Optim Lett (2013) 7:1373–1386
DOI 10.1007/s11590-012-0450-3

ORIGINAL PAPER

On the set-semidefinite representation of nonconvex quadratic programs over arbitrary feasible sets

Gabriele Eichfelder · Janez Povh

Received: 30 June 2011 / Accepted: 26 January 2012 / Published online: 16 February 2012

Abstract In the paper we prove that any nonconvex quadratic problem over some set $K \subseteq \mathbb{R}^n$ with additional linear and binary constraints can be rewritten as a linear problem over the cone, dual to the cone of K-semidefinite matrices. We show that when K is defined by one quadratic constraint or by one concave quadratic constraint and one linear inequality, then the resulting K-semidefinite problem is actually a semidefinite programming problem. This generalizes results obtained by Sturm and Zhang (Math Oper Res 28:246–267, 2003). Our result also generalizes the well-known completely positive representation result from Burer (Math Program 120:479–495, 2009), which is actually a special instance of our result with $K = \mathbb{R}^n_+$.

Keywords Set-positivity · Semidefinite programming · Copositive programming · Mixed integer programming

1 Introduction

In [2,15,17,18] several hard problems from combinatorial optimization have been reformulated as linear programs over the cone of copositive or completely positive matrices. In [5], Burer generalized these results as follows: under rather weak assumptions any nonconvex quadratic problem over the nonnegative orthant with some additional linear and binary constraints can be rewritten as a linear problem over the cone of completely positive matrices. This result has been generalized to capture

G. Eichfelder (✉)
Department of Mathematics, University of Erlangen-Nuremberg, Martensstr. 3,
91058 Erlangen, Germany
e-mail: Gabriele.Eichfelder@am.uni-erlangen.de

J. Povh
Faculty of information studies in Novo mesto, Novi trg 5, 8000 Novo mesto, Slovenia
e-mail: janez.povh@fis.unm.si

the case when the nonnegative orthant is replaces by an arbitrary closed convex cone, see e.g. [6,13].

The main contribution of this paper consists of the following result: optimization problems with a nonconvex quadratic objective function where the feasible set consists of all vectors from a given set K which satisfy given linear and binary constraints have a conic linear programming formulation, i.e. the optimal value of such problems is equal to the optimal value of a linear function over the domain of symmetric matrices which satisfy a bunch of linear constraints and are contained in the cone, dual to the K-semidefinite cone. We also explain the relations between the original feasible set and the feasible set of the conic linear problem. If K is the nonnegative orthant $\mathbb{R}^n_+$, than our result is essentially the same as the well-known Burer's completely positive representation result [5]. If K is a closed convex cone, then this result coincides with results from [6,13].

Our result also captures and generalizes the quadratic cases from Sturm and Zhang [19]: if K is defined by a single quadratic constraint or by one concave quadratic constraint and by one linear inequality, then the resulting conic linear program is actually a semidefinite programming problem, a result that cannot be obtained straightforwardly from the approach in [19] because we allow in the original problem linear constraints and binary constraints. Since the set K can be arbitrary set we can handle also sets K defined by more than one quadratic constraint. Unfortunately, in these cases the resulting conic linear problem is usually no longer a semidefinite programming problem.

Recently Burer and Dong [8] presented a method how to rewrite under some assumptions a non-convex quadratic program over the product of second order cones as a linear program over the dual of the K-semidefinite cone, denoted there as the set of generalized completely positive matrices. This result is complementary to the results in this paper because it is about optimization problems with quadratic constraints where the underlaying set is the product of second order cones, while we consider linear and binary constraints and the underlaying set can be an arbitrary set (not necessarily a cone).

1.1 Notation

In [5] the reformulation is done over the cone of *completely positive* matrices

$$\mathcal{C}^*_{\mathbb{R}^n_+} := \left\{ \sum_i x^i (x^i)^\top : \; x^i \in \mathbb{R}^n_+ \right\} \tag{1}$$

which is the dual cone of the cone of *copositive* matrices defined by

$$\mathcal{C}_{\mathbb{R}^n_+} := \left\{ A \in \mathscr{S}^n : x^\top A x \geq 0 \;\text{ for all }\; x \in \mathbb{R}^n_+ \right\}. \tag{2}$$

Here, $\mathscr{S}^n$ denotes the space of real symmetric $n \times n$ matrices equipped with the inner product defined by $\langle A, B \rangle := \operatorname{trace}(AB)$ for all $A, B \in \mathscr{S}^n$. Recall that the dual cone of a cone C in a topological space X is in general defined by

$$C^* := \left\{x^* \in X^* : x^*(x) \geq 0 \text{ for all } x \in C\right\}$$

with X^* denoting the topological dual space, i.e. the space of all continuous linear maps from X to $\mathbb{R}$.

Replacing $\mathbb{R}^n_+$ in (1) and (2) by an arbitrary nonempty set $K \subseteq \mathbb{R}^n$ we obtain

$$C_K := \{A \in \mathscr{S}^n : x^\top Ax \geq 0 \text{ for all } x \in K\}$$

which is called the K*-semidefinite* (or *set-semidefinite*) cone. In opposition to [11,12] we define here the K-semidefinite cone in the subspace of symmetric matrices instead of in the whole space of linear maps mapping from the Euclidean space $\mathbb{R}^n$ to $\mathbb{R}^n$. The K-semidefinite cone is a convex cone and hence defines a partial ordering in the space of symmetric matrices.

