Optim Lett (2013) 7:1929–1955
DOI 10.1007/s11590-012-0516-2

ORIGINAL PAPER

Algorithms for solutions of extended general mixed variational inequalities and fixed points

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Received: 26 October 2011 / Accepted: 12 June 2012 / Published online: 4 July 2012

Abstract In this paper, we introduce and study a new class of extended general nonlinear mixed variational inequalities and a new class of extended general resolvent equations and establish the equivalence between the extended general nonlinear mixed variational inequalities and implicit fixed point problems as well as the extended general resolvent equations. Then by using this equivalent formulation, we discuss the existence and uniqueness of solution of the problem of extended general nonlinear mixed variational inequalities. Applying the aforesaid equivalent alternative formulation and a nearly uniformly Lipschitzian mapping S, we construct some new resolvent iterative algorithms for finding an element of set of the fixed points of nearly uniformly Lipschitzian mapping S which is the unique solution of the problem of extended general nonlinear mixed variational inequalities. We study convergence analysis of the suggested iterative schemes under some suitable conditions. We also suggest and analyze a class of extended general resolvent dynamical systems associated with the extended general nonlinear mixed variational inequalities and show that the trajectory of the solution of the extended general resolvent dynamical system converges globally exponentially to the unique solution of the extended general nonlinear mixed variational inequalities. The results presented in this paper extend and improve some known results in the literature.

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Keywords Variational inequalities · Nearly uniformly Lipschitzian mappings · Fixed point problems · Resolvent operator · Dynamical systems · Convergence

Mathematics Subject Classification (2010) Primary 47H05; Secondary 47J20 · 49J40 · 90C33

1 Introduction

Variational inequalities theory, which was introduced by Stampacchia [53], arise in various models for a large number of mathematical, physical, regional, social, engineering and other problems. The ideas and techniques of the variational inequalities are being applied in a variety of diverse areas of sciences and proved to be productive and innovative. It has been shown that this theory provides a simple, natural and unified framework for a general treatment of unrelated problems. In recent years, considerable interest has been shown in developing various extensions and generalizations of variational inequalities, both for their own sake and for their applications. Variational inequalities have been generalized and extended in different directions using the novel and innovative techniques. An important and useful generalization is called the mixed variational inequality, or the variational inequality of the second kind, involving the nonlinear term. For the application and numerical methods, see [1,3,4,6–9,11,14,19,20,22,27,40–42,48,55] and references therein. In recent years, much attention has been given to develop efficient and implementable numerical methods including projection method and its variant forms, Wiener–Hopf (normal) equations, linear approximation, auxiliary principle and descent framework for solving equilibrium problems, variational inequalities and related optimization problems (see, for example, [10,12,13,60–62]). It is well known that the projection methods, its variant forms and Wiener–Hopf equations technique cannot be used to suggest and analyze iterative methods for solving mixed variational inequalities due to the presence of the nonlinear term. These facts motivated us to use the technique of resolvent operators, the origin of which can be traced back to Martinet [25] and Brézis [4]. In this technique, the given operator is decomposed into the sum of two (or more) maximal monotone operators, whose resolvent are easier to evaluate than the resolvent of the original operator. Such a method is known as the operator splitting method. This can lead to development of very efficient methods, since one can treat each part of the original operator independently. The operator splitting methods and related techniques have been analyzed and studied by many authors including Peaceman and Rachford [49], Lions and Mercier [23], Glowinski and Tallec [17], and Tseng [54]. For an excellent account of the alternating direction implicit (splitting) methods, see [2]. In the context of the mixed variational inequalities, Noor [37,40,47] has used the resolvent operator technique to suggest some splitting type methods. A useful feature of the forward–backward splitting method for solving the mixed variational inequalities is that the resolvent step involves the subdifferential of the proper, convex and lower semicontinuous part only and the other part facilitates the problem decomposition.

Equally important is the area of mathematical sciences known as the resolvent equations, which was introduced by Noor [30]. Noor [30] has established the equivalence between the mixed variational inequalities and the resolvent equations using essentially the resolvent operator technique. The resolvent equations are being used to develop powerful and efficient numerical methods for solving the mixed variational inequalities and related optimization problems, see [40,46,47] and the references therein. It is worth mentioning that if the nonlinear term involving the mixed variational inequalities is the indicator function of a closed convex set in a Hilbert space, then the resolvent operator is equal to the projection operator.

Also, in recent years, much attention has been given to consider and analyze the projected dynamical systems associated with variational inequalities and nonlinear programming problems, in which the right-hand side of the ordinary differential equation is a projection operator. Such types of the projected dynamical systems were introduced and studied by Dupuis and Nagurney [15]. Projected dynamical systems are characterized by a discontinuous right-hand side. The discontinuity arises from the constraint governing the question. The innovative and novel feature of a projected dynamical system is that its set of stationary points corresponds to the set of solutions of the corresponding variational inequality problems. Hence, the equilibrium and nonlinear problems arising in various branches of pure and applied sciences, which can be formulated in the setting of variational inequalities, can now be studied in the more general setting of the projected dynamical systems. It has been shown in [14–16,27,58,59,63] that the dynamical systems are useful in developing efficient and powerful numerical technique for solving variational inequalities and related optimization problems. Xia and Wang [58,59], Zhang and Nagurney [63] and Nagurney and Zhang [27] have studied the globally asymptotic stability of these projected dynamical systems. Noor [28,37,47] has also suggested and analyzed similar resolvent dynamical systems for mixed variational inequalities by extending and modifying their techniques.

On the other hand, related to the variational inequalities, we have the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis. It is natural to consider a unified approach to these two different problems. Motivated and inspired by the research going in this direction, Noor and Huang [44] considered the problem of finding the common element of the set of the solutions of variational inequalities and the set of the fixed points of the nonexpansive mappings. Noor [32] suggested and analyzed some three-step iterative algorithms for finding the common elements of the set of the solutions of the Noor variational inequalities and the set of the fixed points of nonexpansive mappings. He also proved the convergence analysis of the suggested iterative algorithms under some suitable conditions.

It is well known that every nonexpansive mapping is a Lipschitzian mapping. Lipschitzian mappings have been generalized by various authors. Sahu [51] introduced and investigated nearly uniformly Lipschitzian mappings as a generalization of Lipschitzian mappings.

Motivated by recent works going in this direction, in this paper, we introduce and study a new class of the extended general nonlinear mixed variational inequalities and a new class of the extended general resolvent equations. We prove the equivalence between the extended general nonlinear mixed variational inequalities and the fixed point problems as well as the extended general resolvent equations. Then by using this equivalent formulation, we discuss the existence and uniqueness of solution of the problem of extended general nonlinear mixed variational inequalities. Applying the equivalent alternative formulation and a nearly uniformly Lipschitzian mapping S, we construct some new resolvent iterative algorithms for finding an element of set of the fixed points of nearly uniformly Lipschitzian mapping S which is the unique solution of the problem of extended general nonlinear mixed variational inequalities. The convergence analysis of the suggested iterative methods under some suitable conditions are proved. We also suggest and analyze a class of extended general resolvent dynamical systems associated with the extended general nonlinear mixed variational inequalities and show that the trajectory of the solution of the extended general resolvent dynamical system converges globally exponentially to the unique solution of the extended general nonlinear mixed variational inequalities. Our results improve and extend the corresponding results of [36,47] and many other recent works.

2 Formulations and basic facts

Throughout this article, we will let $\mathcal{H}$ be a real Hilbert space which is equipped with an inner product $\langle .,.\rangle$ and corresponding norm $\|.\|$. Let $T, g, h : \mathcal{H} \to \mathcal{H}$ be three nonlinear single-valued operators and let $\partial\varphi$ denote the subdifferential of function φ, where $\varphi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semicontinuous function on $\mathcal{H}$. For any given constant $\rho > 0$, we consider the problem of finding $u \in \mathcal{H}$ such that

$$\langle \rho T(u) + h(u) - g(u), g(v) - h(u)\rangle \geq \rho\varphi(u) - \rho\varphi(g(v)), \quad \forall v \in \mathcal{H}, \tag{2.1}$$

which is called the *extended general nonlinear mixed variational inequality involving three different nonlinear operators* (EGNMVID).

If $h \equiv I$, the identity operator, then the problem (2.1) reduces to the problem of finding $u \in \mathcal{H}$ such that

$$\langle \rho T(u) + u - g(u), g(v) - u\rangle \geq \rho\varphi(u) - \rho\varphi(g(v)), \quad \forall v \in \mathcal{H}. \tag{2.2}$$

The problem (2.2) is called the *general nonlinear mixed variational inequality* and has been introduced and studied by Noor et al. [47].

Some special cases of the problem (2.1) are introduced and studied by Noor [30,31,33,36], Noor et al. [47] and Stampacchia [53].

Definition 2.1 A set-valued operator $T : \mathcal{H} \multimap \mathcal{H}$ is said to be *monotone* if, for any $x, y \in \mathcal{H}$

$$\langle u - v, x - y\rangle \geq 0, \quad \forall u \in T(x), v \in T(y).$$

A monotone set-valued operator T is called *maximal* if its graph, $\text{Gph}(T) := \{(x, y) \in \mathcal{H} \times \mathcal{H} : y \in T(x)\}$, is not properly contained in the graph of any other monotone operator. It is well-known that T is a maximal monotone operator if and only if $(I + \lambda T)(\mathcal{H}) = \mathcal{H}$, for all $\lambda > 0$, where I denotes the identity operator on $\mathcal{H}$.

Definition 2.2 [4] For any maximal monotone operator T, the resolvent operator associated with T of parameter λ is defined as follows:

$$J_T^\lambda(u) = (I + \lambda T)^{-1}(u), \quad \forall u \in \mathcal{H}.$$

It is single-valued and nonexpansive, that is,

$$\|J_T^\lambda(u) - J_T^\lambda(v)\| \leq \|u - v\|, \quad \forall u, v \in \mathcal{H}.$$

If φ is a proper, convex and lower-semicontinuous function, then its subdifferential $\partial\varphi$ is a maximal monotone operator. In this case, we can define the resolvent operator associated with the subdifferential $\partial\varphi$ of parameter λ as follows:

$$J_\varphi^\lambda(u) = (I + \lambda\partial\varphi)^{-1}(u), \quad \forall u \in \mathcal{H}.$$

The resolvent operator J_φ^λ has the following useful characterization.

