a way that out-of-stock situations never occur. In this problem, customer demands in each time period are deterministic, but may vary from one period to the next. Inventory holding costs are incurred when goods are stored at customer's sites. Like other multi-period inventory problems, PIRP assumes deliveries arrive at customers at the beginning of time periods.

Our research focuses on perishable goods with a fixed lifetime, for example goods with expiration dates such as medications. Such goods have a value that stays essentially constant for a fixed amount of time and then drops to zero. In our study, perishable goods have a fixed shelf-life that corresponds to the number of time periods over which goods can stay in good condition at the customers' sites. When unused goods have exceeded their shelf-life, they will be discarded. Although our model is designed for perishable goods with a fixed lifetime, it can be extended to perishable goods that decrease their value gradually throughout their lifetime. In this case, the cost of quality loss of goods over a time period can be included in the model like inventory costs.

Our PIRP model is similar to standard IRP. The largest difference between PIRP and standard IRP is the way in which the upper bound inventory levels of customers at the end of each time period are defined. While the upper bound inventory levels in standard IRP depend only on the physical storage capacity at the customer's site, the upper bound is restricted by both perishability and the physical storage capacity in PIRP. In our research, perishability is assumed to dominate the physical storage capacity. As a result, the upper bound inventory levels of customers in PIRP are determined only by the perishability constraints.

We define a direct route in PIRP as a route in which a vehicle starts from the depot, visits one customer and then returns to the depot. In case the demand of a customer in any time period is greater than the vehicle capacity, we transform the demand so that it is less than the vehicle capacity by using full-truckload direct routes and modifying the demand d_{it} to the remaining partial load $(= (d_{it} - C\lfloor d_{it}/C \rfloor))$. Consequently, we only consider situations in which the customer demand per time period is less than the vehicle capacity. Also, we make two additional assumptions: (1) Vehicles travel at most one route in any time period; (2) Customers have at most one delivery per time period. The first assumption is very common in other transportation problems such as VRP, while the second assumption does not allow split deliveries in any time period.

We now propose a mathematical model for PIRP using the concept of a feasible route. A feasible route is a route that starts from the depot, visits a subset of customers at most one time and then returns to the depot. Note that this is different from the popular notion of a feasible route in VRP, where a feasible route is defined as a subtour for which the sum of demands of customers on the route are less than the vehicle capacity.

We introduce the following notation

N Set of customers $N = 1, \ldots, |N|$.
V Set of nodes $V = \{0\} \cup N$, where node 0 represents the depot.
T Set of time periods $T = 1, \ldots, |T|$.
K Set of homogeneous vehicles $K = 1, \ldots, |K|$.
C Vehicle capacity.
R Set of all feasible routes.
$R^{'}$ Set of feasible routes in Restricted Master Problem (RMP).
τ_{max} Maximum shelf-life.
d_{it} Demand of customer $i \in N$ in time period $t = 1, \ldots, T, \ldots, T + \tau_{max} - 1$.
u_{it} Upper bound inventory level at customer $i \in N$ in time period $t \in T$, $u_{it} = \left(\sum_{\substack{t < \tau \\ \tau \le t + \tau_{max}}} d_{i\tau} \right)$.
d_{it}^{max} Maximum quantity delivered to customer $i \in N$ in time period $t \in T$, $d_{it}^{max} = u_{it} + d_{it}$.
d_{tl}^{i} Demand of customer $i \in N$ from time period t to l.
h_{it} Inventory holding cost of customer $i \in N$ in time period t.
I_{i0} Inventory level at customer $i \in N$ at the beginning of time period $t = 1$.
$$\alpha_{ir} = \begin{cases} 1 & \text{if route } r \in R \text{ visits customer } i \in N \\ 0 & \text{otherwise} \end{cases}$$
c_{ij} Transportation cost from node $i \in V$ to node $j \in V$.
c_r Transportation cost of route $r \in R$.

The decision variables of the model are defined as follows:
I_{it} = Inventory level of customer $i \in N$ at the end of time period $t \in T$.

$$\theta_{rt} = \begin{cases} 1 & \text{if route } r \text{ is selected in time period } t \in T \\ 0 & \text{otherwise} \end{cases}$$

a_{irt} = Quantity delivered to customer $i \in N$ by route $r \in R$ in time period $t \in T$.

The optimal solution of PIRP is obtained by solving the following mixed integer programming model:

$$Z^* = \min \sum_{t \in T} \left(\sum_{r \in R} c_r \theta_{rt} + \sum_{i \in N} h_{it} I_{it} \right) \tag{1}$$

Subject to

$$\sum_{r \in R} \alpha_{ir} \theta_{rt} \le 1 \qquad \forall i \in N, t \in T \qquad (\varphi) \tag{2}$$

$$\sum_{i \in N} \alpha_{ir} a_{irt} \le C\theta_{rt} \qquad \forall r \in R, t \in T \qquad (\mu) \tag{3}$$

$$I_{it-1} + \sum_{r \in R} \alpha_{ir} a_{irt} = d_{it} + I_{it} \qquad \forall i \in N, t \in T \qquad (\pi) \tag{4}$$

$$I_{it} \le u_{it} \qquad \forall i \in N, t \in T \qquad (\omega) \tag{5}$$

$$\sum_{r \in R} \theta_{rt} \le |K| \qquad \forall t \in T \qquad (\upsilon) \tag{6}$$

$$\theta_{rt} \in \{0, 1\} \qquad \forall r \in R, t \in T \tag{7}$$

$$a_{irt}, I_{it} \ge 0 \qquad \forall i \in N, r \in R, t \in T \tag{8}$$

The objective function (1) represents the minimization of the sum of transportation cost and inventory cost. The first term in the objective function represents the transportation cost while the second term represents inventory holding cost. Constraints (2) ensure that a customer is visited at most once per time period. Constraints (3) require that the vehicle capacity be respected. Constraints (4) are inventory balance equations that relate customer demands, incoming deliveries and inventories. Constraints (5) guarantee that a customer never has an inventory level that is greater than the total demands in the next $(\tau_{max} - 1)$ consecutive time periods. This, in turn, imposes the constraint that perishable goods will never be discarded. Constraints (6) require the maximum number of different routes selected in a time period to be less than the number of vehicles. Lastly, Constraints (7) require the variables θ_{rt} to be binary and Constraints (8) require that inventory levels and quantities of goods delivered to customers be non-negative. In our model, non-negative inventory levels guarantee that no stock-outs occur at any customer during the planning time horizon.