Remark 1 If $K = \mathbb{R}^n$ then C_K and C_K^* are exactly the cone of positive semidefinite matrices denoted by $\mathscr{S}_n^+$.

In this paper, cone(Ω) for some set Ω denotes the cone generated by the set, conv(Ω) is the convex hull and cone conv(Ω) denotes the convex cone generated by the set Ω, i.e. cone conv$(\Omega) = \{\sum_i \alpha_i x^i : \alpha_i \geq 0,\ x^i \in \Omega\}$, and cl$(\Omega)$ is the closure of the set Ω. Further, we assume K to be a nonempty subset of $\mathbb{R}^n$.

1.2 Technical preliminaries

Under the assumptions here, i.e. $K \subseteq \mathbb{R}^n$, the dual cone of the K-semidefinite cone was given in [19, Prop. 1, Lemma 1]:

Lemma 2 *Let $K \subseteq \mathbb{R}^n$ be a nonempty set, then*

$$C_K^* = \text{cl cone conv}\{xx^\top : x \in K\}.$$

Since C_K^* is the closure of a convex cone generated by $\{xx^\top : x \in K\}$, using Carathéodory's theorem we can represent the dual cone by

$$C_K^* = \text{cl}\left(\left\{\sum_{i=1}^{n(n+1)/2} \alpha_i x^i (x^i)^\top \ :\ \alpha_i \geq 0,\ x^i \in K,\ \forall i = 1, \ldots, \frac{n(n+1)}{2}\right\}\right).$$

For shortness of the representation we omit the upper limit $p := n(n+1)/2$ in the sum above and write instead in the following $C_K^* = \text{cl}(\{\sum_i \alpha_i x^i (x^i)^\top \ :\ \alpha_i \geq 0,\ x^i \in K\})$. We would like to add, that in [19, Lemma 1] it was shown that

$$\text{cl cone conv}\{xx^\top : x \in K\} = \text{cone conv}\{xx^\top : x \in \text{cl}(K)\}, \tag{3}$$

for an arbitrary set $K \subseteq \mathbb{R}^n$. We have realized that there is a mistake in the proof of this lemma which is based on the fact that cl cone$(K) \neq$ cone cl(K) for some set K (e.g. for $K = \{1\} \times [0, \infty)$), see the next example.

Example 3 Consider the closed set $K = \{1\} \times [0, \infty)$ and let $x > 0$ be given. Then

$$Y^* := \begin{pmatrix} 0 & 0 \\ 0 & x^2 \end{pmatrix} \in \mathrm{cl}\left(\left\{\sum_i \alpha_i \begin{pmatrix} 1 \\ x^i \end{pmatrix}\begin{pmatrix} 1 \\ x^i \end{pmatrix}^\top : \alpha_i \geq 0,\ x^i \geq 0\right\}\right)$$
$$= \mathrm{cl\ cone\ conv}\{yy^\top : y \in K\}$$

as for all $k = 1, 2, 3, \ldots,$

$$Y_k := \frac{1}{k^2}\begin{pmatrix} 1 \\ kx \end{pmatrix}\begin{pmatrix} 1 \\ kx \end{pmatrix}^\top \in \mathrm{cone\ conv}\{yy^\top : y \in K\}$$

and $\lim_{k\to\infty} Y_k = Y^*$, but

$$Y^* \notin \left\{\sum_i \alpha_i \begin{pmatrix} 1 \\ x^i \end{pmatrix}\begin{pmatrix} 1 \\ x^i \end{pmatrix}^\top : \alpha_i \geq 0,\ x^i \geq 0\right\}$$
$$= \mathrm{cone\ conv}\{yy^\top : y \in \mathrm{cl}\, K\} = \mathrm{cone\ conv}\{yy^\top : y \in K\}.$$

So when $K \subseteq \mathbb{R}^n$ is an arbitrary set then result (3) is not necessarily true. However, if K is a cone, we retrieve the result gained in [19, Lemma 1].

Lemma 4 *Let $K \subseteq \mathbb{R}^n$ be a nonempty set, then*

$$C_K^* = \mathrm{cl\ cone\ conv}\{xx^\top : x \in K\} = \mathrm{conv}\{xx^\top : x \in \mathrm{cl\,cone}\, K\}.$$

Proof First, suppose $Z \in \mathrm{cl\ cone\ conv}\{xx^\top : x \in K\}$. Then $Z = \lim_{k\to\infty} Z_k$ with $Z_k \in \mathrm{cone\ conv}\{xx^\top : x \in K\}$ and

$$Z_k = \sum_i \alpha_{k,i}^2 x^{k,i}(x^{k,i})^\top$$

for nonnegative scalars $\alpha_{k,i}$ and vectors $x^{k,i} \in K$ for all $i \leq n(n+1)/2$. By defining Y_k to be the matrix with columns $\alpha_{k,i}x^{k,i}$, we can write $Z_k = Y_k Y_k^\top$. We have

$$\lim_{k\to\infty} \|Y_k\|^2 = \lim_{k\to\infty} \mathrm{trace}(Y_k Y_k^\top) = \lim_{k\to\infty} \mathrm{trace}(Z_k) = \mathrm{trace}(Z)$$

(with the norm denoting the Frobenius norm). Thus, the sequence Y_k is bounded and has a cluster point Y^*. Each column of Y^* is thus a limit for $k \to \infty$ of a sequence of elements $\alpha_{k,i}x^{k,i}$ of the cone generated by K. Hence, the columns of Y^* belong to the closure of the cone generated by K. So, $Z = Y^*(Y^*)^\top \in \mathrm{conv}\{xx^\top : x \in \mathrm{cl\,cone}\, K\}$.

