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ORIGINAL PAPER

Optimal solutions for the double row layout problem

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Abstract The double row layout problem is how to allocate a given set of n machines on both sides of a straight line corridor so that the total cost of transporting materials between machines is minimized. This is a very difficult combinatorial optimization problem with important applications in industry. We formulate the problem as a mixed-integer program. Computational tests show that the proposed formulation presents a far superior performance than that of a previously published model.

Keywords Double row layout problem · Machine layout problem · Facility layout · Integer programming

1 Introduction

The double row layout problem (DRLP) is a very difficult combinatorial optimization problem. It occurs in automated manufacturing environments, where a material-handling device transports materials among machines arranged in a *double-row layout*, i.e. a layout in which the machines are located on both sides of a straight line corridor (see, for example, [15,17]). An application of the DRLP within a fabrication line that produces *liquid crystal display* (LCD) was described by Chung and Tanchoco [8].

Facility layout problems are generally NP-hard, which means that their exact solution within reasonable computer times is a very difficult task (e.g. for the two-dimensional facility layout problems considered by Sherali et al. [24], the largest instance size that could be solved to optimality has 9 machines).

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The DRLP is NP-hard and is closely related with another NP-hard problem: the single-row facility layout problem (SRFLP). The SRFLP assigns all of the machines to the same side of a corridor and has wide-ranging applications such as room arrangement in office buildings [25], the arrangement of books on a shelf [21], and flexible manufacturing system design [17]. A number of solution methods have been proposed for the SRFLP, either exact (e.g. [1–4,6,20,21,25,26]) or heuristic [10,16,18,19,22,23,27]. Also, lower bounds for the SRFLP are presented by Anjos et al. [5], Amaral and Letchford [4], and by Anjos and Yen [7].

Our earliest reference to double row layouts in the literature dates from more than 20 years ago (i.e. [17]). Even though, the only published exact method for the DRLP is the Mixed Integer Programming (MIP) model of Chung and Tanchoco [8]. This motivates the present paper.

Chung and Tanchoco [8] exactly solved DRLPs with up to about ten machines using CPLEX 10.2. However, their computational tests showed that DRLP instances with more than ten machines exceeded reasonable computational time. In this paper, we propose a new MIP model that presents a much improved performance in regards to Chung and Tanchoco's [8] model, particularly on the largest instances that we tested.

2 The double row layout problem

In the DRLP we consider the following notation:

n	number of machines
$N = \{1, 2, \ldots, n\}$	set of machines
$R = \{\text{lower row, upper row}\}$	set of rows
c_{ij}	amount of flow between machines i and j
ℓ_i	length of machine $i \in N$
$L := \sum_{i=1}^{n} \ell_i$	sum of all machine lengths

Assumptions:

(i) The corridor is situated with its length along the x-axis on the interval $[0, L]$.
(ii) The width of the corridor is negligible.
(iii) The distance between two machines is taken as the x-distance between their centers.

Then, the DRLP is how to assign the n machines to locations on both sides of the corridor so that the total cost of transporting materials between machines is minimized. A mathematical formulation for the DRLP is given by:

$$\min_{\varphi \in \Phi_n} \sum_{1 \le i < j \le n} c_{ij} d_{ij}^{\varphi} \tag{1}$$

where Φ_n is the set of all double row layouts on the set N; and d_{ij}^{φ} is the distance between machines i and j with reference to a layout $\varphi \in \Phi_n$.

3 The proposed mixed-integer programming model

Consider the vector $\alpha = (\alpha_{ij})_{1\leq i,j\leq n;i\neq j} \in \{0, 1\}^{n(n-1)}$ such that $\alpha_{ij} = 1$, if machine i is to the left of machine j and both i and j are at the same row; $\alpha_{ij} = 0$, otherwise. The convex hull of all α-incidence vectors representing a partition of the set of n machines into two linear orders of the machines yields a polytope which we will denote by P_n. Since a complete description of P_n may not be achievable, we shall devote our attention to classes of inequalities that are valid for the convex hull of points defined by P_n. These might be helpful when solving the DRLP with integer programming using a formulation based on the α-incidence vectors.