The PIRP formulation above represents a very large integer programming problem, since the number of feasible routes increases exponentially with the number of customers. Therefore, it becomes difficult to solve optimally even in the case of relatively small problems. However, the advantage of the PIRP formulation is that its LP relaxation typically provides a good lower bound on the optimal value of the problem, so that we can often use the optimal solution of its LP relaxation to find a good integer solution.

4 Algorithm

In this section, we propose a heuristic algorithm which can obtain a good solution for PIRP. Since the column generation method is an important component of this algorithm, we call it the column generation-based heuristic algorithm. By relaxing the binary variables in the PIRP formulation to continuous variables between 0 and 1, we obtain its LP relaxation, denoted as $LP(PIRP)$. Because Constraints (2) of the PIRP formulation indirectly specify that $\theta_{rt} \leq 1$, it is not necessary to include $\theta_{rt} \leq 1$ in $LP(PIRP)$.

4.1 Column generation approach

The main difficulty with solving $LP(PIRP)$ is that its size is very large. To overcome this, we use the column generation method. The general idea of this method is to solve $LP(PIRP)$ with a small but meaningful subset of the feasible routes, forming a so-called Restricted Master Problem (RMP). The optimal dual values of RMP are used to determine whether there exist other feasible routes that can reduce the objective value. If potentially improving routes exist, we add them to RMP and re-solve $LP(PIRP)$. This process is repeated until the optimal solution of $LP(PIRP)$ is found [11].

In order to formulate the pricing problem used to generate the potentially improving routes in the column generation process, we study the dual problem of $LP(PIRP)$. The dual is as follows:

$$(D) \quad \max \sum_{i\in N}\sum_{t\in T}(d_{it}\pi_{it} - \varphi_{it} - u_{it}\omega_{it}) - |K|\sum_{t\in T}\upsilon_t - \sum_{i\in N} I_{i0}\pi_{i1} \tag{9}$$

Subject to

$$C\mu_{rt} - \sum_{i\in N}\alpha_{ir}\varphi_{it} - \upsilon_t \leq c_r \qquad \forall r \in R, t \in T \tag{10}$$

$$\alpha_{ir}\pi_{it} - \alpha_{ir}\mu_{rt} \leq 0 \qquad \forall r \in R, i \in N, t \in T \tag{11}$$

$$\begin{aligned} -\pi_{it} + \pi_{it+1} - \omega_{it} &\leq h_{it} \qquad \forall i \in N, t < |T| - 1 \\ -\pi_{it} - \omega_{it} &\leq h_{it} \qquad \forall i \in N, t = |T| \end{aligned} \tag{12}$$

$$\mu_{rt},\ \varphi_{it},\ \omega_{it},\ \upsilon_t \geq 0 \qquad \forall i \in N, t \in T, r \in R \tag{13}$$

In the above linear programming problem, Constraints (10), (11) and (12) correspond to the primal variables θ_{rt}, a_{irt} and I_{it}, respectively. The dual problem (D)

specifies that when a route is added to RMP, new variables and new constraints must be added to RMP. Consequently, setting up the pricing problem for the PIRP formulation is not straightforward. In general, combining column generation and row generation is a complicated process. Fortunately, in this situation, the dual problem can be simplified so that variables μ_{rt} are not included, a step that makes solving $LP(PIRP)$ by column generation much easier. In particular, we claim that the following reduced dual problem is equivalent to (D).

$$(RD) \quad \max \sum_{i\in N}\sum_{t\in T}(d_{it}\pi_{it} - \varphi_{it} - u_{it}\omega_{it}) - |K|\sum_{t\in T}\upsilon_t - \sum_{i\in N} I_{i0}\pi_{i1} \tag{14}$$

Subject to

$$C\max_{i\in r}(0,\pi_{it}) - \sum_{i\in N}\alpha_{ir}\varphi_{it} - \upsilon_t \le c_r \qquad \forall r \in R, t\in T \tag{15}$$

$$-\pi_{it} + \pi_{it+1} - \omega_{it} \le h_{it} \qquad \forall i \in N, t = 1,\ldots,|T|-1$$

$$-\pi_{it} - \omega_{it} \le h_{it} \qquad \forall i \in N, t = |T| \tag{16}$$

$$\varphi_{it},\ \omega_{it},\ \upsilon_t \ge 0 \qquad \forall i \in N, t \in T \tag{17}$$

In (RD), the constraints involving variables μ_{rt} are eliminated. This projection step introduces Constraints (15), which are a combination of Constraints (10) and (11) in (D).