To prove the converse, we assume $Z \in \mathrm{conv}\{xx^\top : x \in \mathrm{cl\,cone}\, K\}$. Then

$$Z = \sum_i \alpha_i \bar{x}^i(\bar{x}^i)^\top$$

with nonnegative scalars $\alpha_i, \bar{x}^i \in \mathrm{cl\,cone}\, K$ for all $i \leq n(n+1)/2$. Thus, there exist sequences $\lambda_{i,k} \geq 0$ and $x^{i,k} \in K$ with $\bar{x}^i = \lim_{k\to\infty} \lambda_{i,k} x^{i,k}$. Then

$$\begin{aligned} Z &= \sum_i \alpha_i \left(\lim_{k\to\infty} \lambda_{i,k} x^{i,k}\right)\left(\lim_{k\to\infty} \lambda_{i,k} x^{i,k}\right)^\top \\ &= \lim_{k\to\infty} \sum_i \left(\alpha_i \lambda_{i,k}^2\right) x^{i,k} (x^{i,k})^\top \end{aligned}$$

and $Z \in \mathrm{cl\ cone\ conv}\{xx^\top : x \in K\}$.

Corollary 5 *If $K \subseteq \mathbb{R}^n$ is a nonempty cone, then*

$$C_K^* = \mathrm{cl\,conv}\{xx^\top : x \in K\} = \mathrm{conv}\{xx^\top : x \in \mathrm{cl}(K)\}.$$

If K is a nonempty closed cone, then the dual cone reduces to

$$C_K^* = \mathrm{conv}\{xx^\top : x \in K\} = \left\{\sum_i x^i (x^i)^\top : x^i \in K\right\}$$

and C_K^* is closed.

We call constraints $X \in C_K$ (or $X \in C_K^*$) *set-semidefinite* constraints. Anstreicher and Burer give in [1] for low dimensions computable representations of C_K^* in terms of matrices that are positive semidefinite and componentwise nonnegative. For $n = 5$ and $K = \mathbb{R}_+^5$ examinations of the cone of completely positive matrices $C_{\mathbb{R}_+^5}^*$ are done by Burer, Anstreicher and Dür in [7]. Jarre and Schmallowsky present in [14] a numerical test for checking whether some matrix is an element of the cone of completely positive matrices $C_{\mathbb{R}_+^n}^*$. Recently, a numerical test for detecting copositivity based on simplical partitions and several sufficient conditions has been developed by Bundfuss and Dür, see [9,10], and Bomze and Eichfelder [3]. We are not aware of results about separation problems for C_K or C_K^* for general K.

The following lemma is the base for our main result and states that the optimal value of a quadratic function over an arbitrary set S is equal to the optimal value of the corresponding linear function over the convex set generated by dyadic products of elements from this set S.

Lemma 6 *Let a matrix $Q \in \mathscr{S}^n$, a vector $c \in \mathbb{R}^n$ and a nonempty set $S \subseteq \mathbb{R}^n$ be given. Then the following is true*

$$\inf\ \{x^\top Q x + 2c^\top x : x \in S\} \tag{4}$$

$$= \inf\left\{\langle \tilde{Q}, Y\rangle : Y \in \mathrm{conv}\left\{\begin{pmatrix}1\\x\end{pmatrix}\begin{pmatrix}1\\x\end{pmatrix}^\top : x \in S\right\}\right\}, \tag{5}$$

where $\tilde{Q} = \begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}$. *Moreover, if the optimal value of* (5) *is attained then there exists a rank one optimal solution.*

Proof The "$\geq$" part is easy. For any $x \in S$ the matrix $Y = (1\ x^\top)^\top (1\ x^\top)$ is feasible for (5) and gives the same objective value. To prove the converse let us consider $Y = \sum_i \lambda_i \begin{pmatrix} 1 \\ x^i \end{pmatrix} \begin{pmatrix} 1 \\ x^i \end{pmatrix}^\top$, where $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$ and $x^i \in S$, for all i. Let $\bar{x} \in S$ such that

$$\left\langle \tilde{Q}, \begin{pmatrix} 1 \\ \bar{x} \end{pmatrix} \begin{pmatrix} 1 \\ \bar{x} \end{pmatrix}^\top \right\rangle = \min_i \left\{ \left\langle \tilde{Q}, \begin{pmatrix} 1 \\ x^i \end{pmatrix} \begin{pmatrix} 1 \\ x^i \end{pmatrix}^\top \right\rangle \right\}.$$

Then

$$\begin{aligned} \langle \tilde{Q}, Y \rangle &= \sum_i \lambda_i (\langle Q, x^i (x^i)^\top \rangle + 2c^\top x^i) \geq \sum_i \lambda_i (\langle Q, \bar{x}\bar{x}^\top \rangle + 2c^\top \bar{x}) \\ &= \langle Q, \bar{x}\bar{x}^\top \rangle + 2c^\top \bar{x} = \bar{x}^\top Q \bar{x} + 2c^\top \bar{x}. \end{aligned}$$

It follows that the optimal value of (4) is less or equal to the optimal value of (5). Together with the first part we have the equality. The last assertion is trivial.

