Optim Lett (2012) 6:1657–1669
DOI 10.1007/s11590-011-0357-4

ORIGINAL PAPER

Minimization of vectors of curvilinear functionals on the second order jet bundle: sufficient efficiency conditions

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Received: 21 February 2011 / Accepted: 1 June 2011 / Published online: 23 June 2011

Abstract Strongly motivated by its possible applications in Mechanics, in our previous work (Pitea and Postolache (Optim. Lett. doi:10.1007/s11590-010-0272-0, 2011)), we initiated an optimization theory for the second order jet bundle. We considered the problem of minimization of vectors of curvilinear functionals (well known as mechanical work), thought as multi-time multi-objective variational problems, subject to PDE and/or PDI constraints. Within this framework, we introduced necessary conditions. As natural continuation of our results in Pitea and Postolache (Optim. Lett. doi:10.1007/s11590-010-0272-0, 2011), the present work introduces a study of sufficient efficiency conditions. While the background in Sect. 2 is introductory, the theory in Sect. 3 is new as a whole, containing our results.

Keywords Lagrange 1-form density · Multi-objective variational problem · Quasiinvexity · Efficiency

Mathematics Subject Classification (2000) 49J35 · 58E17

1 Introduction

According to Chinchuluun and Pardalos [1], most of the optimization problems arising in practice have several objectives which have to be optimized simultaneously. This kind of problems, of considerable interest, includes various branches of mathematical sciences, engineering design, portfolio selection, game theory, decision problems in management science, web access problems, query optimization in databases etc.

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Also, it is known that such kind of optimization problems arise in wide areas of research for new technology as well. First of all, we have in mind the material sciences where many times optimal estimation of material parameters is required, either non-destructive determination of faults is needed. Next, chemistry which provides a huge class of constrained optimization problems such as the determination of contamination sources given the flow model and the variance of the source. Last, but not least, games theory where the main study is finding optimal wining strategies. For descriptions of the web access problem, the portfolio selection problem and capital budgeting problem, see [1] and some references therein.

In time, several authors have been interested in the study of (sufficient) optimality conditions for vector programming in connection with generalized convexity. To quote illustrative sources, see [2] by Hachimi and Aghezzaf, [3] by Hanson, [4] by Kanniappan, [6] by Mititelu, [8] by Mond and Husain, [11], [12] by Pitea and collaborators, [14] by Preda, [17] by Wang.

Despite of all these important advances optimization theory, our multitime multiobjective problem—imposed by practical reasons—had not been studied so far. In the problem of our study the objective vector function is of curvilinear integral type, the integrand depending both on velocities and accelerations, that is why we have chosen as framework the second order jet bundle [15]. Our study is encouraged by its possible application, especially in Mechanical Engineering, where curvilinear integral objectives are extensively used due to their physical meaning as mechanical work. These objectives play an essential role in mathematical modeling of certain processes in relation with Robotics, Tribology, Engines etc.

This paper aims to establish some new results on nonlinear optimization on the second order jet bundle. It is organized as follows. Next, in Sect. 2 our framework is introduced, while in Sect. 3 sufficient efficiency conditions for our problem are given. Finally, we conclude the paper and suggest possible further development.

2 Our framework

Let (T, h) and (M, g) be Riemannian manifolds of dimensions p and n, respectively. The local coordinates on T and M will be written $t = (t^\alpha)$ and $x = (x^i)$, respectively. Let $J^2(T, M)$ be the second order jet bundle associated to T and M, see [15].

Throughout this work, we use the customary relations between two vectors of the same dimension, [11]. Having in mind the product order relation on $\mathbb{R}^p$, the hyperparallelepiped Ω_{t_0,t_1}, in $\mathbb{R}^p$, with the diagonal opposite points $t_0 = \left(t_0^1, \dots, t_0^p\right)$ and $t_1 = \left(t_1^1, \dots, t_1^p\right)$, can be written as being the interval $[t_0, t_1]$. Suppose γ_{t_0,t_1} is a piecewise C^2-class curve joining the points t_0 and t_1.

2.1 On the second order jet bundle

To make complete our presentation, we recall a background on the second order jet bundle, $J^2(T, M)$. Its elements are the 2-jets $j_t^2\phi$ of the local sections $\phi \in \Gamma_t(\varpi)$. A 2-jet at the point t is an equivalence class which contains the sections having, at the point t, the same value and the same partial derivatives up to the second order.