Proposition 1 (RD) *has the following properties:*

1. *If* $(\varphi^*,\ \pi^*,\ \omega^*,\ \upsilon^*)$ *is an optimal solution of* (RD)*, then there exists* μ^* *such that* $(\varphi^*,\ \pi^*,\ \omega^*,\ \upsilon^*,\ \mu^*)$ *is an optimal solution of* (D).
2. *If* $(\varphi^*,\ \pi^*,\ \omega^*,\ \upsilon^*,\ \mu^*)$ *is an optimal solution of* (D)*, then* $(\varphi^*,\ \pi^*,\ \omega^*,\ \upsilon^*)$ *is an optimal solution of* (RD).
3. $Z^*_D = Z^*_{RD}$ *where* $Z^*_D,\ Z^*_{RD}$ *are the optimal objective values of* (D) *and* (RD)*, respectively.*

Proof In the system of the linear inequalities of (D), we have the following inequalities:

$$0 \le \mu_{rt} \qquad \forall\, r,\ t$$

$$\pi_{it} \le \mu_{rt} \qquad \forall\, i,\ r,\ t \text{ if } \alpha_{ir} = 1$$

$$\frac{\sum_{i\in N}\alpha_{ir}\varphi_{it} + \upsilon_t + c_r}{C} \ge \mu_{rt} \qquad \forall\, r,\ t$$

As a result of Fourier–Motzkin elimination, the above system of inequalities are equivalent to inequalities $\max_{i\in r}\{0,\pi_{it}\} \le \frac{\sum_{i\in N}\alpha_{ir}\varphi_{it}+\upsilon_t+c_r}{C}$ $\forall r,\ t$. This implies that the linear inequalities of (D) and (RD) are equivalent. Since (D) and (RD) have the same objective function, we obtain Proposition 1.1 and 1.2. Proposition 1.3 is a direct result of Propositions 1.1 and 1.2.

Proposition 1 implies that (RD) and (D) are equivalent. Because constraints are added in (RD) only when a new route r is introduced into RMP, we exploit (RD) to construct the pricing problem. In particular, let $(\hat{\varphi}, \hat{\mu}, \hat{\pi}, \hat{\omega}, \hat{\upsilon})$ denote a dual optimal solution of RMP. If $(\hat{\varphi}, \hat{\mu}, \hat{\pi}, \hat{\omega}, \hat{\upsilon})$ satisfies Constraints (15), (16) and (17), solving RMP provides an optimal solution of $LP(PIRP)$. Otherwise, there must exists one route r and one time period $t \in T$ such that $C \max_{i \in r} (0, \hat{\pi}_{it}) - \sum_{i \in r} \hat{\varphi}_{it} - \hat{\upsilon}_t > c_r$. The pricing problem is to find one route r in one time period $t \in T$ that achieves the optimal value of the following:

$$c^* = \min_{r \in R,\, t \in T} \left\{ c_r + \sum_{i \in r} \hat{\varphi}_{it} - C \max_{i \in r} (0, \hat{\pi}_{it}) + \hat{\upsilon}_t \right\} \tag{18}$$

Next, we show in Proposition 2 that the lower bound of the PIRP formulation obtained by solving $LP(PIRP)$ is not as strong as that which appears in the column generation literature for VRP. This lower bound is obtained as the optimal objective value of RMP over the set of direct routes.

Proposition 2 *When transportation costs satisfy the triangle inequality, solving RMP with the set of all direct routes gives us an optimal solution of* $LP(PIRP)$.

Proof Let $DR = \{(0, i, 0) : i \in N\}$ denote the set of all direct routes, where node 0 represents the depot. We claim that $c^* \geq 0$ where c^* is the optimal objective value of the pricing problem (18) when the subset of feasible routes $\bar{R}$ of RMP is equal to DR.

Consider a fixed time t. Given a route $r \in R$, we define $k = \operatorname{argmax}_{i \in r}(\hat{\pi}_{it})$ and consider r_k to be a direct route from the depot to customer k. Since $c_r \geq c_{r_k}$ by the triangle inequality, $\varphi_{it} \geq 0$, and $\max_{i \in r} (0, \hat{\pi}_{it}) = \max (0, \hat{\pi}_{kt})$ by definition, we obtain that $c_r + \sum_{i \in r} \hat{\varphi}_{it} - C \max_{i \in r} (0, \hat{\pi}_{it}) + \hat{\upsilon}_t \geq c_{r_k} + \hat{\varphi}_{kt} - C \max (0, \hat{\pi}_{kt}) + \hat{\upsilon}_t$. Thus if $r_k \in \bar{R}$, we conclude that $c_{r_k} + \hat{\varphi}_{kt} - C \max (0, \hat{\pi}_{kt}) + \hat{\upsilon}_t \geq 0$ and therefore $c_r + \sum_{i \in r} \hat{\varphi}_{it} - C \max_{i \in r} (0, \hat{\pi}_{it}) + \hat{\upsilon}_t \geq 0 \quad \forall r \in R$. This implies that $c^* \geq 0$. We conclude that RMP yields an optimal solution of $LP(PIRP)$ when $\bar{R} = DR$.

4.2 Strengthening the PIRP formulation

We now introduce the following notation:

x_{it}: Number of visits to customer $i \in N$ in time period $t \in T$, $x_{it} = \sum_{r \in R} \alpha_{ir} \theta_{rt}$.
y_{it}: Quantity to deliver to customer $i \in N$ in time period $t \in T$, $y_{it} = \sum_{r \in R} \alpha_{ir} a_{irt}$.

Proposition 2 implies that we must strengthen the PIRP formulation to yield a better lower bound. Naturally, we can tighten the PIRP formulation by using the following valid inequalities:

1. There must be at least one delivery to a customer during τ_{max} time periods. Therefore, the following inequalities are valid for the PIRP formulation:

$$\sum_{t \leq \tau < t + \tau_{max}} x_{i\tau} \geq 1 \; \forall i \in N, t \leq |T| - \tau_{max} + 1 \tag{19}$$

2. It is impossible for a vehicle to deliver more than the total demand of the customers it visits during τ_{max} consecutive time periods. Therefore, Constraints (3) can be strengthened as follows:

$$\sum_{i \in N} \alpha_{ir} a_{irt} \leq \min\left\{C, \sum_{i \in N} \alpha_{ir} d_{it}^{max}\right\} \theta_{rt} \qquad \forall r \in R, t \in T \tag{20}$$

Next, we derive three families of valid inequalities for the PIRP formulation and show that valid inequalities (19) and (20) are dominated by these families.

Proposition 3 *The inequalities*

$$d_{it}\theta_{rt} + I_{it} \geq \alpha_{ir} a_{irt} \qquad \forall r \in R\,, i \in N, t \in T \tag{21}$$

are valid for the PIRP formulation.

