Optim Lett (2013) 7:1893–1911
DOI 10.1007/s11590-012-0530-4

ORIGINAL PAPER

The optimal solution set of the interval linear programming problems

M. Allahdadi · H. Mishmast Nehi

Received: 12 April 2011 / Accepted: 18 July 2012 / Published online: 7 August 2012

Abstract Several methods exist for solving the interval linear programming (ILP) problem. In most of these methods, we can only obtain the optimal value of the objective function of the ILP problem. In this paper we determine the optimal solution set of the ILP as the intersection of some regions, by the best and the worst case (BWC) methods, when the feasible solution components of the best problem are positive. First, we convert the ILP problem to the convex combination problem by coefficients $0 \leq \lambda_j, \mu_{ij}, \mu_i \leq 1$, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. If for each i, j, $\mu_{ij} = \mu_i = \lambda_j = 0$, then the best problem has been obtained (in case of minimization problem). We move from the best problem towards the worst problem by tiny variations of λ_j, μ_{ij} and μ_i from 0 to 1. Then we solve each of the obtained problems. All of the optimal solutions form a region that we call the optimal solution set of the ILP. Our aim is to determine this optimal solution set by the best and the worst problem constraints. We show that some theorems to validity of this optimal solution set.

Keywords Convex combination problem · Interval linear programming · Interval equations system · Solution set

1 Introduction

In the real world, many problems are uncertain (interval) problems. These problems have usually been expressed by certain numbers. A certain number can represent one of an infinite values of an interval number. Therefore, using certain numbers to express uncertain problems is not complete. The interval linear programming (ILP) is a method for decision making under uncertainty.

M. Allahdadi · H. Mishmast Nehi (✉)
Mathematics Faculty, University of Sistan and Baluchestan, Zahedan, Iran
e-mail: hmnehi@hamoon.usb.ac.ir

One of the essential points in the ILP problems is basis stability. Basic difficulty in the ILP problem such as calculating the optimal solution set may be computationally very expensive. However, by holding basis stability, the difficulty becomes much easier to solve and so we can obtain all of the possible solutions set and optimal values range [5,7].

An interval number x is generally represented as $[\underline{x}, \overline{x}]$ where $\underline{x} \leq \overline{x}$. If $\underline{x} = \overline{x}$, then x will be degenerate.

If $\underline{A}$ and $\overline{A}$ are two matrices in $\mathbb{R}^{m\times n}$ and $\underline{A} \leq \overline{A}$, then the set of matrices

$$\mathbf{A} = [\underline{A}, \overline{A}] = \{A |\ \underline{A} \leq A \leq \overline{A}\},$$

is called an interval matrix, and the matrices $\underline{A}$ and $\overline{A}$ are called its bounds. Center and radius matrices have been defined as

$$A^c = \frac{1}{2}(\underline{A} + \overline{A}), \quad \Delta_{\mathbf{A}} = \frac{1}{2}(\overline{A} - \underline{A}).$$

A square interval matrix $\mathbf{A}$ is called regular if each $A \in \mathbf{A}$ is nonsingular.

A special case of an interval matrix is an interval vector which is a one-column interval matrix

$$\mathbf{X} = \{X |\ \underline{X} \leq X \leq \overline{X}\},$$

where $\underline{X}, \overline{X} \in \mathbb{R}^n$.

Interval arithmetic is defined in [1].

A family of ILP problems is defined as follows:

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & A\mathbf{x} \leq (=\geq)\mathbf{b} \\ & \mathbf{x} \geq 0, \end{aligned}$$

where $\mathbf{c} \in \mathbf{c}^I \subseteq \mathbb{R}^n$, $A \in \mathbf{A}^I \subseteq \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbf{b}^I \subseteq \mathbb{R}^m$ and $\mathbf{c}^I$, $\mathbf{A}^I$ and $\mathbf{b}^I$ are interval sets.

Many researchers have worked on several cases of linear programming problem with interval parameters, [2–4,6,11]. Generally, in most of these methods, we obtain the optimal value of objective function of the ILP problem. These methods do not give any information about the optimal solutions of the ILP problem.

In this paper we determine the optimal solution set of the ILP problem by the best and the worst cases (BWC) method, when the feasible solution components of the best problem are positive. Here, we convert the ILP problem to the convex combination problem by coefficients $0 \leq \lambda_j \leq 1, 0 \leq \mu_{ij} \leq 1$ and $0 \leq \mu_i \leq 1$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. If for each i, j, $\mu_{ij} = \mu_i = \lambda_j = 0$, then the best problem has been obtained(in case of minimization problem). We move from the best problem towards the worst problem by tiny variations of λ_j, μ_{ij} and μ_i from 0 to 1. Then we solve each of the obtained problems. All of the optimal solutions form a region that we call the optimal solution set of the ILP. Also, we compare this method with Enhanced-ILP model presented by Zhou et al. [12].

2 Review of ILP

In this section, we review the ILP problem and the optimal value bounds. Also, we define basis stability and its conditions so that under these conditions we can confidently determine the optimal solution set of the ILP.

2.1 Basis stability

B-stability (basis stability) of the ILP problem is important, because it describes the set of all possible optimal solutions [7] and so we can derive the optimal value range. Hence the optimal solution set of the ILP problem will be subset of the optimal solutions generated by B-stability.

Definition 2.1 The problem $\min\{\mathbf{c}^T\mathbf{x} : \ A\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq 0\}$ where $\mathbf{c} \in \mathbf{c}^I \subseteq \mathbb{R}^n$, $A \in \mathbf{A}^I \subseteq \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbf{b}^I \subseteq \mathbb{R}^m$, is called B-stable with basis B if B is an optimal basis for each characteristic problem. Let $B \subseteq \{1, 2, \dots, n\}$ be an index set such that $\mathbf{A}_B$ is non-singular, where $\mathbf{A}_B$ denotes the restriction of A to the columns indexed by B. Similarly, $N = \{1, 2, \dots, n\}\backslash B$ stands for non-basic variables and $\mathbf{A}_N$ denotes the restriction to non-basic indices. B can be computed by solving a version of ILP with the center values.

In general, B-stability conditions have been presented in [5,7]. These conditions are as follows [5]:

(1) (regularity) $\mathbf{A}_B$ is regular.
(2) (feasibility) The solution set of the interval system $\mathbf{A}_B\mathbf{x}_B = \mathbf{b}$ is non-negative.
(3) (optimality) $\mathbf{A}_B$ is optimal.

These conditions can be obtained from the following theorems.