Let us suppose that the local sections satisfy the equality $\phi(t) = \psi(t)$. Consider (t^α, x^i) and $(t^{\alpha'}, x^{i'})$ be two adapted coordinate systems around $\phi(t)$. If the following equalities hold

$$\frac{\partial \phi^i}{\partial t^\alpha}(t) = \frac{\partial \psi^i}{\partial t^\alpha}(t), \quad \frac{\partial^2 \phi^i}{\partial t^\alpha t^\beta}(t) = \frac{\partial^2 \psi^i}{\partial t^\alpha t^\beta}(t),$$

then the following equalities hold too

$$\frac{\partial \phi^{i'}}{\partial t^{\alpha'}}(t) = \frac{\partial \psi^{i'}}{\partial t^{\alpha'}}(t), \quad \frac{\partial^2 \phi^{i'}}{\partial t^{\alpha'} t^{\beta'}}(t) = \frac{\partial^2 \psi^{i'}}{\partial t^{\alpha'} t^{\beta'}}(t).$$

Definition 1 Two local sections $\phi, \psi \in \Gamma_t(\varpi)$ are called 2-*equivalent* at the point t if

$$\phi(t) = \psi(t), \quad \frac{\partial \phi^i}{\partial t^\alpha}(t) = \frac{\partial \psi^i}{\partial t^\alpha}(t), \quad \frac{\partial^2 \phi^i}{\partial t^\alpha t^\beta}(t) = \frac{\partial^2 \psi^i}{\partial t^\alpha t^\beta}(t).$$

The equivalence class containing the section ϕ is called 2-*jet* of the local section ϕ, at the point t, denoted by $j_t^2\phi$.

Definition 2 The set $J^2(T, M) = \{j_t^2\phi \mid t \in T,\ \phi \in \Gamma_t(\varpi)\}$ is called the *second order jet bundle.*

Let $(\mathcal{U}, u)$, $u = (t^\alpha, x^i)$, be an adapted coordinate system on the product manifold $T \times M$. The induced coordinate system, $(\mathcal{U}^2, u^2)$, on $J^2(T, M)$, is defined as

$$\mathcal{U}^2 = \left\{ j_t^2\phi \mid \phi(t) \in \mathcal{U} \right\}, \quad u^2 = \left(t^\alpha, x^i, x^i_\alpha, x^i_{\theta\sigma} \right),$$

where

$$t^\alpha \left(j_t^2\phi \right) = t^\alpha(t), \quad x^i (j_t^2\phi) = x^i\left(\phi(t)\right),$$

$$x^i_\alpha \left(j_t^2\phi \right) = x^i_\alpha \left(j_t^1\phi \right), \quad x^i_{\theta\sigma} \left(j_t^2\phi \right) = \frac{\partial^2 \phi^i}{\partial t^\theta \partial t^\sigma}(t).$$

The pn functions $x^i_\alpha : \mathcal{U}^2 \to \mathbb{R}$ and the $\frac{1}{2}np(p+1)$ functions $x^i_{\theta\sigma} : \mathcal{U}^2 \to \mathbb{R}$ are called *coordinate derivatives*.

Proposition 1 *On the product manifold $T \times M$, consider $(\mathcal{U}, u)$ the atlas of adapted charts. Then, the corresponding charts $(\mathcal{U}^2, u^2)$ form a finite dimensional atlas, of C^∞-class, on the second order jet bundle $J^2(T, M)$.*

Important note. To simplify the presentation, in our subsequent theory, we shall set

$$\pi_x(t) = \left(x, x(t), x_\gamma(t), x_{\theta\sigma}(t)\right), \quad \pi_{x^\circ}(t) = \left(t, x^\circ(t), x^\circ_\gamma(t), x^\circ_{\theta\sigma}(t)\right),$$

where $x_\gamma(t) = \frac{\partial x}{\partial t^\gamma}(t)$, $\gamma = \overline{1, p}$, and $x_{\theta\sigma}(t) = \frac{\partial^2 x}{\partial t^\theta \partial t^\sigma}(t)$, $\theta, \sigma = \overline{1, p}$, are partial velocities and partial accelerations respectively.

2.2 Lagrange 1-forms of the 2nd order and their primitives as curvilinear integrals

A Lagrange 1-form of the 2nd order, on the jet space $J^2(T, M)$, has the form

$$\omega = L_\alpha(\pi_x(t))dt^\alpha + M_i(\pi_x(t))dx^i + N_i^\beta(\pi_x(t))dx^i_\beta + P_i^{\alpha\beta}(\pi_x(t))dx^i_{\alpha\beta}.$$