Proof We know that $I_{it} \geq 0 \quad \forall i \in N$ and $\forall t \in T$. If $\theta_{rt} = 0$, then $a_{irt} = 0 \quad \forall i \in r$. Therefore, $d_{it}\theta_{rt} + I_{it} \geq a_{irt} \quad \forall r \in R\,, i \in r, t \in T$. If $\theta_{rt} = 1$, then $d_{it}\theta_{rt} + I_{it} = d_{it} + I_{it} = \sum_{r \in R} \alpha_{ir} a_{irt} + I_{it-1} \geq \sum_{r \in R} \alpha_{ir} a_{irt} \geq \alpha_{ir} a_{irt}$.

Since $d_{it}\theta_{rt} + I_{it} \leq d_{it} + u_{it} = d_{it}^{max}$, we obtain $a_{irt} \leq d_{it}^{max}\ \forall i \in N,\ \forall r \in R,\ \forall t \in T$ and $\sum_{i \in N} \alpha_{ir} a_{irt} \leq \sum_{i \in N} d_{it}^{max}$. Therefore, it follows that valid inequalities (21) and Constraints (3) dominate inequalities (20).

Proposition 4 *For any $l \in T,\ L = \{1, \ldots, l\},\ S \subseteq L$, the inequalities*

$$\sum_{t \in S} y_{it} \leq \sum_{t \in S} d_{tl}^{i} x_{it} + I_{it} \tag{22}$$

are valid for the PIRP formulation.

Proof Inequalities (22) are derived directly from *(l, S)* inequality [4] when the PIRP formulation is relaxed to the uncapacitated lot sizing problem.

Observing that upper inventory levels at the customer sites are limited by the perishability constraints, we develop the third family of valid inequalities in Proposition 5. It is easy to verify that valid inequalities (23) are stronger than inequalities (20).

Proposition 5 *The inequalities*

$$\begin{aligned} I_{ik-1} + \sum_{t \in S} y_{it} \leq u_{ik-1} + \sum_{t \in S} \min\left\{d_{kt}^{i} + u_{it} - u_{ik-1}, d_{kl}^{i} - u_{ik-1}, d_{tl}^{i}\right\} x_{it} \\ + I_{it} \forall k,\ l \in T,\ l \geq k + \tau_{max} - 2,\ S \subseteq [k, l], i \in N \end{aligned} \tag{23}$$

are valid for the PIRP formulation.

Proof Inequalities (23) are derived from the valid inequalities of the lot sizing problem with inventory bounds [3] when the PIRP formulation is relaxed to the lot sizing problem with inventory bounds.

A result similar to Proposition 1 can be proved when introducing Constraints (22) and (23) to the PIRP formulation. However, the coefficients of pricing problem (18) are changed to reflect the addition of new dual variables associated with such constraints. The general form of the pricing problem that results when these valid inequalities are added is as follows:

$$c^* = \min_{r \in R,\, t \in T} \left\{ c_r - \sum_{i \in r} \Phi_{it} - \max_{i \in r} (0, \Pi_{it}) + \Upsilon_t \right\} \tag{24}$$

where Φ_{it}, Π_{it} and Υ_t are functions of the optimal dual variables of $LP(PIRP)$ (See Appendix A). This form of the pricing problem is equivalent to the following minimization problem:

$$c^* = \min_{t \in T} \left\{ c_t^* + \Upsilon_t \right\} \text{ where } c_t^* = \min_{r \in R, t \in T} \left\{ c_r - \sum_{i \in r} \Phi_{it} - \max_{i \in r} (0, \Pi_{it}) \right\}$$

The problem of finding c_t^* reduces to a profitable tour problem (PTP) with profit Φ_{it} at customer i when $\Pi_{it} = 0$ for all $i \in r$ [10]. Therefore, pricing problem (24) is at least as difficult as PTP. In addition, since PTP is a generalization of TSP, the general form of the pricing problem is an NP-hard problem.

Now we discuss separation algorithms to find the inequalities (21), (22) and (23) that violate a fractional optimal solution of $LP(PIRP)$. Because constraints (21) are satisfied whenever $\theta_{rt} = 0$, the associated separation problem is equivalent to finding violated constraints with $\theta_{rt} > 0$. Since the number of routes with $\theta_{rt} > 0$ is small, the separation algorithm can be based on an enumeration procedure. For constraints (22) and (23), we adopt the polynomial-time separation algorithms in [4] and [3].

4.3 Implementation

The column generation-based heuristic algorithm consists of two steps. In the first step, $LP(PIRP)$ with valid inequalities (22) and (23) is solved optimally by the column generation method. The valid inequalities (21) are only added to $LP(PIRP)$ when we do no use the column generation method to solve LP relaxation. The valid inequalities that cut off the current fractional optimal solution of $LP(PIRP)$ are added repeatedly in the LP relaxation until no violated inequalities are found. The second step tries to find a good solution (upper bound) for PIRP by using CPLEX to solve the PIRP formulation with $\bar{R}$, where $\bar{R}$ is a subset of feasible routes in RMP. Because the number of routes in $\bar{R}$ is small, the computational workload is typically tractable. The heuristic algorithm described above was implemented in C++ using ILOG Concert 2 and the CPLEX 10.1 solver [21].

In this implementation, we consider two strategies for the addition of new columns to RMP: (1) adding a route with the most negative reduced cost to $\bar{R}$ as in Dantzig's rule; (2) adding the most negative route in each time period to $\bar{R}$. Our preliminary computational experiments showed that under option (2) RMP converges to an optimal solution of $LP(PIRP)$ faster. Therefore, option (2) was selected in our algorithm.