Theorem 2.1 [8] *If* $\rho(|(A^c)_B^{-1}|\Delta_{A_B}) < 1$, *then* $\mathbf{A}_B$ *is regular, that* $\rho(.)$ *denotes the spectral radius.*

Theorem 2.2 [8] *If* $\max_{1\leq i\leq n}(|(A^c)_B^{-1}|\Delta_{A_B})_{ii} \geq 1$, *then* $\mathbf{A}_B$ *is not regular.*

Theorem 2.3 [9] *If the interval vector* $\mathbf{r}$ *is an enclosure to the solution set of system* $\mathbf{A}_B\mathbf{x}_B = \mathbf{b}$, *then:*

$$\underline{r}_i = \min\left\{-\mathbf{x}_i^* + (\mathbf{x}_i^c + |\mathbf{x}_i^c|)M_{ii}, \frac{1}{2M_{ii}-1}(-\mathbf{x}_i^* + (\mathbf{x}_i^c + |\mathbf{x}_i^c|)M_{ii})\right\},$$
$$\overline{r}_i = \max\left\{\mathbf{x}_i^* + (\mathbf{x}_i^c - |\mathbf{x}_i^c|)M_{ii}, \frac{1}{2M_{ii}-1}(\mathbf{x}_i^* + (\mathbf{x}_i^c - |\mathbf{x}_i^c|)M_{ii})\right\},$$

where

$$M = (I - |(A_B^c)^{-1}|\Delta_{A_B})^{-1}, \quad \mathbf{x}^c = (A_B^c)^{-1}b^c, \quad \mathbf{x}^* = M(|\mathbf{x}^c| + |(A_B^c)^{-1}|\Delta_b),$$

and A_B^c *is non-singular and* $\rho(|(A_B^c)^{-1}|\Delta_A) < 1$.

Theorem 2.4 [5] *Let diag(q) denotes the diagonal matrix with entries* $q_1, \ldots, q_m$. *If for each* $q \in \{\pm 1\}^m$, *the solution set of system*

$$\begin{cases} ((A_B^c)^T - (\Delta_{A_B})^T diag(q))\mathbf{y} \le \overline{\mathbf{c}}_B \\ -((A_B^c)^T + (\Delta_{A_B})^T diag(q))\mathbf{y} \le -\underline{\mathbf{c}}_B \\ diag(q)\mathbf{y} \ge 0 \end{cases} \tag{2.1}$$

lies in the solution set of system

$$\begin{cases} ((A_N^c)^T + (\Delta_{A_N})^T diag(q))\mathbf{y} \le \underline{\mathbf{c}}_N \\ diag(q)\mathbf{y} \ge 0. \end{cases} \tag{2.2}$$

Then, optimality condition is valid.

Remark 2.1 Since there are no dependencies, the constraints $A\mathbf{x} \le \mathbf{b}, \mathbf{x} \ge 0$ are equivalent to $A\mathbf{x} + I\mathbf{y} = \mathbf{b}, \mathbf{x}, \mathbf{y} \ge 0$.

2.2 ILP and convex combination problem

In this section, we define the ILP, the convex combination problems and the interval hull.

Definition 2.2 An ILP problem is defined as

$$\begin{aligned} \min \quad & z = \sum_{j=1}^{n} [\underline{c}_j, \overline{c}_j] x_j \\ s.t. \quad & \sum_{j=1}^{n} [\underline{a}_{ij}, \overline{a}_{ij}] x_j \le [\underline{b}_i, \overline{b}_i] \quad i = 1, 2, \ldots, m \\ & x_j \ge 0 \qquad j = 1, 2, \ldots, n. \end{aligned} \tag{2.3}$$

We call problems (2.4) and (2.5) as characteristic version and convex combination problem of (2.3):

$$\begin{aligned} \min \quad & z_1 = \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad i = 1, 2, \ldots, m \\ & x_j \ge 0 \quad j = 1, 2, \ldots, n, \end{aligned} \tag{2.4}$$

$$\min \quad z_2 = \sum_{j=1}^{n} [(1-\lambda_j)\underline{c}_j + \lambda_j \overline{c}_j] x_j \tag{2.5}$$

$$s.t. \quad \sum_{j=1}^{n} [(1-\mu_{ij})\underline{a}_{ij} + \mu_{ij}\overline{a}_{ij}] x_j \leq (1-\mu_i)\overline{b}_i + \mu_i \underline{b}_i \quad i = 1, 2, \ldots, m$$

$$x_j \geq 0 \qquad j = 1, 2, \ldots, n,$$

where $c_j \in [\underline{c}_j, \overline{c}_j]$, $a_{ij} \in [\underline{a}_{ij}, \overline{a}_{ij}]$, $b_i \in [\underline{b}_i, \overline{b}_i]$ and $0 \leq \lambda_j \leq 1$, $0 \leq \mu_{ij} \leq 1$, $0 \leq \mu_i \leq 1$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Lemma 2.1 *Two problems (2.4) and (2.5) are equivalent.*

Proof Its Proof is straightforward and we omit it.

Theorem 2.5 [10] *Let $x_j \geq 0$ for each j, then for interval inequality*

$$\sum_{j=1}^{n} [\underline{a}_j, \overline{a}_j] x_j \leq [\underline{b}, \overline{b}],$$

$\sum_{j=1}^{n} \underline{a}_j x_j \leq \overline{b}$ *and* $\sum_{j=1}^{n} \overline{a}_j x_j \leq \underline{b}$ *are the largest and smallest feasible regions respectively.*

Theorem 2.6 [2] *For the ILP problem (2.3), the best and the worst optimal values of the objective function are obtained by solving the following problems respectively.*

$$\min \quad \underline{z} = \sum_{j=1}^{n} \underline{c}_j x_j$$

$$s.t. \quad \sum_{j=1}^{n} \underline{a}_{ij} x_j \leq \overline{b}_i \qquad i = 1, 2, \ldots, m$$

$$x_j \geq 0 \qquad j = 1, 2, \ldots, n,$$

$$\min \quad \overline{z} = \sum_{j=1}^{n} \overline{c}_j x_j$$

$$s.t. \quad \sum_{j=1}^{n} \overline{a}_{ij} x_j \leq \underline{b}_i \qquad i = 1, 2, \ldots, m$$

$$x_j \geq 0 \qquad j = 1, 2, \ldots, n.$$

Definition 2.3 Let $\mathbf{A}^I$ be a $n \times n$ interval matrix and $\mathbf{b}^I$ is an interval vector. For an interval linear system $\mathbf{A}^I\mathbf{x} \leq \mathbf{b}^I$, we define the set S as

$$S = \{\mathbf{x} |\ A\mathbf{x} \leq b,\ A \in \mathbf{A}^I, b \in \mathbf{b}^I\}.$$

the narrowest interval vector containing the set S is called the interval hull of S. This set is generally not an interval vector. It is usually difficult to describe S. Because S

is generally so complicated in shape [4, 2nd chapter], it is usually impractical to use it. Instead, it is a common practice to seek the interval vector $\mathbf{x}^I$ containing S that the narrowest possible interval components. This interval vector is called the hull of the solution set or simply the hull.