Here L_α, M_i, N_i^β and $P_i^{\alpha\beta}$ are Lagrangians of the second order. This one, has the pullback

$$x^*\omega = \left(L_\alpha + M_i x^i_\alpha + N_i^\beta x^i_{\beta\alpha} + P_i^{\beta\gamma} x^i_{\alpha\beta\gamma}\right) dt^\alpha$$

which is a Lagrange 1-form of the third order on M. The coefficients

$$L_\alpha + M_i x^i_\alpha + N_i^\beta x^i_{\beta\alpha} + P_i^{\beta\gamma} x^i_{\alpha\beta\gamma}$$

are third order Lagrangians, which are linear in the third order derivatives. To the form ω one attaches the Pfaff equation $\omega = 0$ and the partial differential equations

$$L_\alpha + M_i x^i_\alpha + N_i^\beta x^i_{\beta\alpha} + P_i^{\beta\gamma} x^i_{\alpha\beta\gamma} = 0.$$

Let $L_\beta(\pi_x(t))dt^\beta$ be a closed Lagrange 1-form (completely integrable), that is $D_\beta L_\alpha = D_\alpha L_\beta$.

A closed 1-form in a simple-connected domain is an exact one. Its primitive can be expressed as a curvilinear integral

$$\phi(t) = \int_{\Gamma_{t_0,t}} L_\alpha\left(\pi_x(s)\right) ds^\alpha, \quad \phi(t_0) = 0,$$

or as a system of PDEs,

$$\frac{\partial \phi}{\partial t^\alpha}(t) = L_\alpha\left(\pi_x(t)\right), \quad \phi(t_0) = 0.$$

If would exist a Lagrangian-like primitive

$$L(\pi_x(t)) = \int_{\Gamma_{t_0,t}} L_\alpha\left(\pi_x(s)\right) ds^\alpha, \quad L\left(\pi_x(t_0)\right)) = 0$$

or $D_\alpha L = L_\alpha$ (the foregoing pullback is the given closed 1-form),

$$\frac{\partial L}{\partial t^\beta} + \frac{\partial L}{\partial x^i}\frac{\partial x^i}{\partial t^\beta} + \frac{\partial L}{\partial x^i_\gamma}\frac{\partial x^i_\gamma}{\partial t^\beta} + \frac{\partial L}{\partial x^i_{\mu\nu}}\frac{\partial x^i_{\mu\nu}}{\partial t^\beta} = L_\beta,$$

relation which can be understood as a completely integrable system of PDEs (of the second order) with the unknown function $x(t)$, too.

Any *smooth* Lagrangian $L\left(\pi_x(t)\right),\ t\in\mathbb{R}^m_+$, produces two smooth closed (completely integrable) 1-forms:

- the differential

$$dL=\frac{\partial L}{\partial t^\gamma}dt^\gamma+\frac{\partial L}{\partial x^i}dx^i+\frac{\partial L}{\partial x^i_\gamma}dx^i_\gamma+\frac{\partial L}{\partial x^i_{\mu\nu}}dx^i_{\mu\nu}$$

having the components $\left(\frac{\partial L}{\partial t^\gamma},\ \frac{\partial L}{\partial x^i},\ \frac{\partial L}{\partial x^i_\gamma},\ \frac{\partial L}{\partial x^i_{\mu\nu}}\right)$, with respect to the corresponding basis $\left(dt^\gamma,\ dx^i,\ dx^i_\gamma,\ dx^i_{\mu\nu}\right)$;

- the restriction of dL to $\pi_x(t)$, that is the pullback

$$dL\bigg|_{\pi_x(t)}=\left(\frac{\partial L}{\partial t^\beta}+\frac{\partial L}{\partial x^i}\frac{\partial x^i}{\partial t^\beta}+\frac{\partial L}{\partial x^i_\gamma}\frac{\partial x^i_\gamma}{\partial t^\beta}+\frac{\partial L}{\partial x^i_{\mu\nu}}\frac{\partial x^i_{\mu\nu}}{\partial t^\beta}\right)dt^\beta,$$

of components

$$D_\beta L=\frac{\partial L}{\partial t^\beta}(\pi_x(t))+\frac{\partial L}{\partial x^i}\left(\pi_x(t)\right)\frac{\partial x^i}{\partial t^\beta}(t)+\frac{\partial L}{\partial x^i_\gamma}(\pi_x(t))\frac{\partial x^i_\gamma}{\partial t^\beta}(t)$$
$$+\frac{\partial L}{\partial x^i_{\mu\nu}}(\pi_x(t))\frac{\partial x^i_{\mu\nu}}{\partial t^\beta}(t),$$

with respect to the basis dt^β (for other significant ideas, see [13], by Ariana Pitea).