The initial subset of feasible routes of RMP should be able to produce a feasible solution for PIRP. Otherwise, the valid inequalities added to PIRP may cause $LP(PIRP)$ to be infeasible. In order to avoid this problem, we must develop a heuristic algorithm to generate a feasible solution for PIRP and employ this solution to initialize $\bar{R}$. The details of the heuristic algorithm are presented in Appendix B. Note that the main focus of the heuristic algorithm is to find a feasible solution quickly. We observe that while the objective value of the initial feasible solution provided by this algorithm is far from the optimal objective value of PIRP, this does not cause a problem because the initial solution does not have much impact on the final result produced by the column generation-based heuristic algorithm.

5 Computational experiments

In this section, we perform a numerical evaluation of the column generation-based heuristic algorithm. The data used in our computational experiments is generated similarly to that in [6]. However, we have to introduce or modify some parameters to fit PIRP. In particular, the data was generated as follows:

- Number of customers: 8, 30, 40 and 50.
- Demand of a customer per time period: random integer in the interval [10, 100].
- Transportation cost per distance unit p: random number in the interval [0.1, 0.2].
- Location (X_i, Y_i) of customer i: random integer numbers in the interval [0, 500].
- Transportation cost: $c_{ij} = p\left(\sqrt{\left(X_i - X_j\right)^2 + \left(Y_i - Y_j\right)^2}\right)$.
- Inventory cost for customer i: random number in the interval [4.6, 5].
- Capacity of the fleet of vehicles: 110 % of total demand of customers, which is $\sum_{i \in N,\, t \in T} d_{it}$.
- Number of vehicles: $\left\lceil \frac{1.1}{C} \left\{\sum_{i \in N,\, t \in T} d_{it}\right\}\right\rceil$.
- Beginning inventory I_{i0}: random integer in the interval [0, $d_{i1} + d_{i2}$].
- Number of time periods: 5.

In all situations, the random data was generated in accordance to a uniform distribution. Ten datasets were generated randomly for the cases of 8 customers, 30 customers, 40 customers and 50 customers, resulting in 40 datasets overall. These datasets were combined with the following parameters, resulting in a total of 320 instances for experimental studies.

- Shelf-life τ_{max}: 2 or 3.
- Vehicle capacity C: d_{max} or $1.5d_{max}$ where $d_{max} = \max\{d_{it} : t \in T, i \in N\}$.

– Objective function of PIRP: (1) Both transportation cost and inventory cost or (2) Transportation cost only.

In the AMPATH project, goods are delivered weekly. Therefore, we use five working days as a planning time horizon. Other problems may require a planning time horizon that is either longer or shorter than this.

5.1 The effectiveness of valid inequalities

Regarding the effectiveness of the three valid inequalities when adding them to the algorithms, we observe an improvement in the integrality gaps due to each valid inequality for problems with 8 customers. A summary of these experiments is reported in Tables 1 and 2. In the third column of these tables, we report the integrality gaps when no valid inequalities have been added to the LP relaxation, which is $100 \times \left(Z^* - Z^{LB}\right)/Z^*$ where Z^* is the objective value of the best integer solution and Z^{LB} is the objective value of the original LP relaxation ($LP\ (PIRP)$). In the fourth column, we present the improvement of the integrality gaps when strengthening the

Table 1 Computational results for problems with 8 customers and shelf-life = 2

Dataset	C	$LPGap$ (%)	Gap improvement				Number of cuts			
			Fal1	Fal2	Fal3	AllFals	Fal1	Fal2	Fal3	AllFals
1	d_{max}	12.66	2.86	1.54	3.40	3.61	502	52	93	647
	$1.5d_{max}$	18.97	2.77	1.76	3.64	3.75	501	74	131	706
2	d_{max}	12.66	4.06	1.86	4.04	4.63	280	54	87	421
	$1.5d_{max}$	20.01	3.79	1.84	5.27	5.35	505	77	162	744
3	d_{max}	13.97	3.08	1.54	4.90	6.03	503	57	102	662
	$1.5d_{max}$	22.26	2.53	1.55	4.48	5.16	505	76	163	744
4	d_{max}	21.97	4.95	1.99	5.79	7.82	501	61	115	677
	$1.5d_{max}$	37.80	3.55	2.02	6.21	6.79	505	84	156	744
5	d_{max}	13.54	3.28	1.49	3.66	3.78	501	56	97	654
	$1.5d_{max}$	22.82	3.58	1.73	5.23	5.82	505	83	161	749
6	d_{max}	15.22	3.09	1.43	4.36	4.80	502	53	85	640
	$1.5d_{max}$	23.21	2.98	1.61	6.10	6.98	505	75	164	744
7	d_{max}	18.39	3.82	1.74	3.92	3.84	451	51	85	587
	$1.5d_{max}$	24.81	4.98	2.12	6.43	6.49	505	75	162	742
8	d_{max}	16.37	3.55	1.78	3.69	4.26	276	53	91	420
	$1.5d_{max}$	22.24	4.31	1.95	5.05	5.10	502	81	160	743
9	d_{max}	17.34	3.88	1.71	4.83	5.54	502	52	98	652
	$1.5d_{max}$	24.21	3.14	1.90	4.27	4.57	505	83	172	760
10	d_{max}	20.36	4.18	1.86	5.06	5.81	502	58	116	676
	$1.5d_{max}$	28.17	3.20	2.04	6.31	6.94	505	89	163	757