Theorem 2.7 *If $\mathbf{x}^\circ = (x_1^\circ, x_2^\circ, \ldots, x_n^\circ)^t$ is an optimal solution and z° is corresponding the objective function value of the problem (2.4), then z° is placed on $[\underline{z}^*, \overline{z}^*]$, where $\underline{z}^*$ and $\overline{z}^*$ are the best and the worst optimal values of the objective function.*

Proof Let $\underline{\mathbf{x}}^* = (\underline{x}_1^*, \underline{x}_2^*, \ldots, \underline{x}_n^*)^t$ and $\overline{\mathbf{x}}^* = (\overline{x}_1^*, \overline{x}_2^*, \ldots, \overline{x}_n^*)^t$ be optimal solutions of the best and the worst problems respectively.

Since the feasible region of the best problem is the largest feasible region, then $\mathbf{x}^\circ$ is a feasible solution of the best problem and furthermore $\underline{\mathbf{x}}^*$ is the optimal solution of the best problem, therefore $\sum_{j=1}^n \underline{c}_j x_j^\circ \geq \sum_{j=1}^n \underline{c}_j \underline{x}_j^* = \underline{z}^*$.

For all j, $c_j \geq \underline{c}_j$, therefore $z^\circ = \sum_{j=1}^n c_j x_j^\circ \geq \sum_{j=1}^n \underline{c}_j x_j^\circ$, and so $z^\circ \geq \underline{z}^*$.

Since the feasible region of the worst problem is the smallest feasible region, then $\overline{\mathbf{x}}^*$ is a feasible solution of the problem (2.4) and furthermore $\mathbf{x}^\circ$ is the optimal solution of the problem (2.4), therefore $z^\circ = \sum_{j=1}^n c_j x_j^\circ \leq \sum_{j=1}^n c_j \overline{x}_j^*$.

For all j, $c_j \leq \overline{c}_j$, therefore $\sum_{j=1}^n c_j \overline{x}_j^* \leq \sum_{j=1}^n \overline{c}_j \overline{x}_j^* = \overline{z}^*$, and so $z^\circ \leq \overline{z}^*$. Hence $\underline{z}^* \leq z^\circ \leq \overline{z}^*$.

Remark 2.2 Let z° be an optimal value of the problem (2.4) such that $z^\circ \in [\underline{z}^*, \overline{z}^*]$ and $\mathbf{x}^\circ$ is corresponding optimal solution, then its possible that $\mathbf{x}^\circ$ does not set in hull. To show this, consider the following interval linear programming problem:

$$\begin{aligned} \min \quad & z = [1,5]x_1 + [3,4]x_2 \\ s.t. \quad & [1,3]x_1 + [5,6]x_2 \geq [3,5] \\ & [2,4]x_1 + [1,4]x_2 \geq [1,4] \\ & x_1, x_2 \geq 0. \end{aligned}$$

Optimal values of the best and the worst objective functions consist of $\underline{z}^* = 1$ and $\overline{z}^* = 11$ and $\underline{\mathbf{x}}^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\overline{\mathbf{x}}^* = \begin{pmatrix} \frac{5}{3} \\ \frac{2}{3} \end{pmatrix}$, so the hull set is equal to $\begin{pmatrix} [1,\frac{5}{3}] \\ [0,\frac{2}{3}] \end{pmatrix}$. One of the characteristic problems is as follows:

$$\begin{aligned} \min \quad & z = 5x_1 + 4x_2 \\ s.t. \quad & x_1 + 5x_2 \geq 5 \\ & 2.4x_1 + 1.6x_2 \geq 3.4 \\ & x_1, x_2 \geq 0. \end{aligned}$$

The optimal solution is equal to $\begin{pmatrix} 0.8654 \\ 0.8269 \end{pmatrix}$ with $z^* = 7.6346$. Note that $z^* \in [\underline{z}^*, \overline{z}^*] = [1, 11]$, whereas $\mathbf{x}^* \notin \begin{pmatrix} [1,\frac{5}{3}] \\ [0,\frac{2}{3}] \end{pmatrix}$.

3 Optimal solution set of the ILP

In this section, we obtain the optimal solution set of the interval linear programming problem by the best and the worst problem constraints, when the optimal solutions of the ILP problem are positive. This assumption can be derived from solving the best problem. If all of the feasible solution components of the best problem, which has the largest feasible region, are positive, then the feasible solution components and hence the optimal solution components in all of the convex combination problems and so the ILP problem are positive. Note that positivity of the feasible solution components of the best problem can easily be concluded by checking the infeasibility of the system of inequalities resulted from making each variable zero each time. Firstly, we give some definitions and theorems in which we require to introduce the optimal solution set of the ILP.

Definition 3.1 For each $1 \leq i \leq m$, we define S_i and T_i as follows:

$$S_i = \left\{ \mathbf{x} : \sum_{j=1}^{n} \overline{a}_{ij} x_j \geq \underline{b}_i \right\}, \quad T_i = \left\{ \mathbf{x} : \sum_{j=1}^{n} \underline{a}_{ij} x_j \leq \overline{b}_i \right\}.$$

Note that T_i and S_i are a set of $\mathbf{x}$ in which satisfies the best problem constraints and the worst problem constraints with inverse sign, respectively. For specifying the optimal solution set of the ILP, we consider two following cases and theorems.

Case 1 $m = n$. In this case, we can prove the following necessary and sufficient condition, which characterize the optimal solution set of the ILP.