Now, we can continue to set our framework. In this respect, consider the closed Lagrange 1-form densities of C^∞-class

$$f_\alpha=(f^\ell_\alpha)\colon J^2(T,M)\to\mathbb{R}^r,\quad \ell=\overline{1,r},\quad \alpha=\overline{1,p},$$

which determine the following path independent functionals

$$F^\ell(x(\cdot))=\int\limits_{\gamma_{t_0,t_1}} f^\ell_\alpha(\pi_x(t))\,dt^\alpha,$$

We accept that the Lagrange matrix densities

$$g=(g^b_a)\colon J^2(T,M)\to\mathbb{R}^{md},\quad a=\overline{1,d},\quad b=\overline{1,m},\quad m<n,$$

of C^∞-class defines the partial differential inequations (PDI) (of evolution)

$$g(\pi_x(t))\ \geqq\ 0,\quad t\in\Omega_{t_0,t_1},\tag{2.1}$$

and the Lagrange matrix densities

$$h = (h_a^b) \colon J^2(T, M) \to \mathbb{R}^{qd}, \quad a = \overline{1, s}, \quad b = \overline{1, q}, \quad q < n,$$

defines the partial differential equations (PDE) (of evolution)

$$h(\pi_x(t)) = 0, \quad t \in \Omega_{t_0,t_1}. \tag{2.2}$$

Let $C^\infty(\Omega_{t_0,t_1}, M)$ be the space of all functions $x \colon \Omega_{t_0,t_1} \to M$ of C^∞-class, with the norm

$$\|x\| = \|x\|_\infty + \sum_{\alpha=1}^{p} \|x_\alpha\|_\infty + \sum_{\theta,\sigma=1}^{p} \|x_{\theta\sigma}\|_\infty.$$

We consider the vector of functionals

$$F(x(\cdot)) = \int\limits_{\gamma_{t_0,t_1}} f_\alpha(\pi_x(t))\, dt^\alpha = \left(F^1(x(\cdot)), \ldots, F^r(x(\cdot))\right),$$

and, with the constraints (2.1) and (2.2), denote by

$$\mathcal{F}(\Omega_{t_0,t_1}) = \{x \in C^\infty(\Omega_{t_0,t_1}, M) \mid x(t_0) = x_0,\ x(t_1) = x_1,\ g(\pi_x(t)) \leqq 0,$$
$$h(\pi_x(t)) = 0,\ t \in \Omega_{t_0,t_1}\}$$

the *set of all feasible solutions* of the problem (MP), introduced right now:

$$(\text{MP}) \begin{cases} \min F(x) \\ \text{subject to} \quad x(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1}). \end{cases}$$

Using the terminology from analytical mechanics, in (MP) there are given a number of r sources producing mechanical work, which have to be minimized on a set of limited resources, namely $\mathcal{F}(\Omega_{t_0,t_1})$.

In our previous work [10], we found necessary conditions for the optimum of problem (MP). We wolud like to further develop these results by introducing sufficient efficiency conditions for problem (MP) and this is the aim of the next section.

3 Main results

First we remind several definitions, then we shall introduce our new results.

Definition 3 A feasible solution $x^\circ(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is called *efficient* for problem (MP) if and only if for any solution $x(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$, we have the implication

$$F(x(\cdot)) \leqq F(x^\circ(\cdot)) \Rightarrow F(x(\cdot)) = F(x^\circ(\cdot)).$$

Definition 4 Let $x^\circ(\cdot)$ be an optimal solution of the problem (MP). Suppose there are the vector λ in $\mathbb{R}^r$, having all components nonnegative but at least one positive, and the smooth matrix functions $\mu\colon \Omega_{t_0,t_1} \to \mathbb{R}^{msp}$ and $\nu\colon \Omega_{t_0,t_1} \to \mathbb{R}^{qsp}$ such that

$$
\begin{aligned}
&< \lambda, \frac{\partial f}{\partial x}(\pi_{x^\circ}(t)) > + < \mu_\alpha(t), \frac{\partial g}{\partial x}(\pi_{x^\circ}(t)) > + < \nu_\alpha(t), \frac{\partial h}{\partial x}(\pi_{x^\circ}(t)) > \\
&\quad -D\gamma\left(< \lambda, \frac{\partial f}{\partial x_\gamma}(\pi_{x^\circ}(t)) > + < \mu_\alpha(t), \frac{\partial g}{\partial x_\gamma}(\pi_{x^\circ}(t)) > + < \nu_\alpha(t), \frac{\partial h}{\partial x_\gamma}(\pi_{x^\circ}(t)) > \right) \\
&\quad +D^2_{\theta\sigma}\left(< \lambda, \frac{\partial f_\alpha}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) >+ < \mu_\alpha(t), \frac{\partial g}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) > \right. \\
&\qquad\qquad \left. + < \nu_\alpha(t), \frac{\partial h}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) > \right) \\
&\quad = 0, \quad t \in \Omega_{t_0,t_1}, \ \alpha = \overline{1,p} \quad \text{(Euler–Lagrange PDEs).}
\end{aligned}
$$

Then $x^\circ(\cdot)$ is called *normal optimal solution* of problem (MP).