Fal1 Family 1, *Fal2* Family 2, *Fal3* Family 3, *AllFals* all Families

Table 2 Computational results for problems with 8 customers and shelf-life = 3

Dataset	C	$LPGap$ (%)	Gap improvement				Number of cuts			
			Fal1	Fal2	Fal3	AllFals	Fal1	Fal2	Fal3	AllFals
1	d_{max}	13.31	3.52	1.65	2.55	4.08	305	52	51	408
	$1.5d_{max}$	21.32	3.03	1.75	3.07	3.93	501	74	83	658
2	d_{max}	14.10	4.55	1.90	2.58	5.07	505	54	49	608
	$1.5d_{max}$	24.10	4.17	2.04	2.93	4.82	505	77	85	667
3	d_{max}	14.73	2.96	1.59	2.40	3.67	503	56	56	615
	$1.5d_{max}$	22.41	3.81	1.76	1.15	6.26	505	76	80	661
4	d_{max}	24.69	4.27	2.04	2.73	4.95	481	61	57	599
	$1.5d_{max}$	38.26	4.06	2.20	4.28	7.56	485	84	82	651
5	d_{max}	13.77	3.60	1.53	2.30	4.03	503	56	51	610
	$1.5d_{max}$	24.50	3.63	1.70	3.05	4.83	505	84	86	675
6	d_{max}	15.08	3.84	1.46	3.47	5.52	503	52	46	601
	$1.5d_{max}$	24.97	3.65	1.61	5.48	8.66	505	76	82	663
7	d_{max}	20.32	4.11	1.78	2.62	4.40	451	51	43	545
	$1.5d_{max}$	29.41	5.88	2.26	3.98	6.66	455	75	77	607
8	d_{max}	19.41	4.32	1.86	2.83	5.10	249	53	46	348
	$1.5d_{max}$	29.75	4.97	2.07	3.63	5.93	501	81	84	666
9	d_{max}	17.81	3.89	1.65	2.61	4.65	501	52	98	651
	$1.5d_{max}$	27.92	4.09	1.91	3.37	4.85	505	83	87	675
10	d_{max}	22.78	4.19	1.86	3.14	5.99	501	58	116	675
	$1.5d_{max}$	34.30	3.79	1.95	4.73	6.43	505	89	94	688

Fal1 Family 1, *Fal2* Family 2, *Fal3* Family 3, *AllFals* all Families

LP relaxation by inequalities (21), which is $\left(Z^* - Z^{LB}\right)/\left(Z^* - Z^{LB}_{family1}\right)$ where $Z^{LB}_{family1}$ is the objective value of the LP relaxation with valid inequalities (21). Similarly, we report the improvement of the integrality gaps when including inequalities (22) (Family 2), inequalities (23) (Family 3) and all inequalities of the three families in the LP relaxation in the fifth, sixth, seventh column, respectively. Columns 8–10 show the number of valid inequalities added to the LP relaxation for valid inequalities of Family 1, 2 and 3, respectively. Column 11 reports the number of valid inequalities added to the LP relaxation when we apply the valid inequalities of all families to the PIRP formulation.

In Tables 1 and 2, we observe that the integrality gaps in Column 3 increase with the shelf-life. These gaps are also higher with the vehicle capacity = $1.5d_{max}$. The computational results show that the valid inequalities improve substantially the lower bounds on the optimal objective value of PIRP, especially when the vehicle capacity is $1.5d_{max}$.

A comparison of the effectiveness of the three families of valid inequalities shows that the largest improvement in the integrality gaps for shelf-life = 2 is made by valid inequalities (23). However, for shelf-life = 3, valid inequalities (21) play the most

Table 3 Computational results for problems with 8 customers

Dataset	C	Objective with inv. cost				Objective without inv. cost			
		Shelf-life = 2		Shelf-life = 3		Shelf-life = 2		Shelf-life = 3	
		$\frac{Z^{col}-LB}{Z^{col}}$ (%)	$\frac{Z^{col}-Z^*}{Z^{col}}$ (%)	$\frac{Z^{col}-LB}{Z^{col}}$ (%)	$\frac{Z^{col}-Z^*}{Z^{col}}$ (%)	$\frac{Z^{col}-LB}{Z^{col}}$ (%)	$\frac{Z^{col}-Z^*}{Z^{col}}$ (%)	$\frac{Z^{col}-LB}{Z^{col}}$ (%)	$\frac{Z^{col}-Z^*}{Z^*}$ (%)
1	d_{max}	3.52	0.28	3.90	0.02	6.34	0.88	5.80	1.14
	$1.5d_{max}$	4.97	2.13	4.99	2.10	4.04	3.27	3.85	2.08
2	d_{max}	8.58	0.00	6.57	0.01	11.73	0.53	9.74	0.00
	$1.5d_{max}$	8.46	0.64	8.77	1.01	10.55	3.67	9.52	3.51
3	d_{max}	4.57	1.19	5.11	0.00	4.60	1.79	4.81	0.03
	$1.5d_{max}$	6.27	1.80	6.46	1.98	5.93	3.42	4.86	3.00
4	d_{max}	7.15	3.76	7.49	0.00	10.14	1.42	11.09	0.01
	$1.5d_{max}$	6.34	3.81	5.12	1.36	3.44	3.05	5.86	3.70
5	d_{max}	6.87	1.83	4.16	1.55	8.12	2.16	2.39	0.00
	$1.5d_{max}$	5.00	0.16	5.26	0.55	7.88	3.39	5.34	2.77

important role in reducing the integrality gap. In all situations, the impact of valid inequalities (22) on strengthening the lower bounds is the least significant.

We also observe that when adding all three families of valid inequalities to the PIRP formulation concurrently, the LBs are stronger but the degree of improvement depends on the specific testing problem. In general, the gap between this LB and Z^* is improved a little when compared with the best one among columns 4, 5 and 6. However, for some special cases such as datasets 3, 4 or 6 in Table 2, a significant improvement is noted when valid inequalities of all three families are included in the PIRP formulation.

5.2 Algorithm performance for small problems

We first examine the quality of the best feasible solution Z^{col} obtained through our heuristic by comparing it with the optimal solution Z^* for the datasets with 8 customers. Although the number of feasible routes in the case of 8 customers is only 255, the time for CPLEX to solve PIRP to optimality is long for some of the instances. In the tables below, Z^* represents the solution obtained by the CPLEX Solver within 6,000 s.