Lemma 2.3 *For any $z \in \mathcal{H}$, $x \in \mathcal{H}$ satisfies the inequality*

$$\langle x - z, y - x\rangle + \lambda\varphi(y) - \lambda\varphi(x) \geq 0, \quad \forall y \in \mathcal{H},$$

if and only if $x = J_\varphi^\lambda(z)$, where J_φ^λ is the resolvent operator associated with $\partial\varphi$ of parameter $\lambda > 0$.

It is well known that J_φ^λ is nonexpansive, that is,

$$\|J_\varphi^\lambda(u) - J_\varphi^\lambda(v)\| \leq \|u - v\|, \quad \forall u, v \in \mathcal{H}.$$

Definition 2.4 A single-valued operator $T : \mathcal{H} \to \mathcal{H}$ is called:

(a) *monotone* if

$$\langle T(x) - T(y), x - y\rangle \geq 0, \quad \forall x, y \in \mathcal{H};$$

(b) *r-strongly monotone* if there exists a constant $r > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq r\|x - y\|^2, \quad \forall x, y \in \mathcal{H};$$

(c) *k-strongly monotone with respect to g* if there exists a constant $k > 0$ such that

$$\langle T(x) - T(y), g(x) - g(y)\rangle \geq k\|x - y\|^2, \quad \forall x, y \in \mathcal{H};$$

(d) *γ-Lipschitz continuous* if there exists a constant $\gamma > 0$ such that

$$\|T(x) - T(y)\| \leq \gamma\|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

3 Existence of solution and uniqueness

This section is concerned with the establish of the existence and uniqueness theorem for solution of the extended general nonlinear mixed variational inequality (2.1). For this end, we need the following lemma in which the equivalence between the extended general nonlinear mixed variational inequality (2.1) and fixed point problem is stated.

Lemma 3.1 *Let T, g, h and $\rho > 0$ be the same as in the problem (2.1). Then $u \in \mathcal{H}$ is a solution of the problem (2.1) if and only if*

$$h(u) = J_{\varphi}^{\rho}(g(u) - \rho T(u)), \tag{3.1}$$

where J_{φ}^{ρ} is the resolvent operator associated with $\partial\varphi$ of parameter ρ.

In view of Lemma 3.1, the extended general mixed variational inequality (2.1) and the fixed point problem (3.1) are equivalent. This equivalent formulation is very useful from numerical and theoretical points of view. In the next theorem, by using this alternative fixed point formulation, we discuss the existence and uniqueness of solution of the extended general mixed variational inequality (2.1).

Theorem 3.2 *Let T, g, h and ρ be the same as in the problem (2.1) and suppose further that T is κ-strongly monotone with respect to g and ξ-Lipschitz continuous, h is π-strongly monotone and ϱ-Lipschitz continuous and g is θ-Lipschitz continuous. If the constant $\rho > 0$ satisfies the following condition:*

$$\begin{cases} |\rho - \frac{\kappa}{\xi^2}| < \frac{\sqrt{\kappa^2 - \xi^2(\theta^2 - (1-\mu)^2)}}{\xi^2}, \\ \kappa > \xi\sqrt{\theta^2 - (1-\mu)^2}, \\ \mu = \sqrt{1 - (2\pi - \varrho^2)} < 1, \\ 2\pi < 1 + \varrho^2, \quad \theta + \mu > 1, \end{cases} \tag{3.2}$$

then the problem (2.1) admits a unique solution.

Proof Define the mapping $F : \mathcal{H} \to \mathcal{H}$ by

$$F(x) = x - h(x) + J_{\varphi}^{\rho}(g(x) - \rho T(x)), \quad \forall x \in \mathcal{H}. \tag{3.3}$$

Now, we establish that the mapping F is a contraction. For this end, let $x, x' \in \mathcal{H}$ be given. By using (3.3), since the resolvent operator J_{φ}^{ρ} is nonexpansive, we get

$$\begin{aligned} &\|F(x) - F(x')\| \\ &\quad \le \|x - x' - (h(x) - h(x'))\| + \|J_{\varphi}^{\rho}(g(x) - \rho T(x)) - J_{\varphi}^{\rho}(g(x') - \rho T(x'))\| \\ &\quad \le \|x - x' - (h(x) - h(x'))\| + \|g(x) - g(x') - \rho(T(x) - T(x'))\|. \end{aligned} \tag{3.4}$$

It follows from π-strongly monotonicity and ϱ-Lipschtz continuity of h that

$$\begin{aligned}
&\|x - x' - (h(x) - h(x'))\|^2 \\
&\quad = \|x - x'\|^2 - 2\langle h(x) - h(x'), x - x'\rangle + \|h(x) - h(x'\|^2 \\
&\quad \leq (1 - 2\pi)\|x - x'\|^2 + \|h(x) - h(x')\|^2 \\
&\quad \leq (1 - 2\pi + \varrho^2)\|x - x'\|^2. \qquad (3.5)
\end{aligned}$$

Since T is κ-strongly monotone with respect to g and ξ-Lipschitz continuous and g is θ-Lipschitz continuous, we have

$$\begin{aligned}
&\|g(x) - g(x') - \rho(T(x) - T(x'))\|^2 \\
&\quad = \|g(x) - g(x')\|^2 - 2\rho\langle T(x) - T(x'), g(x) - g(x')\rangle + \rho^2\|T(x) - T(x')\|^2 \\
&\quad \leq (\theta^2 - 2\rho\kappa + \rho^2\xi^2)\|x - x'\|^2. \qquad (3.6)
\end{aligned}$$

Combining (3.4)–(3.6), we obtain

$$\|F(x) - F(x')\| \leq \psi\|x - x'\|, \qquad (3.7)$$

where

$$\psi = \sqrt{1 - 2\pi + \varrho^2} + \sqrt{\theta^2 - 2\rho\kappa + \rho^2\xi^2}. \qquad (3.8)$$

Using the condition (3.2), we note that $0 \leq \psi < 1$ and so the inequality (3.7) implies that the mapping F is contraction. By Banach's fixed point theorem, F has a unique fixed point in $\mathcal{H}$, that is, there exists a unique point $u \in \mathcal{H}$ such that $F(u) = u$. From (3.3), it follows that $h(u) = J_{\varphi}^{\rho}(g(u) - \rho T(u))$. Now, Lemma 3.1 guarantees that $u \in \mathcal{H}$ is a unique solution of the problem (2.1). This completes the proof.

4 Nearly uniformly Lipschitzian mappings and resolvent iterative schemes

In recent years, the nonexpansive mappings have been generalized and investigated by various authors. One of these generalizations is class of the nearly uniformly Lipschitzian mappings. In this section, we first recall some generalizations of the nonexpansive mappings which have been introduced in recent years and present some new and interesting examples to show relations between these mappings. Then, we use a nearly uniformly Lipschitzian mapping S and the equivalent alternative formulation (2.1) to suggest and analyze some new resolvent iterative algorithms for finding an element of the set of the fixed points S which is the unique solution of the problem of extended general nonlinear mixed variational inequality (2.1). In two next definitions, some generalizations of the nonexpansive mappings are stated.

Definition 4.1 A nonlinear mapping $T : \mathcal{H} \to \mathcal{H}$ is called

(a) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in \mathcal{H}$;

(b) *L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in \mathcal{H};$$

(c) *generalized Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L(\|x - y\| + 1), \quad \forall x, y \in \mathcal{H};$$

(d) *generalized (L, M)-Lipschitzian* [51] if there exist two constants $L, M > 0$ such that

$$\|Tx - Ty\| \leq L(\|x - y\| + M), \quad \forall x, y \in \mathcal{H};$$

(e) *asymptotically nonexpansive* [18] if there exists a sequence $\{k_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for each $n \in \mathbb{N}$,

$$\|T^n x - T^n y\| \leq k_n\|x - y\|, \quad \forall x, y \in \mathcal{H};$$

(f) *pointwise asymptotically nonexpansive* [21] if, for each integer $n \geq 1$,

$$\|T^n x - T^n y\| \leq \alpha_n(x)\|x - y\|, \quad x, y \in \mathcal{H},$$

where $\alpha_n \to 1$ pointwise on X;

(g) *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that for each $n \in \mathbb{N}$,

$$\|T^n x - T^n y\| \leq L\|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

Definition 4.2 [51] A nonlinear mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be:

(a) *nearly Lipschitzian* with respect to the sequence $\{a_n\}$ if, for each $n \in \mathbb{N}$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n), \quad \forall x, y \in \mathcal{H}, \tag{4.1}$$

where $\{a_n\}$ is a fix sequence in $[0, \infty)$ with $a_n \to 0$ as $n \to \infty$.
For an arbitrary, but fixed $n \in \mathbb{N}$, the infimum of constants k_n in (4.1) is called *nearly Lipschitz constant* and it is denoted by $\eta(T^n)$. Notice that

$$\eta(T^n) = \sup\left\{\frac{\|T^n x - T^n y\|}{\|x - y\| + a_n} : x, y \in \mathcal{H}, x \neq y\right\}.$$

A nearly Lipschitzian mapping T with the sequence $\{(a_n, \eta(T^n))\}$ is said to be:

(b) *nearly nonexpansive* if $\eta(T^n) = 1$ for all $n \in \mathbb{N}$, that is,

$$\|T^n x - T^n y\| \leq \|x - y\| + a_n, \quad \forall x, y \in \mathcal{H};$$

(c) *nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T^n) = 1$, in other words, $k_n \geq 1$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} k_n = 1$;

(d) *nearly uniformly L-Lipschitzian* if $\eta(T^n) \leq L$ for all $n \in \mathbb{N}$, in other words, $k_n = L$ for all $n \in \mathbb{N}$.

Remark 4.3 It should be pointed that

(1) Every nonexpansive mapping is a asymptotically nonexpansive mapping and every asymptotically nonexpansive mapping is a pointwise asymptotically nonexpansive mapping. Also, the class of Lipschitzian mappings properly includes the class of pointwise asymptotically nonexpansive mappings.