Theorem 3.1 *Suppose all of the feasible solution components of the best problem are positive. Then the optimal solution set of the ILP is equal to the intersection of the region generated by the best problem constraints and the worst problem constraints with inverse sign.*

Proof Let $\mathbf{x}^*$ be optimal solution of problem (2.5). Since all of the feasible solution components of the best problem are positive and this problem has the largest feasible region, so all of the feasible and hence optimal solution components of the convex combination problems are positive too. In view of the fact that $m = n$, we conclude that all of the convex combination problem constraints are active at $\mathbf{x}^*$. So for each $i = 1, 2, \ldots, m = n$

$$\sum_{j=1}^{n} [(1 - \mu_{ij})\underline{a}_{ij} + \mu_{ij}\overline{a}_{ij}] x_j^* = (1 - \mu_i)\overline{b}_i + \mu_i \underline{b}_i,$$

therefore

$$\sum_{j=1}^{n} \underline{a}_{ij} x_j^* \leq \sum_{j=1}^{n} [(1 - \mu_{ij})\underline{a}_{ij} + \mu_{ij}\overline{a}_{ij}] x_j^* = (1 - \mu_i)\overline{b}_i + \mu_i \underline{b}_i \leq \overline{b}_i,$$

hence $\mathbf{x}^* \in T_i$. Also

$$\sum_{j=1}^{n} \overline{a}_{ij} x_j^* \geq \sum_{j=1}^{n} [(1-\mu_{ij})\underline{a}_{ij} + \mu_{ij}\overline{a}_{ij}] x_j^* = (1-\mu_i)\overline{b}_i + \mu_i \underline{b}_i \geq \underline{b}_i,$$

hence $\mathbf{x}^* \in S_i$. Therefore

$$\mathbf{x}^* \in \bigcap_{i=1}^{m} T_i, \quad \mathbf{x}^* \in \bigcap_{i=1}^{m} S_i,$$

then

$$\mathbf{x}^* \in \left(\bigcap_{i=1}^{m} T_i\right) \bigcap \left(\bigcap_{i=1}^{m} S_i\right).$$

Conversely, suppose

$$\mathbf{x}^* \in \left(\bigcap_{i=1}^{m} T_i\right) \bigcap \left(\bigcap_{i=1}^{m} S_i\right),$$

so for each $i = 1, 2, \ldots, m = n$, $\mathbf{x}^* \in S_i$ and $\mathbf{x}^* \in T_i$, therefore

$$\sum_{j=1}^{n} \overline{a}_{ij} x_j^* \geq \underline{b}_i, \quad \sum_{j=1}^{n} \underline{a}_{ij} x_j^* \leq \overline{b}_i \quad i = 1, 2, \ldots, m.$$

By definition of center and radius which was introduced in the introduction, we have,

$$\sum_{j=1}^{n} (a_{ij}^c + \Delta_{a_{ij}}) x_j^* \geq b_i^c - \Delta_{b_i}, \quad \sum_{j=1}^{n} (a_{ij}^c - \Delta_{a_{ij}}) x_j^* \leq b_i^c + \Delta_{b_i},$$

therefore

$$-\left(\sum_{j=1}^{n} \Delta_{a_{ij}} x_j^* + \Delta_{b_i}\right) \leq \sum_{j=1}^{n} a_{ij}^c x_j^* - b_i^c \leq \sum_{j=1}^{n} \Delta_{a_{ij}} x_j^* + \Delta_{b_i},$$

then

$$\left|\sum_{j=1}^{n} a_{ij}^c x_j^* - b_i^c\right| \leq \sum_{j=1}^{n} \Delta_{a_{ij}} x_j^* + \Delta_{b_i} \quad i = 1, 2, \ldots, m.$$

We define $\mathbf{y} \in \mathbb{R}^m$ by

$$y_i = \begin{cases} \frac{\sum_{j=1}^n a_{ij}^c x_j^* - b_i^c}{\sum_{j=1}^n \Delta_{a_{ij}} x_j^* + \Delta_{b_i}} & \sum_{j=1}^n \Delta_{a_{ij}} x_j^* + \Delta_{b_i} > 0 \\ 1 & \sum_{j=1}^n \Delta_{a_{ij}} x_j^* + \Delta_{b_i} = 0, \end{cases}$$

for $i = 1, 2, \ldots, m$. Then $|y_i| \leq 1$ and

$$\sum_{j=1}^n a_{ij}^c x_j^* - b_i^c = y_i \left(\sum_{j=1}^n \Delta_{a_{ij}} x_j^* + \Delta_{b_i} \right) \quad i = 1, 2, \ldots, m,$$

therefore

$$\sum_{j=1}^n (a_{ij}^c - y_i \Delta_{a_{ij}}) x_j^* = b_i^c + y_i \Delta_{b_i} \quad i = 1, 2, \ldots, m.$$

Since $|y_i| \leq 1$, so $a_{ij}^\circ = a_{ij}^c - y_i \Delta_{a_{ij}} \in [\underline{a}_{ij}, \overline{a}_{ij}]$ and $b_i^\circ = b_i^c + y_i \Delta_{b_i} \in [\underline{b}_i, \overline{b}_i]$ and hence for each i and j, there exist $\mu_{ij}^\circ, \mu_i^\circ \in [0, 1]$ such that

$$\sum_{j=1}^n [(1 - \mu_{ij}^\circ)\underline{a}_{ij} + \mu_{ij}^\circ \overline{a}_{ij}] x_j^* = (1 - \mu_i^\circ)\overline{b}_i + \mu_i^\circ \underline{b}_i, \quad i = 1, 2, \ldots, m.$$

Therefore $\mathbf{x}^*$ is a feasible solution of problem (2.5) such that all of the constraints of the problem (2.5) are active at it and for each j, $x_j^* > 0$, then $\mathbf{x}^*$ is an unique extreme point and so there exists $\lambda_j^\circ \in [0, 1]$ such that $\mathbf{x}^*$ is an optimal solution.

Case 2 $m > n$. In this case, since the number of the constraints are more than the variables, then maybe it will happen the different cases such as existence redundant constraints, degenerate extreme points, alternative solutions and etc, which cause difficulty for obtaining the optimal solution set of the ILP. Therefore generally, we can not prove a necessary and sufficient condition, but we can only prove the following sufficient condition for obtaining the optimal solution set. The obtained set of this sufficient condition is a subset of the optimal solution set of the ILP problem.