2 Set-semidefinite reformulation of quadratic programs

In this section we examine the equivalence between a quadratic optimization problem with linear constraints, a *set* constraint and binary variables, and the reformulation of this problem as a linear program over the dual cone of set-semidefinite matrices. Let $Q \in \mathcal{S}^n$ be a symmetric matrix, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $K \subseteq \mathbb{R}^n$ a nonempty set and $B \subseteq \{1, \ldots, n\}$ an index set. We study the following quadratic optimization problem

$$\begin{aligned} \mathrm{OPT}_{QP} := \inf\ & x^\top Q x + 2c^\top x \\ \text{such that}\ & \\ & Ax = b, \\ & x_j \in \{0, 1\} \text{ for all } j \in B, \\ & x \in K. \end{aligned} \tag{QP}$$

We will refer to the following notation

$$\begin{aligned} \mathrm{Feas}(QP) &:= \{x : x \text{ feasible for } (QP)\}, \\ \mathrm{Feas}^+(QP) &:= \mathrm{conv} \left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^\top : x \in \mathrm{Feas}(QP) \right\}. \end{aligned}$$

We follow the line of [5] and assume in the following:

Assumption 1 If $Ax = b$ and $x \in K$, then $x_j \in [0, 1]$ for all $j \in B$.

Remark 7 Assumption 1 is not very restrictive, if we are allowed to change the set K. Suppose that it does not hold for some $j \in B$, e.g. we have $Ax = b$ but this does not imply $x_j \in [0, 1]$. Then we can add two more equations $x_j + y_j = 1$, $x_j - z_j = 0$ and two sign constraints: $y_j, z_j \geq 0$. Hence, by using $K' := K \times \mathbb{R}^2_+$ with constraints $\{Ax = b, x_j + y_j = 1, x_j - z_j = 0\}$, we fulfill the assumption.

The described method is especially of interest for K a cone, as $K' = K \times \mathbb{R}^2_+$ remains to be a cone and the special structure is not destroyed. Otherwise, we can of course simply replace K by $K' = K \cap \{x \in \mathbb{R}^n : x_j \in [0, 1] \text{ for all } j \in B\}$.

However, if we can not change the set K, as is the case in (12) below, then Assumption 1 is very restrictive. If the set B is empty this assumption is trivial.

If $\text{Feas}(QP)$ is unbounded then $\text{Feas}^+(QP)$ might not be closed. Using the following definitions,

$$L_\infty := \{d \in \mathbb{R}^n_+ : Ad = 0\}, \quad L^+_\infty := \text{conv}\left\{\begin{pmatrix}0\\d\end{pmatrix}\begin{pmatrix}0\\d\end{pmatrix}^\top : d \in L_\infty\right\},$$

Bomze and Jarre [4] proved for the case $K = \mathbb{R}^n_+$ that $\text{cl}(\text{Feas}^+(QP)) = \text{Feas}^+(QP) + L^+_\infty$, under the assumption that the constraint $Ax = b$ together with $x \in \mathbb{R}^n_+ = K$ implies that x_j is bounded for all $j \in B$ (this assumption is implied by Assumption 1).

We cannot extend these result to a general K since Bomze and Jarre used polyhedron-based arguments which are true for $K = \mathbb{R}^n_+$ and do not hold for a general K.

Lemma 8 *Let us consider*

$$OPT_{P1} = \inf\left\{\left\langle\begin{pmatrix}0 & c^\top\\ c & Q\end{pmatrix}, Y\right\rangle : Y \in \text{cl}(Feas^+(QP))\right\}.$$

Then $OPT_{QP} = OPT_{P1}$.

Proof This lemma is a straightforward corollary of Lemma 6. Indeed, the infima of $\left\langle\begin{pmatrix}0 & c^\top\\ c & Q\end{pmatrix}, Y\right\rangle$ over $\text{cl}(\text{Feas}^+(QP))$ and $\text{Feas}^+(QP)$ are the same due to linearity of the objective function. Combining this with Lemma 6 yields the result.

We can further rewrite the feasible set $\text{cl}(\text{Feas}^+(QP))$ as an intersection of the cone $C^*_{1\times K}$ with an affine space defined by the other constraints from (QP). Here, by $1 \times K$ we shortcut the set $\{1\} \times K$. First, we point out that any matrix $Y = \begin{pmatrix}Y_{11} & x^\top\\ x & X\end{pmatrix} \in \text{Feas}^+(QP)$ is feasible for:

$$Y_{11} = 1, \tag{6}$$
$$Ax = b, \tag{7}$$
$$\mathrm{Diag}(AXA^\top) = b \circ b := (b_1^2,\, b_2^2, \dots, b_m^2)^\top, \tag{8}$$
$$x_j = X_{jj} \ \text{ for all } j \in B. \tag{9}$$

We consider the dual cone of the $1 \times K$-semidefinite cone:

$$C^*_{1\times K} = \mathrm{cl}\left(\left\{\sum_i \lambda_i \begin{pmatrix} 1 \\ x^i \end{pmatrix}\begin{pmatrix} 1 \\ x^i \end{pmatrix}^\top \ : \ \lambda_i \ge 0,\ x^i \in K\right\}\right).$$

Note that the cone $C^*_{1\times K}$ is a closed convex cone. We have the following equality.