(2) It is obvious that every Lipschitzian mapping is a generalized Lipschitzian mapping. Furthermore, every mapping with a bounded range is a generalized Lipschitzian mapping. It is easy to see that the class of generalized (L, M)-Lipschitzian mappings is more general than the class of generalized Lipschitzian mappings.

(3) Clearly, the class of nearly uniformly L-Lipschitzian mappings properly includes the class of generalized (L, M)-Lipschitzian mappings and that of uniformly L-Lipschitzian mappings. Note that every nearly asymptotically nonexpansive mapping is nearly uniformly L-Lipschitzian.

Now, we present some new examples to investigate relations between these mappings.

Example 4.4 Let $\mathcal{H} = \mathbb{R}$ and define $T : \mathcal{H} \to \mathcal{H}$ as follow:

$$T(x) = \begin{cases} \frac{1}{\gamma} & x \in [0, \gamma], \\ 0 & x \in (-\infty, 0) \cup (\gamma, \infty), \end{cases}$$

where $\gamma > 1$ is a constant real number. Evidently, the mapping T is discontinuous at the points $x = 0, \gamma$. Since every Lipschitzian mapping is continuous, it follows that T is not Lipschitzian. For each $n \in \mathbb{N}$, take $a_n = \frac{1}{\gamma^n}$. Then

$$|Tx - Ty| \leq |x - y| + \frac{1}{\gamma} = |x - y| + a_1, \quad \forall x, y \in \mathbb{R}.$$

Since $T^n z = \frac{1}{\gamma}$ for all $z \in \mathbb{R}$ and $n \geq 2$, it follows that, for all $x, y \in \mathbb{R}$ and $n \geq 2$,

$$|T^n x - T^n y| \leq |x - y| + \frac{1}{\gamma^n} = |x - y| + a_n.$$

Hence T is a nearly nonexpansive mapping with respect to the sequence $\{a_n\} = \{\frac{1}{\gamma^n}\}$.

The following example shows that the nearly uniformly L-Lipschitzian mappings are not necessarily continuous.

Example 4.5 Let $\mathcal{H} = [0, b]$, where $b \in (0, 1]$ is an arbitrary constant real number and let the self-mapping T of $\mathcal{H}$ be defined as follows:

$$T(x) = \begin{cases} \gamma x & x \in [0, b), \\ 0 & x = b, \end{cases}$$

where $\gamma \in (0, 1)$ is also an arbitrary constant real number. It is plain that the mapping T is discontinuous in the point b. Hence, T is not a Lipschitzian mapping. For each $n \in \mathbb{N}$, take $a_n = \gamma^{n-1}$. Then, for all $n \in \mathbb{N}$ and $x, y \in [0, b)$, we have

$$\begin{aligned} |T^n x - T^n y| &= |\gamma^n x - \gamma^n y| = \gamma^n |x - y| \leq \gamma^n |x - y| + \gamma^n \\ &\leq \gamma |x - y| + \gamma^n = \gamma(|x - y| + a_n). \end{aligned}$$

If $x \in [0, b)$ and $y = b$, then, for each $n \in \mathbb{N}$, we have $T^n x = \gamma^n x$ and $T^n y = 0$. Since $0 < |x - y| \leq b \leq 1$, it follows that, for all $n \in \mathbb{N}$,

$$\begin{aligned} |T^n x - T^n y| &= |\gamma^n x - 0| = \gamma^n x \leq \gamma^n b \leq \gamma^n < \gamma^n |x - y| + \gamma^n \\ &\leq \gamma |x - y| + \gamma^n = \gamma(|x - y| + a_n). \end{aligned}$$

Hence T is a nearly uniformly γ-Lipschitzian mapping with respect to the sequence $\{a_n\} = \{\gamma^{n-1}\}$.

Obviously, every nearly nonexpansive mapping is a nearly uniformly Lipschitzian mapping. In the following example, we show that the class of nearly uniformly Lipschitzian mappings properly includes the class of nearly nonexpansive mappings.

Example 4.6 Let $\mathcal{H} = \mathbb{R}$ and the self-mapping T of $\mathcal{H}$ be defined as follows:

$$T(x) = \begin{cases} \frac{1}{2} & x \in [0, 1) \cup \{2\}, \\ 2 & x = 1, \\ 0 & x \in (-\infty, 0) \cup (1, 2) \cup (2, +\infty). \end{cases}$$

Evidently, the mapping T is discontinuous in the points $x = 0, 1, 2$. Hence T is not a Lipschitzian mapping. For each $n \in \mathbb{N}$, take $a_n = \frac{1}{2^n}$. Then T is not a nearly nonexpansive mapping with respect to the sequence $\{\frac{1}{2^n}\}$ since, taking $x = 1$ and $y = \frac{1}{2}$, we have $Tx = 2$, $Ty = \frac{1}{2}$ and

$$|Tx - Ty| > |x - y| + \frac{1}{2} = |x - y| + a_1.$$

However, it follows that

$$|Tx - Ty| \leq 4\left(|x - y| + \frac{1}{2}\right) = 4(|x - y| + a_1), \quad \forall x, y \in \mathbb{R},$$

and, for all $n \geq 2$,

$$|T^n x - T^n y| \leq 4\left(|x - y| + \frac{1}{2^n}\right) = 4(|x - y| + a_n), \quad \forall x, y \in \mathbb{R},$$

since $T^n z = \frac{1}{2}$ for all $z \in \mathbb{R}$ and $n \geq 2$. Hence, for each $L \geq 4$, T is a nearly uniformly L-Lipschitzian mapping with respect to the sequence $\{\frac{1}{2^n}\}$.

It is clear that every uniformly L-Lipschitzian mapping is a nearly uniformly L-Lipschitzian mapping. In the next example, we show that the class nearly uniformly L-Lipschitzian mappings properly includes the class of uniformly L-Lipschitzian mappings.

Example 4.7 Let $\mathcal{H} = \mathbb{R}$ and the self-mapping T of $\mathcal{H}$ be defined the same as in Example 4.6. Then T is not a uniformly 4-Lipschitzian mapping. If $x = 1$ and $y \in (1, \frac{3}{2})$, then we have $|Tx - Ty| > 4|x - y|$ since $0 < |x - y| < \frac{1}{2}$. But, in view of Example 4.6, T is a nearly uniformly 4-Lipschitzian mapping.

The following example shows that the class of generalized Lipschitzian mappings properly includes the class of Lipschitzian mappings and that of mappings with bounded range.

Example 4.8 [5] Let $\mathcal{H} = \mathbb{R}$ and $T : \mathcal{H} \to \mathcal{H}$ be defined by

$$T(x) = \begin{cases} x - 1 & x \in (-\infty, -1), \\ x - \sqrt{1 - (x + 1)^2} & x \in [-1, 0), \\ x + \sqrt{1 - (x - 1)^2} & x \in [0, 1], \\ x + 1 & x \in (1, \infty). \end{cases}$$

Then T is a generalized Lipschitzian mapping which is not Lipschitzian and its range is not bounded.

Let $S : \mathcal{H} \to \mathcal{H}$ be a nearly uniformly Lipschitzian mapping. We denote the set of all the fixed points of S by $\mathrm{Fix}(S)$ and the set of all the solutions of the problem (2.1) by $\mathrm{EGNMVID}(\mathcal{H}, T, g, h)$. We now characterize the problem. If $u \in \mathrm{Fix}(S) \cap \mathrm{EGNMVID}(\mathcal{H}, T, g, h)$, then it follows from Lemma 3.1 that, for each $n \geq 0$,

$$\begin{aligned} u = S^n u &= u - h(u) + J_{\varphi}^{\rho}(g(u) - \rho T(u)) \\ &= S^n\{u - h(u) + J_{\varphi}^{\rho}(g(u) - \rho T(u))\}. \end{aligned} \tag{4.2}$$

The fixed point formulation (4.2) enables us to define the following resolvent iterative algorithms for finding a common element of two different sets of solutions of the fixed points of the nearly uniformly Lipschitzian mapping S and the extended general nonlinear mixed variational inequality (2.1).

Algorithm 4.9 Let T, g, h and ρ be the same as in the problem (2.1). For arbitrary chosen initial point $u_0 \in \mathcal{H}$, compute the iterative sequence $\{u_n\}_{n=0}^{\infty}$ by the iterative process

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n S^n\{u_n - h(u_n) + J_{\varphi}^{\rho}(g(u_n) - \rho T(u_n))\}, \tag{4.3}$$

where $S : \mathcal{H} \to \mathcal{H}$ is a nearly uniformly Lipschitzian mapping and $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in the interval [0, 1] with $\sum_{n=0}^{\infty} \alpha_n = \infty$.

If $S \equiv I$, then Algorithm 4.9 reduces to the following algorithm.

Algorithm 4.10 Assume that T, g, h and ρ are the same as in the problem (2.1). For arbitrary chosen initial point $u_0 \in \mathcal{H}$, compute the iterative sequence $\{u_n\}_{n=0}^{\infty}$ by the iterative process

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n\{u_n - h(u_n) + J_{\varphi}^{\rho}(g(u_n) - \rho T(u_n))\},$$

where the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is the same as in Algorithm 4.9.

If $h \equiv I$, then Algorithm 4.9 collapses to the following algorithm.

Algorithm 4.11 Suppose that T, g and ρ are the same as in the problem (2.2). For arbitrary chosen initial point $u_0 \in \mathcal{H}$, compute the iterative sequence $\{u_n\}_{n=0}^{\infty}$ by the iterative process

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n S^n J_{\varphi}^{\rho}(g(u_n) - \rho T(u_n)),$$

where the mapping S and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ are the same as in Algorithm 4.9.

If $S \equiv I$, then Algorithm 4.11 reduces to the following algorithm.

Algorithm 4.12 Let T, g and ρ be the same as in the problem (2.2). For arbitrary chosen initial point $u_0 \in \mathcal{H}$, compute the iterative sequence $\{u_n\}_{n=0}^{\infty}$ by the iterative process

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n J_{\varphi}^{\rho}(g(u_n) - \rho T(u_n)),$$

where the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is the same as in Algorithm 4.9.