The polytope P_n is closely related to other polytopes in the literature such as the *clique partitioning polytope* [13,14] and the *linear ordering polytope* [11,12]. Recently, Coll et al. [9] studied the partitioning of a complete digraph D (on n vertices) into subgraphs such that each subgraph is a linear ordering of its vertices. They defined the *polytope of partitions into linear orderings*, denoted by $P_{PLO(D)}$, as the convex hull of any partition into linear orders of the vertices of D. Clearly $P_n \subseteq P_{PLO(D)}$. In what follows, we discuss a partial description of P_n.

Proposition 1 *The following inequalities are valid for* P_n*:*

$$-\alpha_{ij} + \alpha_{ik} + \alpha_{jk} - \alpha_{ji} + \alpha_{ki} + \alpha_{kj} \leq 1, \quad (i, j, k \in N; i < j; k \neq i, k \neq j) \tag{2}$$

$$-\alpha_{ij} + \alpha_{ik} - \alpha_{jk} + \alpha_{ji} - \alpha_{ki} + \alpha_{kj} \leq 1, \quad (i, j, k \in N; i < j; k < j; i \neq k) \tag{3}$$

Proof Inequalities (2) and (3) are valid for $P_{PLO(D)}$ (see, [9]) and since $P_n \subseteq P_{PLO(D)}$, these inequalities are also valid for P_n. □

Proposition 2 *The following inequality is valid for* P_n*:*

$$\alpha_{ij} + \alpha_{ik} + \alpha_{jk} + \alpha_{ji} + \alpha_{ki} + \alpha_{kj} \geq 1, \quad (1 \leq i < j < k \leq n) \tag{4}$$

Proof Consider a set $\{i, j, k\} \subseteq N$, $1 \leq i < j < k \leq n$. In any feasible DRLP solution, at least two machines of $\{i, j, k\}$ have to be at the same row and since this is ensured by Inequality (4), it follows that Inequality (4) must be valid for P_n. □

The following trivial inequalities are also valid for P_n:

$$0 \leq \alpha_{ij}, \alpha_{ij} \leq 1, \quad (1 \leq i, j \leq n; i \neq j) \tag{5}$$

In the sequel we shall consider the polytope $Q_n (\supseteq P_n)$ defined by:

$$Q_n = \{\alpha \in R^{n(n-1)} : (2), (3), (4), \text{ and } (5)\}.$$

Remark 1 The integral points in Q_n are precisely the α-incidence vectors, which means that $P_n \equiv conv(\alpha \in Q_n : \alpha \text{ is integral})$.

Now, consider the following continuous variables:
x_i abscissa of the center of machine $i\,(1 \le i \le n)$ in the x-axis
d_{ij} x-distance between (the centers of) machines i and $j\,(1 \le i < j \le n)$.

Our proposed mixed-integer programming formulation of the DRLP is given by:

$$\text{Minimize} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij} d_{ij} \tag{6}$$

$$d_{ij} \ge x_i - x_j \quad (1 \le i < j \le n) \tag{7a}$$

$$d_{ij} \ge x_j - x_i \quad (1 \le i < j \le n) \tag{7b}$$

$$d_{ij} - \left(\frac{\ell_i + \ell_j}{2}\right)\alpha_{ij} - \left(\frac{\ell_i + \ell_j}{2}\right)\alpha_{ji} \ge 0, \quad (1 \le i < j \le n), \tag{8}$$

$$x_i + \left(\frac{\ell_i + \ell_j}{2}\right) \le x_j + L(1 - \alpha_{ij}), \qquad (1 \le i, j \le n; i \ne j). \tag{9}$$

$$x_{i*} \le x_{j*}, \quad (i*, j*) = \arg \min_{1 \le i < j \le n} c_{ij} \tag{10}$$

$$\alpha \in Q_n \tag{11}$$

$$\alpha_{ij} \in \{0, 1\}, \quad (1 \le i, j \le n; i \ne j) \tag{12}$$

$$\frac{\ell_i}{2} \le x_i \le L - \frac{\ell_i}{2}, \quad (1 \le i \le n) \tag{13}$$

The objective function (6) minimizes the total cost of transporting materials between machines; Constraints (7) compute the distance between each pair of machines. Constraint (8) ensures that if machine i is placed at the same row as machine j, the distance between their centers is at least $(\ell_i + \ell_j)/2$. Constraint (9) ensures that machines do not overlap. Constraint (10) aims to eliminate symmetric solutions. The facilities $(i*, j*)$ with the least amount of flow between them tend to have their centers positioned at different abscissas. Then, there is an optimal solution with $x_{i*} > x_{j*}$ and a symmetric one with $x_{i*} < x_{j*}$; and we chose the latter optimal solution. If $x_{i*} = x_{j*}$ in an optimal solution, Constraint (10) is redundant. A similar constraint has been used by Sherali et al. [24] for their type of layout problem. Constraints (11) and (12) characterize the α-incidence vectors.