The result in Theorem 1 states necessary conditions for the efficiency of the solution for problem (MP). For a proof, see [10].

Theorem 1 *Let $x^\circ(\cdot)$ be a point from $\mathcal{F}(\Omega_{t_0,t_1})$. If $x^\circ(\cdot)$ is a normal efficient solution of problem* (MP)*, then there exist a vector $\lambda \in \mathbb{R}^r$ and the smooth matrix functions $\mu(t) = (\mu_\alpha(t))$, $\nu(t) = (\nu_\alpha(t))$, which satisfy the following conditions*

$$
\begin{aligned}
&< \lambda, \frac{\partial f}{\partial x}(\pi_{x^\circ}(t)) > + < \mu_\alpha(t), \frac{\partial g}{\partial x}(\pi_{x^\circ}(t)) > + < \nu_\alpha(t), \frac{\partial h}{\partial x}(\pi_{x^\circ}(t)) > \\
&\quad -D\gamma\left(< \lambda, \frac{\partial f}{\partial x_\gamma}(\pi_{x^\circ}(t)) > + < \mu_\alpha(t), \frac{\partial g}{\partial x_\gamma}(\pi_{x^\circ}(t)) > + < \nu_\alpha(t), \frac{\partial h}{\partial x_\gamma}(\pi_{x^\circ}(t)) > \right) \\
&\quad +D^2_{\theta\sigma}\left(< \lambda, \frac{\partial f_\alpha}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) >+ < \mu_\alpha(t), \frac{\partial g}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) > \right. \\
&\qquad\qquad \left. + < \nu_\alpha(t), \frac{\partial h}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) > \right) \\
&\quad = 0, \quad t \in \Omega_{t_0,t_1}, \ \alpha = \overline{1,p} \quad \text{(Euler–Lagrange PDEs)} \\
&\quad < \mu_\alpha(t), g(\pi_{x^\circ}(t)) >= 0, \quad t \in \Omega_{t_0,t_1}, \quad \alpha = \overline{1,p}, \\
&\quad\quad \mu_\alpha(t) \geqq 0, \quad t \in \Omega_{t_0,t_1}, \quad \alpha = \overline{1,p}, \\
&\quad\quad \lambda \geq 0, \quad < e, \lambda >= 1, \quad e = (1, \ldots, 1) \in \mathbb{R}^r.
\end{aligned}
$$

To develop our theory, we have to introduce an appropriate generalized convexity. Let ρ be a real number, $b\colon C^\infty(\Omega_{t_0,t_1}, M) \times C^\infty(\Omega_{t_0,t_1}, M) \to [0, \infty)$ a functional, and $a = (a_\alpha)$, $\alpha = \overline{1,p}$, a closed 1-form. To a we associate the curvilinear integral

$$
A(x(\cdot)) = \int_{\gamma_{t_0,t_1}} a_\alpha(\pi_x(t))dt^\alpha.
$$

Definition 5 The functional A is called *[strictly] (ρ, b)-quasiinvex at the point $x^\circ(\cdot)$* if there exists a vector function $\eta\colon J^2(\Omega_{t_0,t_1}, M) \times J^2(\Omega_{t_0,t_1}, M) \to \mathbb{R}^n$, vanishing at the point $(\pi_{x^\circ}(t), \pi_{x^\circ}(t))$, and the function θ defined on $C^\infty(\Omega_{t_0,t_1}, M) \times C^\infty(\Omega_{t_0,t_1}, M)$ to $\mathbb{R}^n$, such that for any $x(\cdot)$ $[x(\cdot) \neq x^\circ(\cdot)]$, the following implication holds

$$
\begin{aligned}
&\big(A(x(\cdot)) \leq A(x^\circ(\cdot))\big)\\
&\Rightarrow \Bigg(b(x((\cdot), x^\circ(\cdot)) \int_{\gamma_{t_0,t_1}} \Bigg[< \eta(\pi_x(t), \pi_{x^\circ}(t)), \frac{\partial a_\alpha}{\partial x}(\pi_{x^\circ}(t)) >\\
&\quad + < D_\gamma \eta(\pi_x(t), \pi_{x^\circ}(t)), \frac{\partial a_\alpha}{\partial x_\gamma}(\pi_{x^\circ}(t)) >\\
&\quad + < D^2_{\theta\sigma} \eta(\pi_x(t), \pi_{x^\circ}(t)), \frac{\partial a_\alpha}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) > \Bigg] dt^\alpha\\
&[<] \leq -\rho b(x(\cdot), x^\circ(\cdot)) \|\theta(x(\cdot), x^\circ(\cdot))\|^2\Bigg).
\end{aligned}
$$

The notion of quasiinvexity is used, in appropriate forms, in recent works for studies of some multiobjective programming problems, for example, see [5] by Mititelu, [9] by Nahak and Mohapatra.