If $\varphi(x) = \delta_K(x)$ for all $x \in K$, where δ_K is the indicator function of a nonempty closed convex set K in $\mathcal{H}$ defined by

$$\delta_K(y) = \begin{cases} 0 & y \in K, \\ \infty & y \notin K, \end{cases} \tag{4.4}$$

then $J_{\varphi}^{\rho} \equiv P_K$, the projection of $\mathcal{H}$ onto K. Accordingly, Algorithm 4.9 reduces to the following algorithm.

Algorithm 4.13 Suppose that T, g, h and ρ are the same as in the problem (2.1). For arbitrary chosen initial point $u_0 \in \mathcal{H}$, compute the iterative sequence $\{u_n\}_{n=0}^{\infty}$ by the iterative process

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n S^n\{u_n - h(u_n) + P_K(g(u_n) - \rho T(u_n))\},$$

where the mapping S and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ are the same as in Algorithm 4.9.

Remark 4.14 Algorithms 3.1 and 3.2 in [36] and Algorithms 3.1–3.3 in [47] are special cases of Algorithms 4.9–4.13. In brief, for a suitable and appropriate choice of the operators T, g, h, S and the constant $\rho > 0$, one can obtain a number of new and previously known iterative schemes, see for example the introduced algorithms in [31,35,43]. This clearly shows that Algorithms 4.9–4.13 are quite general and unifying.

5 Convergence analysis

In this section, we verify the convergence analysis of the suggested iterative Algorithm 4.9 under some suitable conditions. For this end, we need the following lemma.

Lemma 5.1 [56] *Let* $\{a_n\}$ *be a nonnegative real sequence and* $\{b_n\}$ *be a real sequence in* $[0, 1]$ *such that* $\sum_{n=0}^{\infty} b_n = \infty$. *If there exists a positive integer* n_0 *such that*

$$a_{n+1} \leq (1 - b_n)a_n + b_n c_n, \quad \forall n \geq n_0,$$

where $c_n \geq 0$ *for all* $n \geq 0$ *and* $\lim_{n\to\infty} c_n = 0$, *then* $\lim_{n\to 0} a_n = 0$.

Theorem 5.2 *Let* T, g, h *and* ρ *be the same as in* Theorem 3.2 *and suppose that all the conditions* Theorem 3.2 *hold. Suppose that* $S : \mathcal{H} \to \mathcal{H}$ *is a nearly uniformly* L*-Lipschitzian mapping with the sequence* $\{b_n\}_{n=0}^{\infty}$ *such that* $\mathrm{Fix}(S) \cap \mathrm{EGNMVID}(\mathcal{H}, T, g, h) \neq \emptyset$. *Further, let* $L\psi < 1$, *where* ψ *is the same as in* (3.8). *Then the iterative sequence* $\{u_n\}_{n=0}^{\infty}$ *generated by* Algorithm 4.9 *converges strongly to the only element* u^* *of* $\mathrm{Fix}(S) \cap \mathrm{EGNMVID}(\mathcal{H}, T, g, h)$.

Proof In view of Theorem 3.2, the problem (2.1) has a unique solution $u^* \in \mathcal{H}$. Hence, from Lemma 3.1, it follows that $h(u^*) = J_{\varphi}^{\rho}(g(u^*)–\rho T(u^*))$. Since $\mathrm{EGNMVID}(\mathcal{H}, T, g, h)$ is a singleton set, it follows from $\mathrm{Fix}(S) \cap \mathrm{EGNMVID}(\mathcal{H}, T, g, h) \neq \emptyset$ that $u^* \in \mathrm{Fix}(S)$. Therefore, for each $n \geq 0$, we can write

$$u^* = (1 - \alpha_n)u^* + \alpha_n S^n\{u^* - h(u^*) + J_{\varphi}^{\rho}(g(u^*) - \rho T(u^*))\}, \tag{5.1}$$

where the sequences $\{\alpha_n\}_{n=0}^{\infty}$ is the same as in Algorithm 4.9. Applying (4.3) and (5.1), since the resolvent operator J_{φ}^{ρ} is nonexpansive, we have

$$\begin{aligned}\|u_{n+1}-u^*\| &\leq (1-\alpha_n)\|u_n-u^*\|+\alpha_n\|S^n\{u_n-h(u_n)+J_{\varphi}^{\rho}(g(u_n)-\rho T(u_n))\} \\ &\quad -S^n\{u^*-h(u^*)+J_{\varphi}^{\rho}(g(u^*)-\rho T(u^*))\}\| \\ &\leq (1-\alpha_n)\|u_n-u^*\|+\alpha_n L(\|u_n-u^*-(h(u_n)-h(u^*))\| \\ &\quad +\|J_{\varphi}^{\rho}(g(u_n)-\rho T(u_n))-J_{\varphi}^{\rho}(g(u^*)-\rho T(u^*))\|+b_n) \\ &\leq (1-\alpha_n)\|u_n-u^*\|+\alpha_n L(\|u_n-u^*-(h(u_n)-h(u^*))\| \\ &\quad +\|g(u_n)-g(u^*)-\rho(T(u_n)-T(u^*))\|+b_n). \end{aligned} \tag{5.2}$$

Since T is κ-strongly monotone with respect to g and ξ-Lipschitz continuous, g is θ-Lipschitz continuous, h is π-strongly monotone and ϱ-Lipschitz continuous, like in the proofs of (3.5) and (3.6), we have

$$\|u_n-u^*-(h(u_n)-h(u^*))\| \leq \sqrt{1-2\pi+\varrho^2}\|u_n-u^*\| \tag{5.3}$$

and

$$\|g(u_n)-g(u^*)-\rho(T(u_n)-T(u^*))\| \leq \sqrt{\theta^2-2\rho\kappa+\rho^2\xi^2}\|u_n-u^*\|. \tag{5.4}$$

Substituting (5.3) and (5.4) in (5.2), it follows that

$$\begin{aligned}\|u_{n+1}-u^*\| &\leq (1-\alpha_n)\|u_n-u^*\|+\alpha_n L\psi\|u_n-u^*\|+\alpha_n Lb_n \\ &= (1-\alpha_n(1-L\psi))\|u_n-u^*\|+\alpha_n(1-L\psi)\frac{Lb_n}{1-L\psi}, \end{aligned} \tag{5.5}$$

where ψ is the same as in (3.8). Since $L\psi < 1$ and $\lim_{n\to\infty} b_n = 0$, we note that all the conditions of Lemma 5.1 are satisfied and so, from Lemma 5.1 and (5.5), it follows that $u_n \to u^*$, as $n \to \infty$. Hence the sequence $\{u_n\}_{n=0}^{\infty}$ generated by Algorithm 4.9 converges strongly to the unique solution u^* of the problem (2.1), that is, the only element u^* of $\mathrm{Fix}(S) \cap \mathrm{EGNMVID}(\mathcal{H}, T, g, h)$. This completes the proof.

Theorem 5.3 *Suppose that T, g, h and ρ are the same as in* Theorem 3.2 *and all the conditions of* Theorem 3.2 *hold. Then the iterative sequence $\{u_n\}_{n=0}^{\infty}$ generated by* Algorithm 4.10 *converges strongly to the unique solution of the problem* (2.1).

Like in the proof of Theorem 5.2, one can prove the convergence of iterative sequences generated by Algorithms 4.11–4.13.

6 Resolvent equation technique and iterative algorithms

This section is concerned with the introduce of a new class of extended general resolvent equations and the establish of equivalence between the aforesaid class and the class of extended general nonlinear mixed variational inequalities (2.1). Also, by using the obtained equivalence, some new perturbed resolvent iterative algorithms for solving the problem (2.1) are suggested and analyzed.

Let T, g, h and ρ be the same as in the problem (2.1) and suppose that the inverse of the operator h exists. Associated with the problem (2.1), the problem of finding $z \in \mathcal{H}$ such that

$$Th^{-1}J_{\varphi}^{\rho}z + \rho^{-1}R_{\varphi}z = 0, \tag{6.1}$$

where $R_{\varphi} = I - gh^{-1}J_{\varphi}^{\rho}$ with I the identity operator is considered.

The problem (6.1) is called the *extended general resolvent equation* (EGRE) associated with the problem of extended general nonlinear mixed variational inequality (2.1). Forward, we denote by EGRE$(\mathcal{H}, T, g, h)$ the set of all the solutions of the extended general resolvent equation (6.1).

Now, we state some special cases of the problem (6.1).

If $h \equiv I$, then the problem (6.1) reduces to the problem of finding $z \in \mathcal{H}$ such that

$$TJ_{\varphi}^{\rho}z + \rho^{-1}R_{\varphi}z = 0, \tag{6.2}$$

where $R_{\varphi} = I - gJ_{\varphi}^{\rho}$ and it is called the *general resolvent equation* associated with the problem of general nonlinear mixed variational inequality (2.2). The problem (6.2) is introduced and studied by Noor et al. [47].

If $g \equiv I$, then the problem (6.2) is equivalent to finding $z \in \mathcal{H}$ such that

$$TJ_{\varphi}^{\rho}z + \rho^{-1}R_{\varphi}z = 0, \tag{6.3}$$

where $R_{\varphi} = I - J_{\varphi}^{\rho}$. The Eq. (6.3) is the original resolvent equation mainly due to Noor [42].

If $\varphi(x) = \delta_K(x)$ for all $x \in K$, where δ_K is the indicator function of a nonempty closed convex set K in $\mathcal{H}$ defined as (4.4), then $J_{\varphi}^{\rho} \equiv P_K$, that is, the projection of $\mathcal{H}$ onto K. Then the problem (6.1) changes into that of finding $z \in \mathcal{H}$ such that

$$Th^{-1}P_K z + \rho^{-1}Q_K z = 0, \tag{6.4}$$

where $Q_K = I - gh^{-1}P_K$. The equations of the type (6.4), which are called the *extended general Wiener–Hopf equations*, were introduced and studied by Noor [36,39].

Some special cases of the problem (6.4) have been introduced and studied by Noor [33,43] and Shi [52].

Remark 6.1 It has been shown that the resolvent equations and the Wiener–Hopf equations have played an important and significant role in developing several numerical techniques for solving mixed variational inequalities/variational inequalities and related optimizations problems (see, for example, [23,24,28,29,33–39,41,42,45,47,50,52] and references therein).