Lemma 9 *Under Assumption 1 we have*

$$\mathrm{cl}\, \mathit{Feas}^+(QP) = C^*_{1\times K} \cap \{Y \in \mathscr{S}^{n+1} \colon Y \textit{ feasible for } (6)-(9)\}$$

Proof The inclusion "⊆" follows from above since the set on the right hand side is closed. To prove the converse inclusion let us consider

$$Y = \sum_i \lambda_i \begin{pmatrix} 1 \\ x^i \end{pmatrix}\begin{pmatrix} 1 \\ x^i \end{pmatrix}^\top \in C^*_{1\times K} \cap \{Y \in \mathscr{S}^{n+1} \colon Y \text{ feasible for } (6)-(9)\}, \tag{10}$$

where $\lambda_i > 0$ and $x^i \in K$. Constraints (6)–(8) imply that $\sum_i \lambda_i = 1$ and for every row a_j of matrix A we have

$$\sum_i \lambda_i a_j^\top x^i = b_j \ \text{ and } \ \sum_i \lambda_i (a_j^\top x^i)^2 = b_j^2$$

It follows that

$$0 = \sum_i \lambda_i (a_j^\top x^i)^2 - \left(\sum_i \lambda_i a_j^\top x^i\right)^2 = \sum_i \lambda_i \left(a_j^\top x^i - \sum_k \lambda_k a_j^\top x^k\right)^2 \ge 0,$$

hence the equality is throughout. This is possible only if $a_j^\top x^i - \sum_k \lambda_k a_j^\top x^k = 0$ for all i, hence $a_j^\top x^i = a_j^\top x^k$ for all i, k and finally $a_j^\top x^i = b_j$, for all i. Constraint (9) is equivalent to

$$\sum_i \lambda_i x_j^i - \sum_i \lambda_i (x_j^i)^2 = \sum_i \lambda_i x_j^i (1 - x_j^i) = 0. \tag{11}$$

Assumption 1 implies that $x_j^i \in [0, 1]$ for all $j \in B$. Then (11) is possible if and only if $x_j^i \in \{0,\ 1\}$, for all i and for all $j \in B$. Therefore $x^i \in \mathrm{Feas}(QP)$ and $Y \in \mathrm{Feas}^+(QP)$.

The set $C^*_{1\times K} \cap \{Y \in \mathscr{S}^{n+1} \colon Y$ feasible for $(6)-(9)\}$ is a closure of matrices which can be decomposed as (10). Since $\mathrm{cl}\,\mathrm{Feas}^+(QP)$ is closed, the inclusion "⊇" follows.

Lemmas 8 and 9 directly imply that

Theorem 10 *Let Assumption 1 be satisfied. The optimal value OPT_{QP} is equal to the optimal value of*

$$\begin{aligned} OPT_C := \inf\ & \left\langle \begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}, Y \right\rangle \\ \textit{such that}\ & \\ & Y \in C^*_{1\times K}, \\ & Y \quad \textit{feasible for } (6)-(9). \end{aligned} \tag{QP$_\text{C}$}$$

Note that (QP$_\text{C}$) is a linear program over the dual cone $C^*_{1\times K}$. We transformed all nonlinearity and nonconvexity into the structure of the closed convex cone $C^*_{1\times K}$.

3 Relations with some existing representation results

Theorem 10 is a generalization of the completely positive representation results by Burer [5,6] and Eichfelder and Povh [13]. Burer presented a completely positive reformulation of (QP) for the case $K = \mathbb{R}^n_+$. This result was independently generalized further by Burer [6] and Eichfelder and Povh [13] to K an arbitrary closed convex cone. Theorem 10 is therefore the most general representation result since we assumed only that K is an arbitrary set.

In this section we show that Theorem 10 generalizes the work from Burer [5] and from Sturm and Zhang [19].

3.1 Optimization over the nonnegative orthant

Burer [5] considered problem (QP) when K is the nonnegative orthant $\mathbb{R}^n_+$ and obtained the following result

$$\begin{aligned} \text{OPT}_{QP} = \inf\ & \left\langle \begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}, Y \right\rangle \\ \text{such that}\ & \\ & Y \in C^*_{\mathbb{R}^{n+1}_+}, \\ & Y \quad \text{feasible for } (6)-(9). \end{aligned}$$

where $C^*_{\mathbb{R}^{n+1}_+}$ is the cone of completely positive matrices. Note that the only difference between this formulation and our formulation from Theorem 10 is the constraint $C^*_{\mathbb{R}^{n+1}_+}$. We can prove

Lemma 11 $C^*_{\mathbb{R}^{n+1}_+} = C^*_{1\times\mathbb{R}^+_n}$.