The next theorem is the main result of this work.

Theorem 2 *Let us consider the feasible solution $x^\circ(\cdot)$, the vector λ and the functions $\mu(\cdot)$ and $\nu(\cdot)$ from Theorem 1.*

Suppose that the following conditions are satisfied:

a) *for each $\ell = \overline{1, r}$, the functional $F^\ell(x(\cdot)) = \int_{\gamma_{t_0,t_1}} f^\ell_\alpha(\pi_x(t)) dt^\alpha$ is (ρ^ℓ_1, b)-quasiinvex at the point $x^\circ(\cdot)$ with respect to η and θ;*

b) *the functional $\int_{\gamma_{t_0,t_1}} < \mu_\alpha(t), g(\pi_x(t)) > dt^\alpha$ is (ρ_2, b) -quasiinvex at the point $x^\circ(\cdot)$ with respect to η and θ;*

c) *the functional $\int_{\gamma_{t_0,t_1}} < \nu_\alpha(t), h(\pi_x(t)) > dt^\alpha$ is (ρ_3, b)-quasiinvex at the point $x^\circ(\cdot)$ with respect to η and θ;*

d) *one of the integrals of* a) - c) *is (ρ^ℓ_1, b), (ρ_2, b) or (ρ_3, b)-strictly quasiinvex at the point $x^\circ(\cdot)$;*

e) $\lambda_\ell \rho^\ell_1 + \rho_2 + \rho_3 \geq 0$.
Then the point $x^\circ(\cdot)$ is an efficient solution of problem (MP).

Proof Let us suppose that the point $x^\circ(\cdot)$ is not an efficient solution for problem (MP). Then, there is a feasible solution $x(\cdot)$ for the problem (MP), such that for each $\ell = \overline{1, r}$, $F^\ell(x(\cdot)) \leq F^\ell(x^\circ(\cdot))$, the case $x(\cdot) = x^\circ(\cdot)$ being excluded.

According to condition a), it follows

$$
\begin{aligned}
&b(x(\cdot), x^\circ(\cdot)) \int_{\gamma_{t_0,t_1}} \Bigg[< \eta(\pi_x(t), \pi_{x^\circ}(t)), \frac{\partial f^\ell_\alpha}{\partial x}(\pi_{x^\circ}(t)) >\\
&\quad + < D_\gamma \eta(\pi_x(t), \pi_{x^\circ}(t)), \frac{\partial f^\ell_\alpha}{\partial x_\gamma}(\pi_{x^\circ}(t)) > + < D^2_{\theta\sigma} \eta(\pi_x(t), \pi_{x^\circ}(t)), \frac{\partial f^\ell_\alpha}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) > \Bigg] dt^\alpha\\
&\leqq -\rho^\ell_1 b(x(\cdot), x^\circ(\cdot)) \|\theta(x(\cdot), x^\circ(\cdot))\|^2, \quad \ell = \overline{1, r}.
\end{aligned}
$$

Multiplying each inequality by $\lambda^\circ_\ell, \ell = \overline{1, r}$ and summing from $\ell = 1$ to r, we obtain

$$b(x(\cdot), x^\circ(\cdot)) \int_{\gamma_{t_0,t_1}} \Bigg[< \eta(\pi_x(t), \pi_{x^\circ}(t)), < \lambda^\circ, \frac{\partial f_\alpha}{\partial x}(\pi_{x^\circ}(t)) >>$$
$$+ < D_\gamma \eta(\pi_x(t), \pi_{x^\circ}(t)), < \lambda^\circ, \frac{\partial f_\alpha}{\partial x_\gamma}(\pi_{x^\circ}(t)) >>$$
$$+ < D^2_{\theta\sigma} \eta(\pi_x(t), \pi_{x^\circ}(t)), < \lambda^\circ, \frac{\partial f_\alpha}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) >> \Bigg] dt^\alpha$$
$$\leqq \; - \lambda^\circ_\ell \rho^\ell_1 b(x(\cdot), x^\circ(\cdot)) \|\theta(x(\cdot), x^\circ(\cdot))\|^2. \tag{3.1}$$