Theorem 3.2 *Suppose all of the feasible solution components of the best problem are positive. Let at least n constraints in the worst problem be active at $\overline{\mathbf{x}}^*$. Without loss of generality, by rearranging the constraints, let the constraints $1, 2, \ldots, l\,(n \leq l \leq m)$ in the worst problem be active at $\overline{\mathbf{x}}^*$ and for $i = l+1, \ldots, m$ and $j = 1, 2, \ldots, n$, $\underline{a}_{ij} = 0$. If $\mathbf{x}^* \in (\bigcap_{i=1}^m T_i) \bigcap (\bigcap_{i=1}^l S_i)$, then $\mathbf{x}^*$ is an optimal solution of the ILP problem.*

Proof Suppose $\mathbf{x}^* \in (\bigcap_{i=1}^m T_i) \bigcap (\bigcap_{i=1}^l S_i)$ is arbitrary. We want to show $\mathbf{x}^*$ is an optimal solution of a convex combination problem. Since

$$\mathbf{x}^* \in \left(\bigcap_{i=1}^m T_i \right) \bigcap \left(\bigcap_{i=1}^l S_i \right),$$

so for each $i = 1, 2, \ldots, m, \mathbf{x}^* \in T_i$ and for each $i = 1, 2, \ldots, l,\ n \leq l \leq m, \mathbf{x}^* \in S_i$, therefore

$$\sum_{j=1}^{n} \underline{a}_{ij} x_j^* \leq \overline{b}_i, \quad i = 1, 2, \ldots, l, l+1, \ldots, m, \tag{3.1}$$
$$\sum_{j=1}^{n} \overline{a}_{ij} x_j^* \geq \underline{b}_i, \quad i = 1, 2, \ldots, l.$$

Similar to proof of the converse case of Theorem 3.1, for $i = 1, 2, \ldots, l$ and $j = 1, 2, \ldots, n$, there exist μ_{ij}° and μ_i° such that

$$\sum_{j=1}^{n} [(1 - \mu_{ij}^\circ)\underline{a}_{ij} + \mu_{ij}^\circ \overline{a}_{ij}] x_j^* = (1 - \mu_i^\circ)\overline{b}_i + \mu_i^\circ \underline{b}_i.$$

Now, consider the following convex combination problem:

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} c_j^\circ x_j \\ s.t. \quad & \sum_{j=1}^{n} a_{ij}^\circ x_j \leq b_i^\circ \quad i = 1, 2, \ldots, l, \\ & \sum_{j=1}^{n} \underline{a}_{ij} x_j \leq \overline{b}_i \quad i = l+1, \ldots, m, \\ & x_j \geq 0 \quad j = 1, 2, \ldots, n, \end{aligned} \tag{3.2}$$

where $a_{ij}^\circ = (1 - \mu_{ij}^\circ)\underline{a}_{ij} + \mu_{ij}^\circ \overline{a}_{ij}$ and $c_j^\circ = (1 - \lambda_j^\circ)\underline{c}_j + \lambda_j^\circ \overline{c}_j$ in which $\lambda_j^\circ \in [0, 1]$ is arbitrary, for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, l$.

First, note that $\mathbf{x}^*$ is a feasible solution of the problem (3.2), since for $i = 1, 2, \ldots, l$, $\sum_{j=1}^{n} a_{ij}^\circ x_j^* = b_i^\circ$ and for $i = l+1, \ldots, m$, $\sum_{j=1}^{n} \underline{a}_{ij} x_j^* \leq \sum_{j=1}^{n} a_{ij}^\circ x_j^* \leq b_i^\circ \leq \overline{b}_i$. For optimality $\mathbf{x}^*$, consider the dual of the problem (3.2):

$$\begin{aligned} \max \quad & \sum_{i=1}^{l} b_i^\circ y_i + \sum_{i=l+1}^{m} \overline{b}_i y_i \\ s.t. \quad & \sum_{i=1}^{l} a_{ij}^\circ y_i + \sum_{i=l+1}^{m} \underline{a}_{ij} y_i \leq c_j^\circ \quad j = 1, 2, \ldots, n, \\ & y_i \leq 0 \quad j = 1, 2, \ldots, n. \end{aligned}$$

According to Theorem2.5, the feasible region of the best problem is the largest feasible region, so all of the feasible regions of the convex combination problems are subset of the feasible region of the best problem. By assumption all of the feasible solution components of the best problem are positive, thus all of the feasible solution components of the convex combination problems are positive too, therefore for $j = 1, 2, \ldots, n$, $c_j^\circ = \sum_{i=1}^{l} a_{ij}^\circ y_i + \sum_{i=l+1}^{m} \underline{a}_{ij} y_i$, also by assumption, $\underline{a}_{ij} = 0$ for all $i = l+1, \ldots, m$ and $j = 1, 2, \ldots, n$, therefore for all j, $c_j^\circ = \sum_{i=1}^{l} a_{ij}^\circ y_i$, and hence $\mathbf{c}^\circ = \sum_{i=1}^{l} y_i \mathbf{a}_i^\circ$, and so the objective function gradient $\mathbf{c}^\circ$ has been presented as a nonpositive linear combination of the gradients of the active constraints at $\mathbf{x}^*$(i.e. $\mathbf{a}_1^\circ, \mathbf{a}_2^\circ, \ldots, \mathbf{a}_l^\circ$), and hence, in view of the Karush–Kuhn–Tucker optimality conditions, $\mathbf{x}^*$ is the optimal solution.

4 Comparison to the EILP model

In this section, we compare our method with the Enhanced-ILP model presented by Zhou et al. [12]. In EILP method, two submodels have been introduced for obtaining feasible decision variables space. Let the ILP problem as follows:

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} \\ s.t. \quad & A\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0, \end{aligned} \tag{4.1}$$

where $\mathbf{c} \in \mathbf{c}^I \subseteq \mathbb{R}^n$, $A \in \mathbf{A}^I \subseteq \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbf{b}^I \subseteq \mathbb{R}^m$ and $\mathbf{x} \in \mathbf{x}^I \subseteq \mathbb{R}^n$.

In the EILP model, the problem (4.1) could be transformed into two submodels as follows [12]:

$$\begin{aligned} \min \quad & \underline{z} = \sum_{j=1}^{k} 0.5(\underline{c}_j + \overline{c}_j)\underline{x}_j + \sum_{j=k+1}^{n} 0.5(\underline{c}_j + \overline{c}_j)\overline{x}_j \\ s.t. \quad & \sum_{j=1}^{k} |[\underline{a}_{ij}, \overline{a}_{ij}]|^+ sign([\underline{a}_{ij}, \overline{a}_{ij}])\underline{x}_j + \sum_{j=k+1}^{n} |[\underline{a}_{ij}, \overline{a}_{ij}]|^- sign([\underline{a}_{ij}, \overline{a}_{ij}])\overline{x}_j \le \overline{b}_i \\ & [\underline{x}_j, \overline{x}_j] \ge 0, \quad j = 1, 2, \ldots, n, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \min \quad & \overline{z} = \sum_{j=1}^{k} 0.5(\underline{c}_j + \overline{c}_j)\overline{x}_j + \sum_{j=k+1}^{n} 0.5(\underline{c}_j + \overline{c}_j)\underline{x}_j \\ s.t. \quad & \sum_{j=1}^{k} |[\underline{a}_{ij}, \overline{a}_{ij}]|^- sign([\underline{a}_{ij}, \overline{a}_{ij}])\overline{x}_j + \sum_{j=k+1}^{n} |[\underline{a}_{ij}, \overline{a}_{ij}]|^+ sign([\underline{a}_{ij}, \overline{a}_{ij}])\underline{x}_j \le \underline{b}_i \\ & [\underline{x}_j, \overline{x}_j] \ge 0, \quad j = 1, 2, \ldots, n \\ & \overline{x}_j \ge \underline{x}_j^* \quad j = 1, \ldots, k \\ & \underline{x}_j \le \overline{x}_j^* \quad j = k+1, \ldots, n, \end{aligned} \tag{4.3}$$

where for $j = 1, 2, \ldots, k$, $[\underline{c}_j, \overline{c}_j] \geq 0$ and for $j = k+1, k+2, \ldots, n$, $[\underline{c}_j, \overline{c}_j] < 0$ and

$$|[\underline{a}_{ij}, \overline{a}_{ij}]|^- = \begin{cases} \underline{a}_{ij} & [\underline{a}_{ij}, \overline{a}_{ij}] \geq 0 \\ -\overline{a}_{ij} & [\underline{a}_{ij}, \overline{a}_{ij}] < 0 \end{cases}, \quad |[\underline{a}_{ij}, \overline{a}_{ij}]|^+ = \begin{cases} \overline{a}_{ij} & [\underline{a}_{ij}, \overline{a}_{ij}] \geq 0 \\ -\underline{a}_{ij} & [\underline{a}_{ij}, \overline{a}_{ij}] < 0 \end{cases},$$
$$sign([\underline{a}_{ij}, \overline{a}_{ij}]) = \begin{cases} 1 & [\underline{a}_{ij}, \overline{a}_{ij}] \geq 0 \\ -1 & [\underline{a}_{ij}, \overline{a}_{ij}] < 0. \end{cases}$$

To ensure that optimal solution $[\underline{\mathbf{x}}^*, \overline{\mathbf{x}}^*]$ is feasible, the following constraints could be added to the problem (4.3).

$$\sum_{j=1}^{k-p} (|[\underline{a}_{\delta j}, \overline{a}_{\delta j}]|^- \overline{x}_j - |[\underline{a}_{\delta j}, \overline{a}_{\delta j}]|^+ \underline{x}_j^*) - \sum_{j=k+1}^{n-q} (|[\underline{a}_{\delta j}, \overline{a}_{\delta j}]|^+ \underline{x}_j - |[\underline{a}_{\delta j}, \overline{a}_{\delta j}]|^- \overline{x}_j^*) \leq 0, \quad \forall \delta \tag{4.4}$$

where δ is the number of active constraints in problem (4.2) as well as its $[\underline{a}_{\delta j}, \overline{a}_{\delta j}] \geq 0$ for $j = 1, \ldots, k-p$ and $[\underline{a}_{\delta j}, \overline{a}_{\delta j}] \leq 0$ for $j = k+1, \ldots, n-q$.

In the EILP model, the obtained solution region of BWC method is restricted so that a range of absolutely feasible solutions is produced. In fact, by embedding the equation (4.4) in the problem (4.3), the final solution of the ILP has no infeasible zone, whereas making the solution interval hull small for obtaining the absolutely feasible solutions causes us to lose some of the optimal solutions of the ILP problem. In our method, in case $m = n$, we do not lose any feasible solution and we obtain all of the optimal solutions of the ILP problem. For more explanations see example 1. In case $m > n$, we derive the sufficient condition in Theorem 3.2 for obtaining the optimal solutions set.

5 Examples

In this section we will solve several examples by using the presented methods in the previous sections.

Example 1 Consider the following interval linear programming problem:

$$\begin{aligned} \min \quad & z = [1, 5]x_1 + [3, 4]x_2 \\ s.t. \quad & [0.5, 1]x_1 + [-2, -1]x_2 \leq [-1, 0] \\ & [-4, -3]x_1 + [0, 1]x_2 \leq [-3, -2] \\ & x_1, x_2 \geq 0. \end{aligned}$$

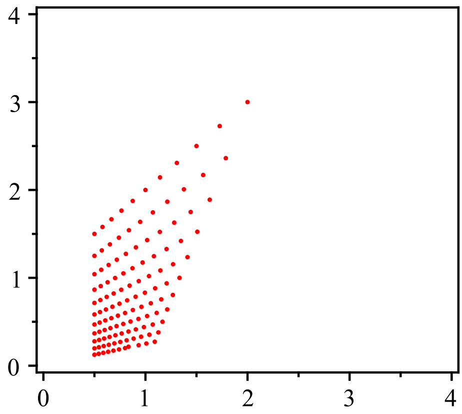


Fig. 1 Some of the optimal solutions of the convex combination problem of Example 1

The above problem is equivalent to the following problem:

$$\begin{aligned} \min \quad & z = [1, 5]x_1 + [3, 4]x_2 \\ s.t. \quad & [0.5, 1]x_1 + [-2, -1]x_2 + x_3 = [-1, 0] \\ & [-4, -3]x_1 + [0, 1]x_2 + x_4 = [-3, -2] \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

A candidate basis for B-stability is $B = (1, 2)$, since its optimal for characteristic problem with center values.

1. According to Theorem 2.1, the spectral radius is equal to 0.57, so $\mathbf{A}_B$ is regular.

$$\mathbf{A}_B = \begin{pmatrix} [0.5, 1] & [-2, -1] \\ [-4, -3] & [0, 1] \end{pmatrix}.$$

2. According to Theorem 2.3, we compute an enclosure to the solution set of $\mathbf{A}_B \mathbf{x}_B = \mathbf{b}$ to be $\mathbf{x}_B = \begin{pmatrix} [0.5, 2] \\ [0.125, 3] \end{pmatrix}$, which is non-negative.