Proof The direction "⊆" is obvious. For the other direction let us consider $Y \in C^*_{\mathbb{R}^{n+1}_+}$. Then there are $\alpha_i \geq 0$ and $x^i \in \mathbb{R}^n_+$ with

$$Y = \sum_i \begin{pmatrix} \alpha_i \\ x^i \end{pmatrix} \begin{pmatrix} \alpha_i \\ x^i \end{pmatrix}^\top = \underbrace{\sum_{i\,:\,\alpha_i \neq 0} \alpha_i^2 \begin{pmatrix} 1 \\ \frac{1}{\alpha_i} x^i \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{\alpha_i} x^i \end{pmatrix}^\top}_{Y_1} + \underbrace{\sum_{i\,:\,\alpha_i = 0} \begin{pmatrix} 0 \\ x^i \end{pmatrix} \begin{pmatrix} 0 \\ x^i \end{pmatrix}^\top}_{Y_2} \in C^*_{\mathbb{R}^{n+1}_+}.$$

Obviously $Y_1 \in C_{1\times\mathbb{R}^n_+}$. Since for all i with $\alpha_i = 0$

$$\begin{pmatrix} 0 \\ x^i \end{pmatrix} \begin{pmatrix} 0 \\ x^i \end{pmatrix}^\top = \lim_{n\to\infty} \underbrace{\frac{1}{n^2} \begin{pmatrix} 1 \\ nx^i \end{pmatrix} \begin{pmatrix} 1 \\ nx^i \end{pmatrix}^\top}_{\in C_{1\times\mathbb{R}^n_+}},$$

we have $Y_2 \in C^*_{1\times\mathbb{R}^n_+}$, too. As $C_{1\times\mathbb{R}^n_+}$ is a convex cone this implies $Y \in C_{1\times\mathbb{R}^n_+}$.

Corollary 12 *For the case $K = \mathbb{R}^n_+$ the set-semidefinite representation of* (QP) *from Theorem* 10 *coincides with the completely positive representation from* [5].

3.2 Optimization problems with one quadratic constraint

Sturm and Zhang [19] considered optimization problems with one quadratic constraint and showed that under some assumptions they have a semidefinite programming representation. In this subsection we study the same type of problems and prove results, slightly more general (or at least more straightforward) as they proved.

Let us consider the case when K is a (nonconvex) nonempty set defined by one quadratic constraint:

$$K = \{x \in \mathbb{R}^n : x^\top P x + 2p^\top x + p_0 \leq 0\} \tag{12}$$

where $p \in \mathbb{R}^n$, $p_0 \in \mathbb{R}$ and $P \in \mathcal{S}^n$. The dual of the $1 \times K$-semidefinite cone is

$$C^*_{1\times K} = \mathrm{cl}\left(\left\{\sum_i \lambda_i \begin{pmatrix} 1 \\ x^i \end{pmatrix} \begin{pmatrix} 1 \\ x^i \end{pmatrix}^\top : \lambda_i \geq 0, \left\langle \begin{pmatrix} p_0 & p^\top \\ p & P \end{pmatrix}, \begin{pmatrix} 1 \\ x^i \end{pmatrix} \begin{pmatrix} 1 \\ x^i \end{pmatrix}^\top \right\rangle \leq 0 \right\}\right).$$

We have the following representation for $C^*_{1\times K}$:

Lemma 13

$$C^*_{1\times K} = \left\{ \begin{pmatrix} Y_{11} & x^\top \\ x & X \end{pmatrix} \in \mathcal{S}^+_{n+1} : \left\langle \begin{pmatrix} p_0 & p^\top \\ p & P \end{pmatrix}, \begin{pmatrix} Y_{11} & x^\top \\ x & X \end{pmatrix} \right\rangle \leq 0 \right\}.$$

Proof The direction "$\subseteq$" is obvious. For the converse let us consider $Y \in \mathscr{S}_{n+1}^{+}$ with

$$\left\langle \begin{pmatrix} p_0 & p^\top \\ p & P \end{pmatrix}, \begin{pmatrix} Y_{11} & x^\top \\ x & X \end{pmatrix} \right\rangle \leq 0.$$

Lemma 2.4 from [16] (see also Proposition 3 from [19]), which is also the crucial result for an alterative proof of the famous S-lemma, implies that there exist $\begin{pmatrix} \alpha_i \\ x^i \end{pmatrix} \in \mathbb{R}^{n+1}$ such that $Y = \sum_i \begin{pmatrix} \alpha_i \\ x^i \end{pmatrix} \begin{pmatrix} \alpha_i \\ x^i \end{pmatrix}^\top$ and

$$\left\langle \begin{pmatrix} p_0 & p^\top \\ p & P \end{pmatrix}, \begin{pmatrix} \alpha_i \\ x^i \end{pmatrix} \begin{pmatrix} \alpha_i \\ x^i \end{pmatrix}^\top \right\rangle = (x^i)^\top P x^i + 2\alpha_i p^\top x^i + \alpha_i^2 p_0 \leq 0, \text{ for all } i.$$

Without loss of generality we may assume that $\alpha_i \geq 0$. If $\alpha_i > 0$ then $\begin{pmatrix} \alpha_i \\ x^i \end{pmatrix} \begin{pmatrix} \alpha_i \\ x^i \end{pmatrix}^\top = \alpha_i^2 \begin{pmatrix} 1 \\ \frac{1}{\alpha_i} x^i \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{\alpha_i} x^i \end{pmatrix}^\top \in C_{1\times K}^*$. The remaining of the proof deals with the case $\alpha_i = 0$. Then $(x^i)^\top P x^i \leq 0$.