Table 3 shows the relative gaps between LB and Z^{col} and the relative gaps between Z^{col} and Z^* for the datasets with 8 customers. In this table, LB represents the lower bound on the optimal objective value of PIRP, which is the optimal solution of $LP\ (PIRP)$ with valid inequalities (22) and (23). These results indicate that the gaps between Z^{col} and Z^* are very small for almost all instances. This suggests that the gaps between LB and Z^{col} are caused mainly by the difference between LB and Z^*.

5.3 Algorithm performance for large problems

The pricing problem is a hard combinatorial problem because it contains a traveling salesman component. Therefore, solving the pricing problem optimally is computationally expensive. The column generation method does not require finding an optimal solution to the pricing problem at each iteration. As long as a route with a negative pricing value is found, the corresponding column can be added to $\bar{R}$. Therefore, we first try to obtain a good solution to the pricing problem heuristically. In practice, it is common to consider only feasible routes whose distance is smaller than a given threshold or that visit a small number of customers. In our study, based on the AMPATH project, we never have a delivery route with more than five customers. Therefore, in order to reduce the number of feasible routes that are generated, we consider feasible routes with no more than five customers in the PIRP formulation. Even with this restriction, the number of feasible routes is still quite large. For instance, in problems with 50 customers, the number of feasible routes under consideration is 2,369,935.

The details of this heuristic algorithm are presented in Appendix C. In case the heuristic fails to find an appropriate route, an enumeration procedure is used to find an optimal solution to the pricing problem. Note that we may extend the heuristic algorithm to find routes with more than 5 customers. By doing this, we allow more general routes to be included in our model.

In the column generation approach, a large amount of time is spent on $LP(PIRP)$, because it has to be solved at each iteration. The size of $LP(PIRP)$ is very large even with a small number of feasible routes. For instance, if PIRP has 50 customers, 5 time periods and 100 routes in $\bar{R}$, the number of variables in $LP(PIRP)$ is approximately 26,000. In order to reduce the computational time of solving $LP(PIRP)$, we implement the following two features in the algorithm. Firstly, instead of solving the primal problem $LP(PIRP)$, we solve its dual problem (RD) and evaluate the pricing problem using the optimal solution of (RD). Secondly, we keep track of inactive routes, which are defined as routes with $\theta_{rt} = 0$ at an optimal solution of $LP(PIRP)$. After a certain number of iterations, if an inactive route in $\bar{R}$ has not become active, it will be removed from RMP.

As mentioned in the implementation section, the heuristic solution is found by solving PIRP over the set of feasible routes $\bar{R}$. In our computational experiments, the amount of time to obtain Z^{col} by CPLEX was restricted to at most 2 h, except for problems involving 50 customers, in which case the computation time was restricted to at most 3 h.

Table 4 shows the results for the heuristic algorithm when the objective function includes inventory cost, while Table 5 shows the results for the heuristic algorithm when the objective function does not include inventory cost. Comparing the results in these tables, it is evident that when the objective function does not include inventory cost, the gaps between LB and Z^{col} are usually smaller. It is also evident that the heuristic algorithm tends to obtain a better solution when the shelf-life is 3. In addition, the results indicate that when the customer demand per time period is small in comparison to the vehicle capacity, the algorithm generates more cuts and takes more time to obtain LB. At the same time, the integrality gaps of these problems are higher.

This suggests that PIRP becomes harder when the customer demand per time period is small in comparison to the vehicle capacity.

6 Conclusions and future research

This paper introduces an inventory routing problem for perishable goods that integrates inventory and routing decisions in a optimization model, and proposes a column generation-based heuristic algorithm to solve it. Computational results demonstrate that our heuristic algorithm is effective. We believe that our solution approach can be used for other inventory routing problems.

This research is a first step toward developing a branch-and-cut-and-price algorithm for this challenging problem. In the future, we will focus on developing more effective families of cuts and finding an adequate branching strategy. We believe that a branch-and-cut-and-price algorithm could work very effectively for problems of small or medium size. For large problems, heuristic algorithms would probably have to be developed. Therefore, another line of future research is to develop effective heuristic algorithms for PIRP that are capable of solving large problems.

Appendix A: The general form of the pricing problem

Let β, σ be the dual variables associated with valid inequalities (22), (23), respectively. Then Φ_{it}, Π_{it}, Υ_t in Eq. (24) are calculated as follows:

$$\begin{aligned}
\Phi_{it} &= -\varphi_{it} + \sum_{l\in T}\sum_{s\in[1,l]}\sum_{t\in s}\sum_{i\in N} d^i_{tl}\beta_{ils} \\
&+ \sum_{l\in T}\sum_{k\in T, k\geq l+\tau_{max}-1}\sum_{s\in[l,k]}\sum_{t\in s}\sum_{i\in N} \min\left\{d^i_{kt} + u_{it} - u_{ik-1}, d^i_{kl} - u_{ik-1}, d^i_{tl}\right\}\sigma_{ilks} \\
\Pi_{it} &= \min\left\{C, \sum_{i\in r} d^{max}_{it}\right\} \\
&\times \left\{\pi_{it} - \sum_{l\in T}\sum_{s\in[1,l]}\sum_{t\in s}\sum_{i\in N}\beta_{ils} - \sum_{l\in T}\sum_{k\in T, k\geq l+\tau_{max}-1}\sum_{s\in[l,k]}\sum_{t\in s}\sum_{i\in N}\sigma_{ilks}\right\} \\
\Upsilon_t &= \upsilon_t
\end{aligned}$$

Appendix B: Procedure to obtain a feasible solution

The main purpose of the heuristic is to create a feasible solution to initialize set R'. The algorithm is described as follows:

Step 1: Initialization: Set $t = 0$ and $\alpha = 0.8$.