In the next lemma, the equivalence between the extended general nonlinear mixed variational inequality (2.1) and the extended general resolvent equation (6.1) is proved.

Lemma 6.2 *Suppose that T, g, h and ρ are the same as in the problem (2.1) and let the inverse of the operator h exists. Then $u \in \mathcal{H}$ is a solution of the problem (2.1) if and only if the extended general resolvent equation (6.1) has a solution $z \in \mathcal{H}$ satisfying*

$$h(u) = J_{\varphi}^{\rho} z, \quad z = g(u) - \rho T(u).$$

Proof Let $u \in \mathcal{H}$ be a solution of the problem (2.1). Then Lemma 3.1 guarantees that

$$h(u) = J_{\varphi}^{\rho}(g(u) - \rho T(u)). \tag{6.5}$$

Taking $z = g(u) - \rho T(u)$, in (6.5), we have $h(u) = J_{\varphi}^{\rho} z$, which leads to

$$u = h^{-1} J_{\varphi}^{\rho} z. \tag{6.6}$$

By using (6.6) and this fact that $z = g(u) - \rho T(u)$, we have

$$z = gh^{-1} J_{\varphi}^{\rho} z - \rho T h^{-1} J_{\varphi}^{\rho} z. \tag{6.7}$$

It is obvious that the equality (6.7) is equivalent to

$$T h^{-1} J_{\varphi}^{\rho} z + \rho^{-1} R_{\varphi} z = 0, \tag{6.8}$$

where R_{φ} is the same as in Eq. (6.1). Now, (6.8) guarantees that $z \in \mathcal{H}$ is a solution of the extended general resolvent equation (6.1).

Conversely, if $z \in \mathcal{H}$ is a solution of the problem (6.1) satisfying

$$h(u) = J_{\varphi}^{\rho} z, \quad z = g(u) - \rho T(u),$$

then it follows from Lemma 3.1 that $u \in \mathcal{H}$ is a solution of the problem (2.1). This completes the proof.

Now, by using the problem (6.1) and Lemma 6.2, we obtain some fixed point formulations for constructing a number of the new perturbed resolvent iterative algorithms for solving the problem (2.1).

(I) Applying (6.1) and Lemma 6.2, we obtain

$$\begin{aligned} T h^{-1} J_{\varphi}^{\rho} z + \rho^{-1} R_{\varphi} z = 0 &\Leftrightarrow \rho T h^{-1} J_{\varphi}^{\rho} z + R_{\varphi} z = 0 \\ &\Leftrightarrow \rho T h^{-1} J_{\varphi}^{\rho} z + z - gh^{-1} J_{\varphi}^{\rho} z = 0 \\ &\Leftrightarrow z = gh^{-1} J_{\varphi}^{\rho} z - \rho T h^{-1} J_{\varphi}^{\rho} z \\ &\Leftrightarrow z = g(u) - \rho T(u). \end{aligned}$$

This fixed point formulation enables us to construct the following resolvent iterative algorithm for solving the problem (2.1).

Algorithm 6.3 Let T, g, h and ρ be the same as in the problem (2.1) such that h is an onto mapping. For arbitrary chosen initial point $z_0 \in \mathcal{H}$, compute the iterative sequence $\{z_n\}_{n=0}^{\infty}$ in the following way:

$$\begin{cases} h(u_n) = S^n J_{\varphi}^{\rho} z_n, \\ z_{n+1} = (1-\alpha_n)z_n + \alpha_n(g(u_n) - \rho T(u_n)), \end{cases} \tag{6.9}$$

where the mapping S and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ are the same as in Algorithm 4.9.

(II) By using (6.1) and Lemma 6.2, we have

$$\begin{aligned} Th^{-1}J_{\varphi}^{\rho}z + \rho^{-1}R_{\varphi}z = 0 &\Leftrightarrow R_{\varphi}z = R_{\varphi}z - Th^{-1}J_{\varphi}^{\rho}z - \rho^{-1}R_{\varphi}z \\ &\Leftrightarrow R_{\varphi}z = -Th^{-1}J_{\varphi}^{\rho}z + (1-\rho^{-1})R_{\varphi}z \\ &\Leftrightarrow z = gh^{-1}J_{\varphi}^{\rho}z - Th^{-1}J_{\varphi}^{\rho}z + (1-\rho^{-1})R_{\varphi}z \\ &\Leftrightarrow z = g(u) - T(u) + (1-\rho^{-1})R_{\varphi}z. \end{aligned}$$

Using this fixed point formulation, we can define the following resolvent iterative algorithm for solving the problem (2.1).

Algorithm 6.4 Assume that T, g, h and ρ are the same as in Algorithm 6.3. For arbitrary chosen initial point $z_0 \in \mathcal{H}$, compute the iterative sequence $\{z_n\}_{n=0}^{\infty}$ in the following way:

$$\begin{cases} h(u_n) = S^n J_{\varphi}^{\rho} z_n, \\ z_{n+1} = (1-\alpha_n)z_n + \alpha_n(g(u_n) - T(u_n) + (1-\rho^{-1})R_{\varphi}z_n), \end{cases}$$

where the mapping S and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ are the same as in Algorithm 4.9.

(III) Let the operators T and h be linear and suppose that the inverses of T and h, that is, T^{-1} and h^{-1} exists. Then (6.1) can be written in the following way:

$$\begin{aligned} Th^{-1}J_{\varphi}^{\rho}z + \rho^{-1}R_{\varphi}z = 0 &\Leftrightarrow T(h^{-1}(z - R_{\varphi}z)) + \rho^{-1}R_{\varphi}z = 0 \\ &\Leftrightarrow h^{-1}(z - R_{\varphi}z) = T^{-1}(-\rho^{-1}R_{\varphi}z) \\ &\Leftrightarrow z - R_{\varphi}z = h(-\rho^{-1}T^{-1}R_{\varphi}z) \\ &\Leftrightarrow z = R_{\varphi}z - \rho^{-1}hT^{-1}R_{\varphi}z \\ &\Leftrightarrow z = (I - \rho^{-1}hT^{-1})R_{\varphi}z. \end{aligned}$$

This fixed point formulation allows us to construct the following projection iterative algorithm for solving the problem (2.1).

Algorithm 6.5 Suppose that T, g, h and ρ are the same as in Algorithm 6.3. For arbitrary chosen initial point $z_0 \in \mathcal{H}$, define the iterative sequence $\{z_n\}_{n=0}^{\infty}$ by the iterative process

$$z_{n+1} = (1-\alpha_n)z_n + \alpha_n(I - \rho^{-1}hT^{-1})R_\varphi z_n,$$

where the sequence $\{\alpha_n\}_{n=0}^\infty$ is the same as in Algorithm 4.9.

If $\varphi(x) = \delta_K(x)$ for all $x \in K$, where δ_K is defined by (4.4), then $J_\varphi^\rho \equiv P_K$. Then Algorithms 6.3–6.5 reduce to the following projection iterative algorithms.

Algorithm 6.6 Let T, g, h and ρ be the same as in Algorithm (6.3). For arbitrary chosen initial point $z_0 \in \mathcal{H}$, compute the iterative sequence $\{z_n\}_{n=0}^\infty$ in the following way:

$$\begin{cases} h(u_n) = S^n P_K z_n, \\ z_{n+1} = (1-\alpha_n)z_n + \alpha_n(g(u_n) - \rho T(u_n)), \end{cases}$$

where the mapping S and the sequence $\{\alpha_n\}_{n=0}^\infty$ are the same as in Algorithm 4.9.

Algorithm 6.7 Suppose that T, g, h and ρ are the same as in Algorithm 6.3. For arbitrary chosen initial point $z_0 \in \mathcal{H}$, compute the iterative sequence $\{z_n\}_{n=0}^\infty$ in the following way:

$$\begin{cases} h(u_n) = S^n P_K z_n, \\ z_{n+1} = (1-\alpha_n)z_n + \alpha_n(g(u_n) - T(u_n) + (1-\rho^{-1})Q_K z_n), \end{cases}$$

where the mapping S and the sequence $\{\alpha_n\}_{n=0}^\infty$ are the same as in Algorithm 4.9.

Algorithm 6.8 Assume that T, g, h and ρ are the same as in Algorithm 6.3. For arbitrary chosen initial point $z_0 \in \mathcal{H}$, define the iterative sequence $\{z_n\}_{n=0}^\infty$ by the iterative process

$$z_{n+1} = (1-\alpha_n)z_n + \alpha_n(I - \rho^{-1}hT^{-1})Q_K z_n,$$

where the sequence $\{\alpha_n\}_{n=0}^\infty$ is the same as in Algorithm 4.9.

Remark 6.9 Algorithms 4.1–4.3 in [47] are special cases of Algorithms 6.3–6.5, respectively. Also, Algorithms 4.1–4.3 in [36] are special cases of Algorithms 6.6–6.8. In brief, for a suitable and appropriate choice of the operators T, g, h, S and the constant ρ, one can obtain a number of new and previously known iterative schemes. See, for example, the suggested algorithms in [35,42]. This clearly shows that Algorithms 6.3–6.8 are quite general and unifying.

7 Strongly convergence theorem

In this section, we study the convergence analysis of iterative sequence generated by perturbed resolvent iterative Algorithm 6.3. Similarly, one can discuss the convergence analysis of iterative sequences generated by Algorithms 6.4–6.8.

Theorem 7.1 *Let T, g, h and ρ be the same as in the problem* (2.1) *and suppose that all the conditions of* Theorem 3.2 *hold. Assume that $S : \mathcal{H} \to \mathcal{H}$ is a nearly uniformly L-Lipschitzian mapping with the sequence $\{b_n\}_{n=0}^{\infty}$ such that, for each $u \in$* EGNMVID$(\mathcal{H}, T, g, h)$, $h(u) \in \text{Fix}(S)$. *Further, assume that $L\psi < 1$, where ψ is the same as in* (3.8). *Then there exists a unique solution u^* of the problem* (2.1) *such that the iterative sequence $\{z_n\}_{n=0}^{\infty}$ generated by* Algorithm 6.3, *converges strongly to the only element z of* EGRE$(\mathcal{H}, T, g, h)$.