By applying property b), the following relation

$$\int_{\gamma_{t_0,t_1}} < \mu_\alpha(t), g(\pi_x(t)) > dt^\alpha \; \leqq \; \int_{\gamma_{t_0,t_1}} < \mu_\alpha(t), g(\pi_{x^\circ}(t)) > dt^\alpha$$

leads us to

$$b(x(\cdot), x^\circ(\cdot)) \int_{\gamma_{t_0,t_1}} \Bigg[< \eta(\pi_x(t), \pi_{x^\circ}(t)), < \mu_\alpha(t), \frac{\partial g}{\partial x}(\pi_{x^\circ}(t)) >>$$
$$+ < D_\gamma \eta(\pi_x(t), \pi_{x^\circ}(t)), < \mu_\alpha(t), \frac{\partial g}{\partial x_\gamma}(\pi_{x^\circ}(t)) >>$$
$$+ < D^2_{\theta\sigma} \eta(\pi_x(t), \pi_{x^\circ}(t)), < \mu_\alpha(t), \frac{\partial g}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) >> \Bigg] dt^\alpha$$
$$\leqq -\rho_2 b(x(\cdot), x^\circ(\cdot)) \|\theta(x(\cdot), x^\circ(\cdot))\|^2. \tag{3.2}$$

Taking into account condition c), the equality

$$\int_{\gamma_{t_0,t_1}} < \nu_\alpha(t), h(\pi_x(t)) > dt^\alpha = \int_{\gamma_{t_0,t_1}} < \nu_\alpha(t), h(\pi_{x^\circ}(t)) > dt^\alpha$$

implies

$$b(x(\cdot), x^\circ(\cdot)) \int_{\gamma_{t_0,t_1}} \Bigg[< \eta(\pi_x(t), \pi_{x^\circ}(t)), < \nu_\alpha(t), \frac{\partial h}{\partial x}(\pi_{x^\circ}(t)) >>$$
$$+ < D_\gamma \eta(\pi_x(t), \pi_{x^\circ}(t)), < \nu_\alpha(t), \frac{\partial h}{\partial x_\gamma}(\pi_{x^\circ}(t)) >>$$
$$+ < D^2_{\theta\sigma} \eta(\pi_x(t), \pi_{x^\circ}(t)), < \nu_\alpha(t), \frac{\partial h}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) >> \Bigg] dt^\alpha$$
$$\leqq -\rho_3 b(x(\cdot), x^\circ(\cdot)) \|\theta(x(\cdot), x^\circ(\cdot))\|^2. \tag{3.3}$$

Summing side by side relations (3.1), (3.2), (3.3) and using condition d), it follows

$$
\begin{aligned}
& b(x(\cdot), x^\circ(\cdot)) \int\limits_{\gamma_{t_0,t_1}} < \eta(\pi_x(t), \pi_{x^\circ}(t)), < \lambda, \frac{\partial f_\alpha}{\partial x}(\pi_{x^\circ}(t)) > \\
&\quad + < \mu_\alpha(t), \frac{\partial g}{\partial x}(\pi_{x^\circ}(t)) > + < \nu_\alpha(t), \frac{\partial h}{\partial x}(\pi_{x^\circ}(t)) >> dt^\alpha \\
&\quad + b(x(\cdot), x^\circ(\cdot)) \int\limits_{\gamma_{t_0,t_1}} < D_\gamma \eta(\pi_x(t), \pi_{x^\circ}(t)), < \lambda, \frac{\partial f_\alpha}{\partial x_\gamma}(\pi_{x^\circ}(t)) > \\
&\quad + < \mu_\alpha(t), \frac{\partial g}{\partial x_\gamma}(\pi_{x^\circ}(t)) > + < \nu_\alpha(t), \frac{\partial h}{\partial x_\gamma}(\pi_{x^\circ}(t)) >> \\
&\quad + < D^2_{\theta\sigma} \eta(\pi_x(t), \pi_{x^\circ}(t)), < \lambda, \frac{\partial f_\alpha}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) > \\
&\quad + < \mu_\alpha(t), \frac{\partial g}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) > + < \nu_\alpha(t), \frac{\partial h}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) >> dt^\alpha \\
& < -\left(\lambda_\ell \rho_1^\ell + \rho_2 + \rho_3\right) b(x(\cdot), x^\circ(\cdot)) \|\theta(x(\cdot), x^\circ(\cdot))\|^2.
\end{aligned} \tag{3.4}
$$