If $(x^i)^\top P x^i < 0$ then $(x^i)^\top P x^i + 2\varepsilon p^\top x^i + \varepsilon^2 p_0 \leq 0$ for ε sufficiently small, hence

$$Y_\varepsilon = \begin{pmatrix} \varepsilon \\ x^i \end{pmatrix} \begin{pmatrix} \varepsilon \\ x^i \end{pmatrix}^\top = \varepsilon^2 \begin{pmatrix} 1 \\ \frac{1}{\varepsilon} x^i \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{\varepsilon} x^i \end{pmatrix}^\top \in C_{1\times K}^*$$

and $\lim_{\varepsilon\to 0} Y_\varepsilon = \begin{pmatrix} 0 \\ x^i \end{pmatrix} \begin{pmatrix} 0 \\ x^i \end{pmatrix}^\top \in C_{1\times K}^*$. By the same line of reasoning we prove the case when $(x^i)^\top P x^i = 0$ and $p^\top x^i \neq 0$. We consider $Y_\varepsilon = \begin{pmatrix} \varepsilon \\ x^i \end{pmatrix} \begin{pmatrix} \varepsilon \\ x^i \end{pmatrix}^\top$ if $p^\top x^i < 0$ and $Y_\varepsilon = \begin{pmatrix} \varepsilon \\ -x^i \end{pmatrix} \begin{pmatrix} \varepsilon \\ -x^i \end{pmatrix}^\top$ if $p^\top x^i > 0$.

If $(x^i)^\top P x^i = 0$ and $p^\top x^i = 0$ then

$$\begin{pmatrix} 0 \\ x^i \end{pmatrix} \begin{pmatrix} 0 \\ x^i \end{pmatrix}^\top = \lim_{\varepsilon\to 0} \varepsilon^2 \begin{pmatrix} 1 \\ z \pm \frac{1}{\varepsilon} x^i \end{pmatrix} \begin{pmatrix} 1 \\ z \pm \frac{1}{\varepsilon} x^i \end{pmatrix}^\top \in C_{1\times K}^*$$

where z is an arbitrary vector from K. Note that we take the sign of $\frac{1}{\varepsilon} x^i$ such that $\pm (x^i)^\top P z \leq 0$.

As $C_{1\times K}^*$ is a convex cone the assertion is proven.

We point out that Lemma 13 contains essentially the same result as Theorem 1 from [19]. However, the proof of the later result is based on Lemma 1 from [19] which we disprove in Section 1. Instead of fixing the original rather complex proof we decided to provide a straightforward proof here.

We have the following corollary.

Corollary 14 *We have the following semidefinite programming representation of* (QP) *for the case when* K *is of the form* (12)*:*

$$\begin{aligned} OPT_{QP} = \inf & \left\langle \begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}, Y \right\rangle \\ \textit{such that} & \\ Y = & \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix} \in \mathscr{S}^+_{n+1}, \\ Ax = & b, \\ Diag(AXA^\top) = & b \circ b, \\ \left\langle \begin{pmatrix} p_0 & p^\top \\ p & P \end{pmatrix}, \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix} \right\rangle \le & 0. \end{aligned}$$

We demonstrate Corollary 14 by the following example.

Example 15 Let us consider the following nonconvex quadratic problem:

$$\inf \{x^2 + xy + y^2 - 2x - 2y : \; y + x/2 = 2, \; x^2 - y^2 - 2xy + 1 \le 0\}.$$

The feasible set is plotted in Fig. 1 as a bold line above the interval $[-\frac{6}{7}, 2]$.

The optimal value is $-\frac{1}{3}$ and is attained at $x = \frac{2}{3}$, $y = \frac{5}{3}$. Theorem 10 and Lemma 13 imply that we can reformulate this optimization problem into

$$\begin{gathered} \inf \; \langle Q, Y \rangle \\ \text{such that} \\ Y \in \mathscr{S}^+_3, Y_{11} = 1, \langle A_1, Y \rangle = 4, \langle A_2, Y \rangle = 4, \langle A_3, Y \rangle \le 0 \end{gathered}$$

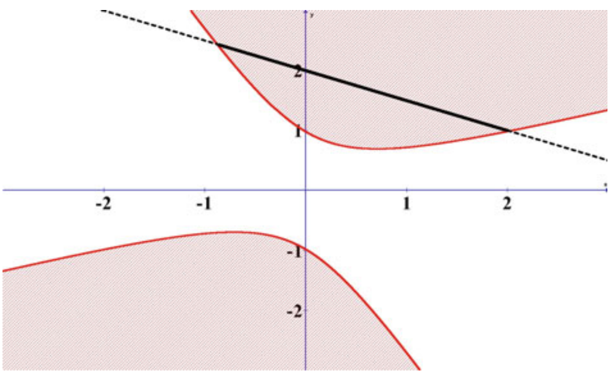

Fig. 1 Feasible set of the quadratic problem of Example 15

where

$$Q = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 1 & 0.5 \\ -1 & 0.5 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0.5 & 1 \\ 0.5 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.25 & 0.5 \\ 0 & 0.5 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & -1 \end{pmatrix}.$$

The optimal solution is a rank one matrix yielding the optimal value $-1/3$:

$$Y_{\text{opt}} = \begin{pmatrix} 1 & \frac{2}{3} & \frac{5}{3} \\ \frac{2}{3} & \frac{4}{9} & \frac{10}{9} \\ \frac{5}{3} & \frac{10}{9} & \frac{25}{9} \end{pmatrix}$$

Remark 16 Our approach generalizes the results from Sturm and Zhang [19] in the following ways:

– beside one quadratic constraint of type (12) we can handle arbitrary many linear constraints in the problem (QP).
– we can also include binary constraints provided that Assumption 1 is satisfied.