Proof Theorem 3.2 guarantees the existence a unique solution $u^* \in \mathcal{H}$ for the problem (2.1). Accordingly, in view of Lemma 6.2, there exists a unique point $z \in \mathcal{H}$ satisfying

$$h(u^*) = J_{\varphi}^{\rho} z, \quad z = g(u^*) - \rho T(u^*). \tag{7.1}$$

Since $h(u^*) \in \text{Fix}(S)$, by using (7.1), it follows that for each $n \geq 0$

$$h(u^*) = S^n J_{\varphi}^{\rho} z, \quad z = g(u^*) - \rho T(u^*). \tag{7.2}$$

Applying (6.9), (7.2) and the assumptions, we get

$$\begin{aligned} \|z_{n+1} - z\| &\leq (1 - \alpha_n)\|z_n - z\| + \alpha_n \|g(u_n) - g(u^*) - \rho(T(u_n) - T(u^*))\| \\ &\leq (1 - \alpha_n)\|z_n - z\| + \alpha_n \sqrt{\theta^2 - 2\rho\kappa + \rho^2\xi^2}\|u_n - u^*\|. \end{aligned} \tag{7.3}$$

To obtain an estimation for $\|u_n - u^*\|$, using (6.9) and (7.2), we find that

$$\begin{aligned} \|u_n - u^*\| &\leq \|u_n - u^* - (h(u_n) - h(u^*))\| + \|S^n J_{\varphi}^{\rho} z_n - S^n J_{\varphi}^{\rho} z\| \\ &\leq \sqrt{1 - 2\pi + \varrho^2}\|u_n - u^*\| + L(\|J_{\varphi}^{\rho} z_n - J_{\varphi}^{\rho} z\| + b_n) \\ &\leq \sqrt{1 - 2\pi + \varrho^2}\|u_n - u^*\| + L(\|z_n - z\| + b_n), \end{aligned}$$

which leads to

$$\|u_n - u^*\| \leq \frac{L}{1 - \sqrt{1 - 2\pi + \varrho^2}}\|z_n - z\| + \frac{Lb_n}{1 - \sqrt{1 - 2\pi + \varrho^2}}. \tag{7.4}$$

Applying (7.3) and (7.4), we have

$$\begin{aligned} \|z_{n+1} - z\| &\leq (1 - \alpha_n)\|z_n - z\| + \alpha_n \frac{L\sqrt{\theta^2 - 2\rho\kappa + \rho^2\xi^2}}{1 - \sqrt{1 - 2\pi + \varrho^2}}\|z_n - z\| \\ &\quad + \alpha_n \frac{L\sqrt{\theta^2 - 2\rho\kappa + \rho^2\xi^2}\, b_n}{1 - \sqrt{1 - 2\pi + \varrho^2}} \\ &= (1 - \alpha_n)\|z_n - z\| + \alpha_n L\omega\|z_n - z\| + \alpha_n L\omega b_n \\ &= (1 - \alpha_n(1 - L\omega))\|z_n - z\| + \alpha_n(1 - L\omega)\frac{L\omega b_n}{1 - L\omega}, \end{aligned} \tag{7.5}$$

where $\omega = \frac{\sqrt{\theta^2-2\rho\kappa+\rho^2\xi^2}}{1-\sqrt{1-2\pi+\varrho^2}}$. Since $\psi < 1$, where ψ is the same as in (3.8), we deduce that $\omega < 1$. If $L \geq 1$ then from the assumption $L\psi < 1$ it follows that

$$\sqrt{1-2\pi+\varrho^2} + L\sqrt{\theta^2-2\rho\kappa+\rho^2\xi^2} < 1,$$

whence we derive that $L\omega < 1$. For the case that $L < 1$, it is obvious that $L\omega < 1$. Since all the conditions of Lemma 5.1 hold from (7.5) and Lemma 5.1 it follows that the sequence $\{z_n\}_{n=0}^{\infty}$ generated by Algorithm 6.3 converges strongly to the unique solution $z \in \mathcal{H}$ of the problem (6.1) and there is nothing to prove.

8 Extended general resolvent dynamical systems

In this section, we consider the dynamical system technique to study the existence and uniqueness of solution of the extended general nonlinear mixed variational inequality of type (2.1). Dupuis and Nagurney [15] introduced and studied the projected dynamical systems associated with variational inequalities, in which the right hand side of the ordinary differential equations is a projection operator. The novel feature of the projected dynamical system is that the its set of stationary points corresponds of the set of the corresponding set of the solutions of the variational inequality problem. Thus the equilibrium and nonlinear programming problems, which can be formulated in the setting of the variational inequalities, can now be studied in the more general framework of the dynamical systems. It has been shown [14–16,27,58,59,63] that these dynamical systems are useful in developing efficient and powerful numerical techniques for solving variational inequalities. Noor [28,37,47] has also suggested and analyzed similar resolvent dynamical systems for mixed variational inequalities by extending and modifying the techniques of Xia and Wang [58,59]. In Sect. 3, we have shown that the extended general nonlinear mixed variational inequalities (2.1) are equivalent to fixed-point problems. We use this equivalent to suggest and analyze a resolvent dynamical system associated with the extended general nonlinear mixed variational inequality (2.1). The fixed point formulation (3.1) enables us to suggest the following dynamical system

$$\frac{du}{dt} = \lambda\{J_{\varphi}^{\rho}(g(u) - \rho T(u)) - h(u)\}, \quad u(t_0) = u_0 \in \mathcal{H}, \tag{8.1}$$

associated with the extended general nonlinear mixed variational inequality (2.1), where $\lambda > 0$ is a constant. The system of type (8.1) is called the *extended general resolvent dynamical system* associated with the extended general nonlinear mixed variational inequality (2.1). Here the right hand is related to the resolvent and is discontinuous on the boundary. It is clear from the definition that the solution to (8.1) always stays in the constraint set. This implies that the qualitative results such as the existence, uniqueness, and continuous dependence of the solution on the given date (8.1) can be studied. The dynamical system describes the adjustment processes which may produce important transient phenomena prior to the achievement of a steady state.

If $h \equiv I$, then the extended general resolvent dynamical system (8.1) reduces to the following system:

$$\frac{du}{dt} = \lambda\{J_{\varphi}^{\rho}(g(u) - \rho T(u)) - u\}, \quad u(t_0) = u_0 \in \mathcal{H}, \tag{8.2}$$

which has been introduced and studied by Noor [47].

If the function φ is the indicator function of a closed convex set K in $\mathcal{H}$, then $J_{\varphi}^{\rho} \equiv P_K$ is the projection of $\mathcal{H}$ onto K. In this case, the extended general resolvent dynamical system (8.1) reduces to the following system:

$$\frac{du}{dt} = \lambda\{P_K(g(u) - \rho T(u)) - h(u)\}, \quad u(t_0) = u_0 \in \mathcal{H}. \tag{8.3}$$

The system of type (8.3) is called the *extended general projection dynamical system* associated with the extended general nonlinear variational inequality introduced by Noor [30,36] and appears to be new one.

If $h \equiv I$, then the extended general projection dynamical system (8.3) collapses to the following system:

$$\frac{du}{dt} = \lambda\{P_K(g(u) - \rho T(u)) - u\}, \quad u(t_0) = u_0 \in \mathcal{H}, \tag{8.4}$$

which is introduced and studied by Noor et al. [47].

To state our results, we need the following well-known concepts.

Definition 8.1 [58] The dynamical system is said to be *converge to the solution set* Ω^* of the problem (2.1), if, irrespective of the initial point, the trajectory of the dynamical system satisfies

$$\lim_{t\to\infty} \operatorname{dist}(u(t), \Omega^*) = 0, \tag{8.5}$$

where $\operatorname{dist}(u(t), \Omega^*) = \inf_{v\in\Omega^*} \|u - v\|$.

It is easy to see that, if the set Ω^* has a unique point u^*, then (8.5) implies that $\lim_{t\to\infty} u(t) = u^*$.

If the dynamical system is still stable at u^* in the Lyapunov sense, then the dynamical system is globally asymptotically stable at u^*.

Definition 8.2 [58] The dynamical system is said to be *globally exponentially stable with degree* η *at* u^*, if, irrespective of the initial point, the trajectory of the dynamical system satisfies

$$\|u(t) - u^*\| \leq c_0\|u(t_0) - u^*\| \exp(-\eta(t - t_0)), \quad \forall t \geq t_0,$$

where c_0 and η are positive constants independent of the initial point. It is evident that globally exponentially stability is necessarily globally asymptotically stable and the dynamical system converges arbitrarily fast.

Lemma 8.3 (Gronwall's inequality in [26]) *Let $\hat{u}$ and $\hat{v}$ be real-valued nonnegative continuous functions with domain $\{t : t \geq t_0\}$ and let $\alpha(t) = \alpha_0(|t - t_0|)$, where α_0 is a monotone increasing function. If, for all $t \geq t_0$,*

$$\hat{u}(t) \leq \alpha(t) + \int_{t_0}^{t} \hat{u}(s)\hat{v}(s)d(s),$$

then

$$\hat{u}(t) \leq \alpha(t) + \exp\left\{\int_{t_0}^{t} \hat{v}(s)d(s)\right\}.$$

In Theorem 3.2, the existence and uniqueness of solution of the extended general nonlinear mixed variational inequality (2.1) is proved. Now, by using Lemma 8.3 and the assumptions of Theorem 3.2, we establish the existence and uniqueness of solution of the extended general resolvent dynamical system of type (8.1) associated with the extended general nonlinear mixed variational inequality (2.1).