In the model of Chung and Tanchoco [8] the number of constraints is $\frac{5}{2}n(n-1)+3n$, the number of continuous variables is $n(n-1)+2n$ and the number of binary variables is $n(n-1)+2n$. In the proposed model the number of constraints is $\frac{n}{2}(2n^2-n-1)+1$, the number of continuous variables is $\frac{n}{2}(n-1)+n$ and the number of binary variables is $n(n-1)$. Thus, the proposed model has a smaller number of binary variables, and a smaller number of continuous variables. It has a larger number of constraints, but most of these are valid inequalities that strengthen the proposed model contributing for its improved performance relatively to the model of Chung and Tanchoco [8].

4 Computational experiments

The MIP models M_{CT} of Chung and Tanchoco [8] and the new MIP model (denoted by $M1$) were solved using CPLEX 12.1.0 on an Intel Core Duo, 1.73 GHz PC with 1 GB of RAM and with the Windows XP operating system.

We consider the four largest instances introduced by Simmons [25] (of size $n \leq 11$), and two larger instances of size $n = 12$ (data for these instances is available from the author). A time limit of three hours was imposed, after which the CPLEX solver is to be aborted.

The results obtained are displayed in Table 1. For each problem instance, the first three columns present: the *reference* to the problem instance data, the *name* of the problem, the number n of machines, and the *optimal value* obtained for the instance. The next columns display for each model: the *amount of computational time* spent (in seconds), the *number of branch-and-bound nodes* consumed, and the *CPLEX optimality gap* attained.

Recall that when the CPLEX solver terminates with a proved optimal solution, the solver reports an *optimality gap* of zero. The larger the *optimality gap*, the more distant the solver is to proving optimality.

Table 1 shows that $M1$ performs far better than model M_{CT} in terms of computational times and branch-and-bound nodes. The computational times increase very rapidly with the size n of the instance. With model M_{CT}, the instance with $n = 10$, requires almost 2 h of computational time. However, with model $M1$ the same instance requires 183.5 s. With model M_{CT}, for the instances with $n > 10$, the specified time limit of 3 h is exceeded with large gaps being reported by CPLEX, while with model $M1$ any of these instances are solved in less than 1 h.

Table 1 Performance of mixed integer programming models of the DRLP

Reference	Problem	n	Optimal value	Model	Gap (%)	Number of nodes	Time (s)
Simmons [25]	S9	9	1,179.0	$M1$	0	38,604	60.7
				M_{CT}	0	201,588	858.5
	S9H	9	2,293.0	$M1$	0	543,646	334.3
				M_{CT}	0	2,497,460	5,877.8
	S10	10	1,351.0	$M1$	0	85,319	183.5
				M_{CT}	0	2,887,550	7,141.0
	S11	11	3,424.5	$M1$	0	1,007,600	2,417.2
				M_{CT}	44.24	2,250,499	12,987.28[a]
This paper	12a	12	1,493.0	$M1$	0	1,011,089	2,940.0
				M_{CT}	79.76	853,451	12,987.6[a]
	12b	12	1,606.5	$M1$	0	788,428	3,244.6
				M_{CT}	54.21	1,291,880	11,687.5[a]

[a] Aborted after exceeding an imposed time limit of 3 h

5 Conclusions

We considered the DRLP, which is the problem of arranging n machines on both sides of a central corridor so as to minimize a weighted sum of the distances between machines. We proposed a mixed-integer programming formulation for the problem. Computational results showed that for larger instances the new formulation leads to much faster optimal solutions than those obtained with a previously published linear mixed-integer programming model.

The formulation introduced here is very interesting from a theoretical perspective. Future research should consider a thorough study of valid inequalities that are useful for the proposed model. After identifying the most useful valid inequalities, one can incorporate them in a branch-and-cut framework, which might further improve the results presented here.

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