3. According to Theorem 2.4, the solution set of the system (2.1) lies in the system (2.2) for each $q \in \{\pm 1\}^2$. Thus the problem is B-stable. The best and the worst values of the objective function is $\frac{7}{8}$ and 22 respectively. The optimal solutions are $\underline{\mathbf{x}}^* = \begin{pmatrix} 0.5 \\ 0.125 \end{pmatrix}$ and $\overline{\mathbf{x}}^* = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. We solve the convex combination problem for different values of $\lambda_j, \mu_i, \mu_{ij}$ for $i, j = 1, 2$ in $[0, 1]$. All of the optimal solution points set solution region of the ILP which is shown in Fig. 1. Since all of the feasible solution components of the best problem are positive and $m = n = 2$, then we can use Theorem 3.1, therefore the optimal solution set of the ILP will be as Fig. 4 that is as Fig. 1. The solution regions of the best and the worst problems with inverse constraints sign have been shown in Figs. 2 and 3. Now, we solve the problem by the EILP model. Two submodels are as follows:

Fig. 2 Solution region of the best problem of Example 1

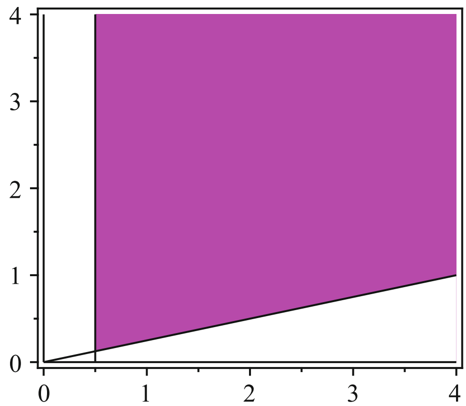


Fig. 3 Solution region of the worst problem with inverse constraints sign of Example 1

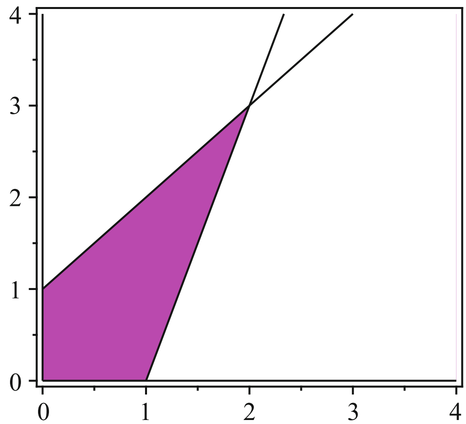
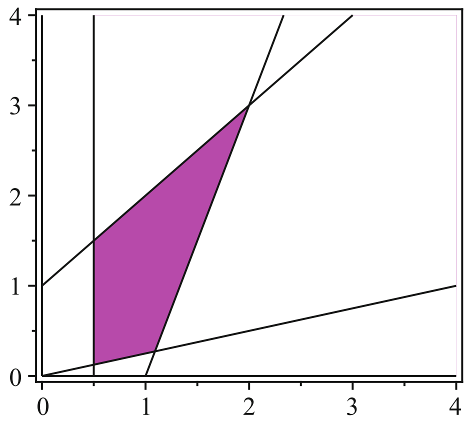


Fig. 4 The optimal solution set of the ILP problem of Example 1



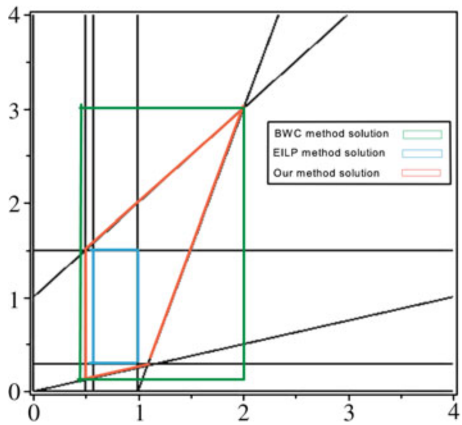


Fig. 5 Solution regions by the BWC, EILP and our methods of Example 1

$$\begin{aligned} \min\ & \underline{z} = 3\underline{x}_1 + 3.5\underline{x}_2 \\ s.t.\ & \underline{x}_1 - 2\underline{x}_2 \leq 0 \\ & -4\underline{x}_1 + \underline{x}_2 \leq -2 \\ & \underline{x}_1, \underline{x}_2 \geq 0 \end{aligned} \qquad \begin{aligned} \min\ & \overline{z} = 3\overline{x}_1 + 3.5\overline{x}_2 \\ s.t.\ & 0.5\overline{x}_1 - \overline{x}_2 \leq -1 \\ & -3\overline{x}_1 + 0\overline{x}_2 \leq -3 \\ & \overline{x}_1 \geq 0.57143, \quad \overline{x}_2 \geq 0.28571 \\ & 0.5\overline{x}_1 \leq 1 \times 0.57143, \quad \overline{x}_1, \overline{x}_2 \geq 0. \end{aligned}$$

The optimal solutions are $[\underline{x}_1^*, \overline{x}_1^*] = [0.57143, 1]$, $[\underline{x}_2^*, \overline{x}_2^*] = [0.28571, 1.5]$ and $[\underline{z}^*, \overline{z}^*] = [2.71, 8.25]$. The solution regions by the BWC, EILP and our methods have been shown in Fig. 5. By comparing these methods we conclude that in the BWC solution region there are infeasible solutions and in the EILP solution region, we lose some of the optimal solutions.

Example 2 Let the ILP problem as follows:

$$\begin{aligned} \min\ & z = 2x_1 + 3x_2 \\ s.t.\ & [0, 1]x_1 \leq 10 \\ & -3x_1 + 2x_2 \leq -5 \\ & [\frac{1}{10}, \frac{9}{10}]x_1 + [-2, -1]x_2 \leq -2 \\ & x_1, x_2 \geq 0. \end{aligned}$$

The best and the worst values of the objective function are 8.190 and 41.250. The optimal solutions are $\underline{\mathbf{x}}^* = \begin{pmatrix} 2.414 \\ 1.121 \end{pmatrix}$ and $\overline{\mathbf{x}}^* = \begin{pmatrix} 7.5 \\ 8.75 \end{pmatrix}$. We solve the above problem for different values of μ_{11} and μ_{3j} in [0, 1], then all of the optimal solution points set solution region of the ILP which is shown in Fig. 6. The second and third constraints of the worst problem are active at $\overline{\mathbf{x}}^*$, so we can use Theorem 3.2, and we derive a region shown in Fig. 9 which all of the its points are optimal and it is a subset of the optimal solution set of the above ILP. The feasible regions of the best and worst