Theorem 8.4 *Let T, g, h and ρ be the same as in* Theorem 3.2 *and suppose that all the conditions of* Theorem 3.2 *hold. Then, for each $u_0 \in \mathcal{H}$, there exists a unique continuous solution $u(t)$ of the extended general resolvent dynamical system* (8.1) *with $u(t_0) = u_0$ over $[t_0, \infty)$.*

Proof According to Theorem 3.2, the extended general nonlinear mixed variational inequality (2.1) has a unique solution $u^* \in \mathcal{H}$. Accordingly, from Lemma 3.1 it follows that $h(u^*) = J_{\varphi}^{\rho}(g(u^*) - \rho T(u^*))$. Define

$$F(u) = \lambda\{J_{\varphi}^{\rho}(g(u) - \rho T(u)) - h(u)\}, \quad \forall u \in \mathcal{H},$$

where $\lambda > 0$ is a constant. Then, for all $u, v \in \mathcal{H}$, we have

$$\begin{aligned}
&\|F(u) - F(v)\| \\
&\quad \leq \lambda\Big(\|J_{\varphi}^{\rho}(g(u) - \rho T(u)) - J_{\varphi}^{\rho}(g(v) - \rho T(v))\| + \|h(u) - h(v)\|\Big) \\
&\quad \leq \lambda\Big(\|u - v\| + \|u - v - (h(u) - h(v))\| + \|J_{\varphi}^{\rho}(g(u) - \rho T(u)) - J_{\varphi}^{\rho}(g(v) - \rho T(v))\|\Big) \\
&\quad \leq \lambda\Big(\|u - v\| + \|u - v - (h(u) - h(v))\| + \|g(u) - g(v) - \rho(T(u) - T(v))\|\Big). \qquad (8.6)
\end{aligned}$$

Since T is κ-strongly monotone with respect to g and ξ-Lipschitz continuous, g is θ-Lipschitz continuous, h is π-strongly monotone and ϱ-Lipschitz continuous, like in the proofs of (3.5) and (3.6), we have

$$\|u - v - (h(u) - h(v))\| \leq \sqrt{1 - 2\pi + \varrho^2}\|u - v\| \qquad (8.7)$$

and

$$\|g(u) - g(v) - \rho(T(u) - T(v))\| \leq \sqrt{\theta^2 - 2\pi\kappa + \rho^2\xi^2}\|u - v\|. \quad (8.8)$$

Substituting (8.7) and (8.8) in (8.6) deduce that

$$\|F(u) - F(v)\| \leq \lambda(1 + \psi)\|u - v\|,$$

where ψ is the same as in (3.8). Hence the operator F is locally Lipschitz continuous in $\mathcal{H}$. Therefore, for each $u_0 \in \mathcal{H}$, there exists a unique and continuous solution $u(t)$ of the extended general resolvent dynamical system (8.1) defined in a interval $t_0 \leq t < \mathcal{T}$ with the initial condition $u(t_0) = u_0$. Let $[t_0, \mathcal{T})$ be its maximal interval of existence. Now, we show that $\mathcal{T} = \infty$. For any $u \in \mathcal{H}$, we have

$$\begin{aligned}
\|F(u)\| &= \lambda\|J_\varphi^\rho(g(u) - \rho T(u)) - h(u)\| \\
&\leq \lambda\Big(\|J_\varphi^\rho(g(u) - \rho T(u)) - h(u^*)\| + \|h(u) - h(u^*)\|\Big) \\
&\leq \lambda\Big(\|u - u^*\| + \|u - u^* - (h(u) - h(u^*))\| \\
&\quad + \|J_\varphi^\rho(g(u) - \rho T(u)) - J_\varphi^\rho(g(u^*) - \rho T(u^*))\|\Big) \\
&\leq \lambda\Big(\|u - u^*\| + \|u - u^* - (h(u) - h(u^*))\| \\
&\quad + \|g(u) - g(u^*) - \rho(T(u) - T(u^*))\|\Big) \\
&\leq \lambda(1 + \psi)\|u - u^*\| \\
&\leq \lambda(1 + \psi)\|u^*\| + \lambda(1 + \psi)\|u\|,
\end{aligned}$$

then

$$\begin{aligned}
\|u(t)\| &\leq \|u_0\| + \int_{t_0}^{t} \|F(u(s))\|\, ds \\
&\leq (\|u_0\| + k_1(t - t_0)) + k_2 \int_{t_0}^{t} \|u(s)\|\, ds,
\end{aligned}$$

where $k_1 = \lambda(1 + \psi)\|u^*\|$ and $k_2 = \lambda(1 + \psi)$. Therefore, using Lemma 8.3, we have

$$\|u(t)\| \leq (\|u_0\| + k_1(t - t_0))e^{k_2(t - t_0)}, \quad \forall t \in [t_0, \mathcal{T}).$$

Hence, the solution is bounded for $t \in [t_0, \mathcal{T})$ if $\mathcal{T}$ is finite. Accordingly, $\mathcal{T} = \infty$.

We now show that the trajectory of solution of the extended general resolvent dynamical system (8.1) converges to a unique solution of the extended general nonlinear mixed variational inequality (2.1) by using the technique of Xia and Wang [57,58].

Theorem 8.5 *Let T, g, h and ρ be the same as in* Theorem 3.2 *and suppose that all the conditions* Theorem 3.2 *hold. If $1-\mu<\pi<\theta$, where μ is the same as in* (3.2)*, then the extended general resolvent dynamical system* (8.1) *converges globally exponentially to the unique solution of the extended general nonlinear mixed variational inequality* (2.1).

Proof Theorem 3.2 guarantees the existence of a unique solution $u^* \in \mathcal{H}$ for the problem (2.1). So Lemma 3.1 implies that $h(u^*) = J_{\varphi}^{\rho}(g(u^*) - \rho T(u^*))$. On the other hand, in view of Theorem 8.4, the extended general resolvent dynamical system (8.1) has a unique solution $u(t)$ over $[t_0, \mathcal{T})$ for any fixed $u_0 \in \mathcal{H}$. Let $u(t) = u(t, t_0 : u_0)$ be the solution of (8.1) with $u(t_0) = u_0$. Now, we consider the Lyapunov function L defined on $\mathcal{H}$ as follows:

$$L(u) = \|u - u^*\|^2, \quad \forall u \in \mathcal{H}. \tag{8.9}$$

Then it follows from (8.1), (8.9) and π-strongly monotonicity of h that

$$\begin{aligned}\frac{dL}{dt} &= \frac{dL}{du}\frac{du}{dt} = 2\left\langle u(t) - u^*, \frac{du}{dt}\right\rangle \\ &= 2\lambda\langle u(t) - u^*, J_{\varphi}^{\rho}(g(u) - \rho T(u)) - h(u)\rangle \\ &= -2\lambda\langle u(t) - u^*, h(u) - h(u^*)\rangle \\ &\quad + 2\lambda\langle u(t) - u^*, J_{\varphi}^{\rho}(g(u) - \rho T(u)) - h(u^*)\rangle \\ &\le -2\lambda\pi\|u(t) - u^*\|^2 + 2\lambda\|u(t) - u^*\|\|J_{\varphi}^{\rho}(g(u) - \rho T(u)) \\ &\quad - J_{\varphi}^{\rho}(g(u^*) - \rho T(u^*))\|.\end{aligned} \tag{8.10}$$

Since T is κ-strongly monotone with respect to g and ξ-Lipschitz continuous, g is θ-Lipschitz continuous, and the resolvent operator J_{φ}^{ρ} is nonexpansive, in similar way to the proof of (3.6), we have

$$\begin{aligned}&\|J_{\varphi}^{\rho}(g(u) - \rho T(u)) - J_{\varphi}^{\rho}(g(u^*) - \rho T(u^*))\| \\ &\quad \le \|g(u) - g(u^*) - \rho(T(u) - T(u^*))\| \\ &\quad \le \sqrt{\theta^2 - 2\rho\kappa + \rho^2\xi^2}\|u - u^*\|.\end{aligned} \tag{8.11}$$

Substituting (8.11) in (8.10), we obtain

$$\begin{aligned}\frac{d}{dt}\|u(t) - u^*\|^2 &\le -2\lambda\left(\pi - \sqrt{\theta^2 - 2\rho\kappa + \rho^2\xi^2}\right)\|u(t) - u^*\|^2 \\ &= -2\lambda\vartheta\|u(t) - u^*\|^2,\end{aligned}$$

where $\vartheta = \pi - \sqrt{\theta^2 - 2\rho\kappa + \rho^2\xi^2}$. Therefore, we have

$$\|u(t) - u^*\| \le \|u(t) - u^*\|e^{-\lambda\vartheta(t - t_0)}.$$

The condition (3.2) and this fact that $1-\mu<\pi<\theta$ guarantee that $\vartheta>0$. Hence the trajectory of the solution of the extended general resolvent dynamical system (8.1) converges globally exponentially to the unique solution of the extended general nonlinear mixed variational inequality (2.1). This completes the proof.

Remark 8.6 Theorem 3.2 improves and extends Theorem 3.1 in [47]. Theorem 3.1 in [36] and Theorem 3.2 in [47] are special cases of Theorems 5.2 and 5.3. Also, Theorem 7.1 generalizes and improves Theorem 4.1 in [36,45].

9 Summary and conclusion

In this paper, we have introduced and considered a new class of extended general nonlinear mixed variational inequalities and a new class of extended general resolvent equations involving three different nonlinear operators. We have proved the equivalence between the extended general nonlinear mixed variational inequalities and the fixed point problems as well as the extended general resolvent equations. Then by this equivalent formulation, we have discussed the existence and uniqueness theorem for solution of the problem of extended general nonlinear mixed variational inequalities. This equivalence and a nearly uniformly Lipschitzian mapping S are used to suggest and analyze some new perturbed resolvent iterative algorithms for finding an element of the set of the fixed points of the nearly uniformly Lipschitzian mapping S which is the unique solution of the problem of extended general nonlinear mixed variational inequalities. We have also suggested and analyzed a class of extended general resolvent dynamical systems associated with the extended general nonlinear mixed variational inequalities. We have shown that the trajectory of the extended general resolvent dynamical system converges globally exponentially to the unique solution of the extended general nonlinear mixed variational inequalities. Several special cases are also discussed. It is expected that the results proved in this paper may simulate further research regarding the numerical methods and their applications in various fields of pure and applied sciences.

Acknowledgments This work was supported by the Korea Research Foundation Grant funded by the Korean Government (KRF-2008-313-C00050). This research was also supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant Number: 2011-0021821).