Thanks to the comment of one of the referees we have realized that the original approach from Sturm and Zhang can also be easily extended to cover linear equality constraints directly. Indeed, by a proper change of variables we can eliminate x and work entirely in the affine subspace $\{x\colon\ Ax = b\}$. The set K from (12) becomes the intersection of K and $\{x\colon\ Ax = b\}$ and keeps the same structure as (12). For instance in Example 15 we can eliminate x by $x = 4 - 2y$ to obtain a quadratically constrained quadratic problem.

The second item from above also needs additional comment. Assumption 1 is very restrictive for this case since we want to keep K as in (12) and we currently do not see a way how to make it hold if it is not satisfied initially.

Remark 17 Theorems 2 and 3 from [19] are true even though they rely on Lemma 1 from [19]. To prove them it is sufficient to use our version of Lemma 1, i.e. Lemma 4 from page 1376. Therefore the problem (QP) where K is defined either by a strictly concave (or a strictly convex) quadratic equality constraint or by a concave and linear inequality constraint also admits semidefinite programming formulations.

4 Conclusions

In the paper we present a result that nontrivially generalizes and connects two important results from Burer [5] and Sturm and Zhang [19]. We show that any quadratic problem where the feasible set is defined by linear and binary constraints and is a subset of some arbitrary set K can be rewritten as a linear program over the cone dual to the K-semidefinite cone. When K is the nonnegative orthant then this result coincides with the completely positive representation result from [5]. When K is defined by one quadratic constraint or by one concave quadratic constraint and one linear inequality then our result generalizes results from Sturm and Zhang [19] since our approach enables direct inclusion of linear equality constraints and under rather restrictive assumption also binary constraints.

Acknowledgments Supported by the Slovenian Research Agency (project no. 1000-08-210518).

References

1. Anstreicher, K.M., Burer, S.: Computable representations for convex hulls of low dimensional quadratic forms. Math. Program. **124**, 33–43 (2010)
2. Bomze, I., Dür, M., de Klerk, E., Roos, C., Quist, A.J., Terlaky, T.: On copositive programming and standard quadratic optimization problems. J. Glob. Optim. **18**, 301–320 (2000)
3. Bomze, I., Eichfelder, G.: Copositivity detection by difference-of-convex decomposition and omega-subdivision, Preprint Series of the Institute of Applied Mathematics, University of Erlangen-Nuremberg, vol. 333 (2010)
4. Bomze, I., Jarre, F.: A note on Burer's copositive representation of mixed-binary QPs. Optim. Lett. **4**, 465–472 (2010)
5. Burer, S.: On the copositive representation of binary and continuous nonconvex quadratic programs. Math. Program. **120**, 479–495 (2009)
6. Burer, S.: Copositive Programming. In: Anjos, M., Lasserre, J.B. (eds.) Handbook of Semidefinite, Conic and Polynomial Optimization, Springer (2011)
7. Burer, S., Anstreicher, K.M., Dür, M.: The difference between 5×5 doubly nonnegative and completely positive matrices. Linear Algebra Appl. **431**, 1539–1552 (2009)
8. Burer, S., Dong, H.: Representing quadratically constrained quadratic programs as generalized copositive programs. http://www.optimization-online.org/DB_HTML/2011/07/3095.html (2011)
9. Bundfuss, S., Dür, M.: Algorithmic copositivity detection by simplicial partition. Linear Algebra Appl. **428**, 1511–1523 (2008)
10. Bundfuss, S., Dür, M.: An adaptive linear approximation algorithm for copositive programs. SIAM J. Optim. **20**, 30–53 (2009)
11. Eichfelder, G., Jahn, J.: Set-Semidefinite Optimization. J. Convex Anal. **15**, 767–801 (2008)
12. Eichfelder, G., Jahn, J.: Foundations of Set-Semidefinite Optimization. In: Pardalos, P., Rassias, Th.M., Khan, A.A. (eds.) Nonlinear Analysis and Variational Problems, Chapter 18, pp. 259–284, Springer (2009)
13. Eichfelder, G., Povh, J.: On the set-semidefinite representation of non-convex quadratic programs with cone constraints. Croat. Oper. Res. Rev. **1**, 26–39 (2011)
14. Jarre, F., Schmallowsky, K.: On the computation of C^* certificates. J. Glob. Optim. **45**, 281–296 (2009)
15. de Klerk, E., Pasechnik, D.V.: Approximation of the stability number of a graph via copositive programming. SIAM J. Optim. **12**, 875–892 (2002)
16. Pólik, I., Terlaky, T.: A survey of the S-lemma. SIAM Rev. **49**, 371–418 (2007)
17. Povh, J., Rendl, F.: A copositive programming approach to graph partitioning. SIAM J. Optim. **18**, 223–241 (2007)
18. Povh, J., Rendl, F.: Copositive and semidefinite relaxations of the quadratic assignment problem. Discret. Optim. **6**, 231–241 (2009)
19. Sturm, J.F., Zhang, S.: On cones of nonnegative quadratic functions. Math. Oper. Res. **28**, 246–267 (2003)