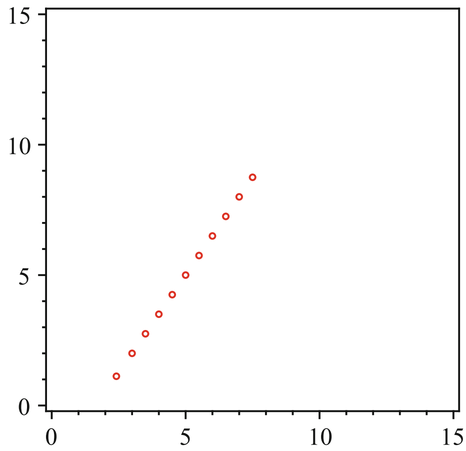


Fig. 6 Some of the optimal solutions of the convex combination problem of Example 2

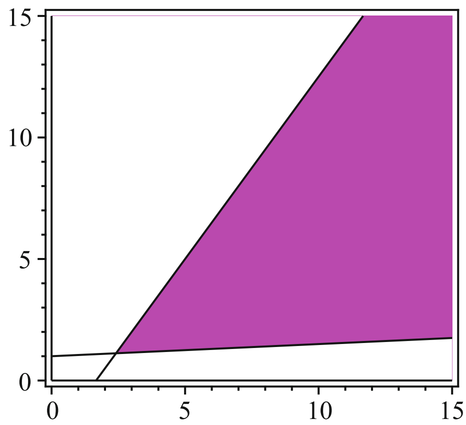


Fig. 7 Solution region of the best problem of Example 2

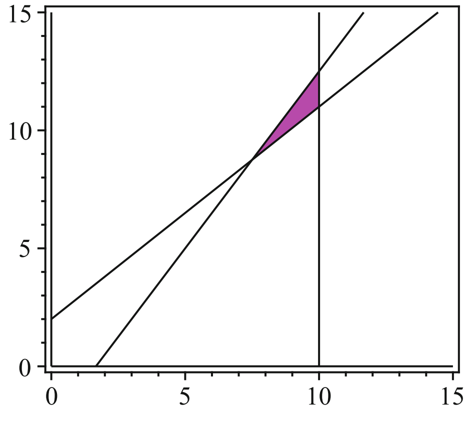


Fig. 8 Solution region of the worst problem of Example 2

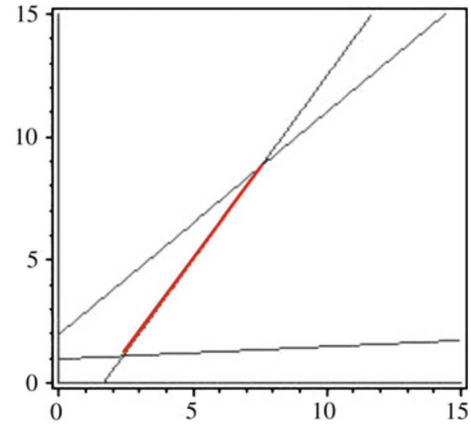


Fig. 9 The subset of the optimal solution set of Example 2 resulted from Theorem 3.2

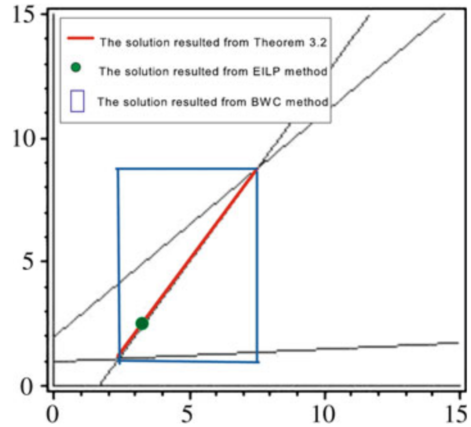


Fig. 10 Solution regions by the BWC, EILP and our methods of Example 2

problems have been shown in Figs. 7 and 8. Now, we solve the problem by the EILP model. Two submodels of the EILP method are as follows:

$$\begin{aligned} \min \quad & \underline{z} = 2\underline{x}_1 + 3\underline{x}_2 \\ s.t. \quad & \underline{x}_1 \le 10 \\ & -3\underline{x}_1 + 2\underline{x}_2 \le -5 \\ & \frac{9}{10}\underline{x}_1 - 2\underline{x}_2 \le -2 \\ & \underline{x}_1, \underline{x}_2 \ge 0 \end{aligned} \qquad \begin{aligned} \min \quad & \overline{z} = 2\overline{x}_1 + 3\overline{x}_2 \\ s.t. \quad & 0\overline{x}_1 \le 10 \\ & -3\overline{x}_1 + 2\overline{x}_2 \le -5 \\ & \frac{1}{10}\overline{x}_1 - \overline{x}_2 \le -2 \\ & \overline{x}_1 \ge 3.333, \quad \overline{x}_2 \ge 2.5 \\ & \overline{x}_1 \le 30, \; \overline{x}_2 \le 2.5, \; \overline{x}_1, \overline{x}_2 \ge 0. \end{aligned}$$

The optimal solutions are $\underline{x}_1^* = \overline{x}_1^* = 3.333$, $\underline{x}_2^* = \overline{x}_2^* = 2.5$ and $\underline{z}^* = \overline{z}^* = 14.167$. The solution region by the BWC, EILP and our method have been shown in Fig 10. By comparing these methods we conclude that in the BWC solution region there are infeasible solutions and in the EILP solution region, we derive a point and we lose some of the optimal solutions.

6 Conclusion

A method has been proposed to determine the optimal solution set of the interval linear programming (ILP) problem. In most of the methods for solving the ILP problem, the optimal value of the objective function of the ILP problem has been obtained. In this paper, we proposed a method to determine the optimal solution set of the ILP problem, by the best and the worst case (BWC) methods, when the feasible solution components of the best problem are positive. Since, B-stability is important in obtaining the optimal values range, then we suppose that the ILP problem is B-stable. Replacing different values of λ_j, μ_{ij} and μ_i in [0, 1] in the convex combination problem, their solving and so obtaining the optimal solution set have been considered. Also, we compared our method with the EILP model and we have shown in the EILP model we lose some of the optimal solutions, whereas in our method, when $m = n$, we can obtain all of the optimal solutions of the ILP problem and when $m > n$, we derive the sufficient condition in Theorem 3.2 for obtaining the optimal solutions set.

Some of the other cases for obtaining the optimal solution set of the ILP which can be studied in next research consists of obtaining the necessary condition in the case $m > n$ and determining the optimal solution set of the ILP when some of the components in the optimal solution of the best or worst problems are zero.

Acknowledgments The authors thank the two anonymous reviewers for the useful remarks and valuable suggestions which helped to improve the first version of the paper.

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