References

1. Agarwal, R.P., Cho, Y.J., Petrot, N.: Systems of general nonlinear set-valued mixed variational inequalities problems in Hilbert spaces. Fixed Point Theory Appl. **2011**, 31 (2011)
2. Ames, W.F.: Numerical Methods for Partial Differential Equations, 3rd edn. Academic Press, New York (1992)
3. Baiocchi, C., Capelo, A.: Variational and Quasi Variational Inequalities. Wiley, University of California (1984)
4. Brézis, H.: Operateurs Maximaux Monotone et Semigroupes de Contractions dans les Espace d'Hilbert. North-Holland, Amsterdam (1973)
5. Chang, S.S., Cho, Y.J., Zhou, H.: Iterative Methods for Nonlinear Operator Equations in Banach Spaces. Nova Science Publishers, Huntington. ISBN: 1-59033-170-2 (2002)

6. Chang, S.S., Cho, Y.J., Kim, J.K.: On the two-step projection methods and applications to variational inequalities. Math. Inequal. Appl. **10**, 755–760 (2007)
7. Cho, Y.J., Argyros, I.K., Petrot, N.: Approximation methods for common solutions of generalized equilibrium, systems of nonlinear variational inequalities and fixed point problems. Comput. Math. Appl. **60**, 2292–2301 (2010)
8. Cho, Y.J., Fang, Y.P., Huang, N.J., Hwang, H.J.: Algorithms for systems of nonlinear variational inequalities. J. Korean Math. Soc. **41**, 489–499 (2004)
9. Cho, Y.J., Kang, S.M., Qin, X.: On systems of generalized nonlinear variational inequalities in Banach spaces. Appl. Math. Comput. **206**, 214–220 (2008)
10. Cho, Y.J., Kim, J.K., Verma, R.U.: A class of nonlinear variational inequalities involving partially relaxed monotone mappings and general auxiliary principle. Dyn. Syst. Appl. **11**, 333–338 (2002)
11. Cho, Y.J., Petrot, N.: Regularization and iterative method for general variational inequality problem in Hilbert spaces. J. Inequal. Appl. **2011**, 21 (2011)
12. Cho, Y.J., Qin, X.: Systems of generalized nonlinear variational inequalities and its projection methods. Nonlinear Anal. **69**, 4443–4451 (2008)
13. Cho, Y.J., Qin, X.: Generalized systems for relaxed cocoercive variational inequalities and projection methods in Hilbert spaces. Math. Inequal. Appl. **12**, 365–375 (2009)
14. Dong, J., Zhang, D., Nagurney, A.: A projected dynamical systems model of general financial equilibrium with stability analysis. Math. Comput. Model. **24**(2), 35–44 (1996)
15. Dupuis, P., Nagurney, A.: Dynamical systems and variational inequalities. Ann. Oper. Res. **44**, 19–42 (1993)
16. Friesz, T.L., Bernstein, D.H., Stough, R.: Dynamical systems, variational inequalities and control theoretical models for predicting time-varying unban network flows. Trans. Sci. **30**, 14–31 (1996)
17. Glowinski, R., Tallec, P.L.: Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics. SIAM, Philadelphia (1989)
18. Goebel, K., Kirk, W.A.: A fixed point theorem for asymptotically nonexpansive mappings. Proc. Am. Math. Soc. **35**, 171–174 (1972)
19. Huang, N.J., Bai, M.R., Cho, Y.J., Kang, S.M.: Generalized nonlinear mixed quasi-variational inequalities. Comput. Math. Appl. **40**, 205–215 (2000)
20. Huang, N.J., Fang, Y.P., Cho, Y.J.: A new class of generalized nonlinear mixed quasi-variational inequalities in Banach spaces. Math. Inequal. Appl. **6**, 125–132 (2003)
21. Kirk, W.A., Xu, H.K.: Asymptotic pointwise contractions. Nonlinear Anal. **69**, 4706–4712 (2008)
22. Kikuchi, N., Oden, J.T.: Contact Problems in Elasticity. SIAM, Philadelphia (1988)
23. Lions, P.L., Mercier, B.: Splitting algorithms for the sum of two nonlinear operators. SIAM J. Numer. Anal. **16**, 964–979 (1979)
24. Lions, J.L., Stampacchia, G.: Variational inequalities. Commun. Pure Appl. Math. **20**, 493–512 (1967)
25. Martinet, B.: Regularization dinequations variationelles par approximations successives. Rev. Francaise DAuto. Et Inform. Rech. Oper **4**, 154–159 (1972)
26. Miller, R.K., Michel, A.N.: Ordinary Differential Equations. Academic Press, New York (1982)
27. Nagurney, A., Zhang, D.: Projected Dynamical Systems and Variational Inequalities with Applications. Kluwer, Dordrecht (1995)
28. Noor, M.A.: A Wiener–Hopf dynamical system for variational inequalities. NZ J. Math. **31**, 173–182 (2002)
29. Noor, M.A.: An iterative schemes for a class of quasi variational inequalities. J. Math. Anal. Appl. **110**, 463–468 (1985)
30. Noor, M.A.: Auxiliary principle technique for extended general variational inequalities. Banach J. Math. Anal. **2**, 33–39 (2008)
31. Noor, M.A.: Differentiable nonconvex functions and general variational inequalities. Appl. Math. Comput. **199**, 623–630 (2008)
32. Noor, M.A.: Existence results for quasi-variational inequalities. Banach J. Math. Anal. **1**, 186–194 (2007)
33. Noor, M.A.: General variational inequalities. Appl. Math. Lett. **1**, 119–121 (1988)
34. Noor, M.A.: Merfunctions for quasi variational inequalities. J. Math. Inequal. **1**, 259–268 (2007)
35. Noor, M.A.: New approximation schemes for general variational inequalities. J. Math. Anal. Appl. **251**, 217–229 (2000)
36. Noor, M.A.: Projection iterative methods for extended general variational inequalities. J. Appl. Math. Comput. **32**, 83–95 (2010)

37. Noor, M.A.: Resolvent dynamical systems for mixed variational inequalities. Korean J. Comput. Appl. Math. **9**, 15–26 (2002)
38. Noor, M.A.: Sensitivity analysis for quasi variational inequalities. J. Optim. Theory Appl. **95**, 399–407 (1997)
39. Noor, M.A.: Sensitivity analysis of extended general variational inequalities. Appl. Math. E-Notes **9**, 17–26 (2009)
40. Noor, M.A.: Some algorithms for general monotone mixed variational inequalities. Math. Comput. Model. **29**(7), 1–9 (1999)
41. Noor, M.A.: Some recent advances in variational inequalities. Part I, basic concepts. NZ J. Math. **26**, 53–80 (1997)
42. Noor, M.A.: Some recent advances in variational inequalities. Part II, other concepts. NZ J. Math. **26**, 229–255 (1997)
43. Noor, M.A.: Some developments in general variational inequalities. Appl. Math. Comput. **152**, 199–277 (2004)
44. Noor, M.A., Huang, Z.: Three-step iterative methods for nonexpansive mappings and variational inequalities. Appl. Math. Comput. **187**, 680–685 (2007)
45. Noor, M.A., Noor, K.I., Said, E.A.: Iterative methods for solving general quasi-variational inequalities. Optim. Lett. **4**(4), 513–530 (2010)
46. Noor, M.A., Noor, K.I., Rassias, T.M.: Some aspects of variational inequalities. Optim. Lett. **4**(4), 285–312 (2010)
47. Noor, M.A., Noor, K.I., Yaqoob, H.: On general mixed variational inequalities. Acta Appl. Math. **110**(1), 227–246 (2010)
48. Patriksson, M.: Nonlinear Programming and Variational Inequalities, A Unified Approach. Kluwer, Dordrecht (1988)
49. Peaceman, D.H., Rachford, H.H.: The numerical solution of parabolic elliptic differential equations. SIAM J. Appl. Math **3**, 28–41 (1955)
50. Qin, X., Noor, M.A.: General Wiener–Hopf equation technique for nonexpansive mappings and general variational inequalities in Hilbert spaces. Appl. Math. Comput. **201**, 716–722 (2008)
51. Sahu, D.R.: Fixed points of demicontinuous nearly Lipschitzian mappings in Banach spaces. Comment. Math. Univ. Carolin **46**(4), 653–666 (2005)
52. Shi, P.: Equivalence of variational inequalities with Wiener–Hopf equations. Proc. Am. Math. Soc. **111**, 339–346 (1991)
53. Stampacchia, G.: Formes bilineaires coercitives sur les ensembles convexes. C. R. Acad. Sci. Paris **258**, 4413–4416 (1964)
54. Tseng, P.: A modified forward-backward splitting method for maximal monotone mappings. SIAM J. Control Optim. **38**, 431–446 (2000)
55. Wang, L., Cho, Y.J., Huang, N.J.: Nonzero solutions for a system of variational inequalities in reflexive Banach spaces. Math. Comput. Model. **49**, 579–585 (2009)
56. Weng, X.L.: Fixed point iteration for local strictly pseudo-contractive mapping. Proc. Am. Math. Soc. **113**, 727–732 (1991)
57. Xia, Y.S.: On convergence conditions of an extended projection neural network. Neural Comput. **17**, 515–525 (2005)
58. Xia, Y.S., Wang, J.: A recurrent neural network for solving linear projection equations. Neural Netw. **137**, 337–350 (2000)
59. Xia, Y.S., Wang, J.: On the stability of globally projected dynamical systems. J. Optim. Theory Appl. **106**, 129–150 (2000)
60. Yao, Y., Cho, Y.J., Chen, R.D.: An iterative algorithm for solving fixed point problems, variational inequality problems and mixed equilibrium problems. Nonlinear Anal. **71**, 3363–3373 (2009)
61. Yao, Y., Cho, Y.J., Liou, Y.: Hierarchical convergence of an implicit double-net algorithm for nonexpansive semigroups and variational inequality problems. Fixed Point Theory Appl. **2011**, 101 (2011)
62. Yao, Y., Cho, Y.J., Liou, Y.: Iterative algorithms for variational inclusions, mixed equilibrium problems and fixed point problems approach to optimization problems. Central Eur. J. Math. **9**, 640–656 (2011)
63. Zhang, D., Nagurney, A.: On the stability of the projected dynamical systems. J. Optim. Theory Appl. **85**, 97–124 (1985)