Table 4 Computational results with inventory cost in the objective

$\|N\|$	C	Shelf-life = 2					Shelf-life = 3				
		Sec	NCuts	AGap (%)	WGap (%)	BGap (%)	Sec	NCuts	AGap (%)	WGap (%)	BGap (%)
30	d_{max}	31.70	805.67	7.14	9.82	4.12	34.38	828.22	5.78	6.70	4.00
30	$1.5d_{max}$	105.73	1,343.89	7.11	8.40	5.91	119.66	1,342.89	6.97	8.12	5.60
40	d_{max}	164.15	1,184.60	7.18	8.98	4.16	163.10	1,212.40	6.47	8.83	5.60
40	$1.5d_{max}$	438.08	2,098.90	8.33	10.88	6.51	454.57	2,085.10	7.85	10.28	4.58
50	d_{max}	545.19	1,553.00	7.06	8.74	5.58	526.74	1,572.80	6.11	6.91	4.95
50	$1.5d_{max}$	1,393.41	2,938.10	9.82	16.37	7.00	1,410.72	2,956.10	9.16	11.98	7.72

AGap average gap, *WGrap* gap of the worst case, *BGap* gap of the best case

Table 5 Computational results without inventory cost in the objective

$\|N\|$	C	Shelf-life = 2					Shelf-life = 3				
		Sec	NCuts	AGap (%)	$WGap$ (%)	BGap (%)	Sec	NCuts	AGap (%)	WGap (%)	BGap (%)
30	d_{max}	25.78	571.70	7.45	10.31	4.23	12.44	388.33	5.75	7.22	3.60
30	$1.5d_{max}$	53.09	783.89	6.09	8.74	4.09	23.29	561.89	5.27	6.70	2.71
40	d_{max}	93.18	811.50	4.67	5.89	2.91	48.56	553.10	3.81	4.74	2.83
40	$1.5d_{max}$	177.99	972.20	6.90	8.58	4.85	83.87	787.00	4.44	6.30	1.87
50	d_{max}	390.77	996.60	4.66	5.57	3.72	224.80	666.90	3.80	5.50	2.33
50	$1.5d_{max}$	492.89	1,281.80	6.87	8.37	5.60	288.01	927.60	4.40	6.00	1.53

AGap average gap, *WGrap* gap of the worst case, *BGap* gap of the best case

Step 2: If $t > |T|$, stop algorithm; Otherwise, set $del_{it}^{min} = \max\{0, d_{it} - I_{it}\}$; $del_{it}^{max} = \max\left\{0, d_{it}^{max} - I_{it}\right\}$; $S_0 = \left\{i \in N : del_{it}^{min} = 0\right\}$: set of customers who have to deliver in time period t; $S_1 = N \backslash S_0$: set of customers who do not have to deliver in time period t.

Step 3: Open new route $r = \{0, 0\}$; Set $C_{left} = C$;

Loop 1: If $C_{left} = 0$ or S_0 is empty, exit Loop 1; Otherwise, without loss of generality, assume that $del_{i_1 t}^{min} < del_{i_2 t}^{min}$ if $i_1 \leq i_2$ Set $i^* = \left\{\max\ i : del_{it}^{min} \leq C_{left}\right\}$; If such an i^* does not exist, exit Loop 1; Set $y_{it} = \min\left\{\max\left\{\alpha del_{it}^{max}, del_{it}^{min}\right\}, C_{left}\right\}$, $S_0 = S_0 \backslash i$, $C_{left} = C_{left} - y_{it}$; Insert customer i in route r If the number of customers in route r is greater than 5, go to Step 3. Otherwise, go to Loop 1.

Loop 2: If $C_{left} = 0$ or S_1 is empty, exit Loop 2; Otherwise, select i randomly from S_1; Set $y_{it} = \min\left\{C_{left}, del_{it}^{max}\right\}$, $S_1 = S_1 \backslash i$, $C_{left} = C_{left} - y_{it}$; Insert customer i in route r; If the number of customers in route r is greater than 5, go to Step 3. Otherwise, go to Loop 2. Attempt to determine a short tour by applying GENIUS to r. Repeat Step 3 until number of routes is greater than $|K|$ or S_1 is empty.

Step 4: If S_0 is not empty, go back to Step 1 and reduce α by 20%. Otherwise, Set $I_{it} = y_{it} - d_{it} + I_{it-1}$; Set $t = t + 1$; Go to Step 2.

Appendix C: Heuristic algorithm for pricing problem

Since the pricing problem is time-consuming to solve optimally, we propose a heuristic algorithm to solve the pricing problem $\min\limits_{r \in R}\left\{c_r - \sum_{i \in r} \Phi_i - \max\limits_{i \in r}(0, \Pi_i)\right\}$. The details of the algorithm are given below.

Step 1 **(Route initiation):** Set $r = \{0, 0\}$;
Find $i \in N$ such that $\{c_{0i} + c_{i0} - \Phi_i - \max(0, \Pi_i)\}$ is minimized; Update $r = \{0, i, 0\}$.

Step 2 **(Insert):** Set $\Pi_{max} = \max\limits_{i \in r}(0, \Pi_i)$;
For customer $i \in N \backslash r$, let $e_i = \min\limits_{l,k}(c_{li} + c_{ik} - c_{lk})$ where l and k are two consecutive customers in route r. Find $i \in N \backslash r$ such that $p_i = \left\{e_i - \Phi_i - [\Pi_i - \Pi_{max}]^+\right\}$ is minimized. If $p_i < 0$, insert customer i in route r and apply GENIUS [17] to find the shortest tour. Otherwise, go to Step 3. If the number of customers in route r is greater than 5, go to Step 3. Otherwise, repeat Step 2.

Step 3 **(Delete):** Find $i \in r$ such that

$$d_i = \left\{c_{ik} + c_{il} - c_{kl} - \Phi_i - [\Pi_i - \Pi_{max}]^+\right\}$$

is maximized where $\Pi_{max} = \max_{j \in r, j \neq i} \{\Pi_j\}$; l and k are two customers before and after customer i in route r. If $d_i > 0$, then remove i from route r and apply GENIUS. Go to Step 2. Otherwise, the heuristic is stopped.

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