This inequality implies that $b(x(\cdot), x^\circ(\cdot)) > 0$, and we obtain

$$
\begin{aligned}
& \int\limits_{\gamma_{t_0,t_1}} < \eta(\pi_x(t), \pi_{x^\circ}(t)), < \lambda, \frac{\partial f_\alpha}{\partial x}(\pi_{x^\circ}(t)) > \\
&\quad + < \mu_\alpha(t), \frac{\partial g}{\partial x}(\pi_{x^\circ}(t)) > + < \nu_\alpha(t), \frac{\partial h}{\partial x}(\pi_{x^\circ}(t)) >> dt^\alpha \\
&\quad + \int\limits_{\gamma_{t_0,t_1}} < D_\gamma \eta(\pi_x(t), \pi_{x^\circ}(t)), < \lambda, \frac{\partial f_\alpha}{\partial x_\gamma}(\pi_{x^\circ}(t)) > \\
&\quad + < \mu_\alpha(t), \frac{\partial g}{\partial x_\gamma}(\pi_{x^\circ}(t)) > + < \nu_\alpha(t), \frac{\partial h}{\partial x_\gamma}(\pi_{x^\circ}(t)) >> dt^\alpha \\
&\quad + \int\limits_{\gamma_{t_0,t_1}} < D^2_{\theta\sigma} \eta(\pi_x(t), \pi_{x^\circ}(t)), < \lambda, \frac{\partial f_\alpha}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) > \\
&\quad + < \mu_\alpha(t), \frac{\partial g}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) > + < \nu_\alpha(t), \frac{\partial h}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) >> dt^\alpha \\
& < -\left(\lambda_\ell \rho_1^\ell + \rho_2 + \rho_3\right) \|\theta(x(\cdot), x^\circ(\cdot))\|^2,
\end{aligned}
$$

that is

$$
\begin{aligned}
& \int\limits_{\gamma_{t_0,t_1}} < \eta(\pi_x(t), \pi_{x^\circ}(t)), < \lambda, \frac{\partial f_\alpha}{\partial x}(\pi_{x^\circ}(t)) > \\
&\quad + < \mu_\alpha(t), \frac{\partial g}{\partial x}(\pi_{x^\circ}(t)) > + < \nu_\alpha(t), \frac{\partial h}{\partial x}(\pi_{x^\circ}(t)) >> dt^\alpha
\end{aligned}
$$

$$+\int_{\gamma_{t_0,t_1}} D_\gamma\Bigg(< \eta(\pi_x(t),\pi_{x^\circ}(t)), < \lambda, \frac{\partial f_\alpha}{\partial x_\gamma}(\pi_{x^\circ}(t)) >$$

$$+ < \mu_\alpha(t), \frac{\partial g}{\partial x_\gamma}(\pi_{x^\circ}(t)) > + < \nu_\alpha(t), \frac{\partial h}{\partial x_\gamma}(\pi_{x^\circ}(t)) >> \Bigg)dt^\alpha$$

$$-\int_{\gamma_{t_0,t_1}} < \eta(\pi_x(t),\pi_{x^\circ}(t)), D_\gamma\Bigg(< \lambda, \frac{\partial f_\alpha}{\partial x_\gamma}(\pi_{x^\circ}(t)) >$$

$$+ < \mu_\alpha(t), \frac{\partial g}{\partial x_\gamma}(\pi_{x^\circ}(t)) > + < \nu_\alpha(t), \frac{\partial h}{\partial x_\gamma}(\pi_{x^\circ}(t)) > \Bigg) > dt^\alpha$$

$$+\int_{\gamma_{t_0,t_1}} < \eta(\pi_x(t),\pi_{x^\circ}(t)), D^2_{\theta\sigma}\Bigg(< \lambda, \frac{\partial f_\alpha}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) >$$

$$+ < \mu_\alpha(t), \frac{\partial g}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) > + < \nu_\alpha(t), \frac{\partial h}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) > \Bigg) > dt^\alpha$$

$$+\int_{\gamma_{t_0,t_1}} D_\theta < D_\sigma\eta(\pi_x(t),\pi_{x^\circ}(t)), < \lambda, \frac{\partial f_\alpha}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) >$$

$$+ < \mu_\alpha(t), \frac{\partial g}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) > + < \nu_\alpha(t), \frac{\partial h}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) >> dt^\alpha$$

$$-\int_{\gamma_{t_0,t_1}} D_\sigma < \eta(\pi_x(t),\pi_{x^\circ}(t)), D_\theta\Bigg(< \lambda, \frac{\partial f_\alpha}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) >$$

$$+ < \mu_\alpha(t), \frac{\partial g}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) > + < \nu_\alpha(t), \frac{\partial h}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) > \Bigg) > dt^\alpha$$

$$< -\Big(\lambda_\ell\rho_1^\ell + \rho_2 + \rho_3\Big)\|\theta(x(\cdot),x^\circ(\cdot))\|^2.$$

Taking into account the conditions in Theorem 1, the previous inequality becomes

$$\int_{\gamma_{t_0,t_1}} D_\gamma\Bigg(< \eta(\pi_x(t),\pi_{x^\circ}(t)), < \lambda, \frac{\partial f_\alpha}{\partial x_\gamma}(\pi_{x^\circ}(t)) >$$

$$+ < \mu_\alpha(t), \frac{\partial g}{\partial x_\gamma}(\pi_{x^\circ}(t)) > + < \nu_\alpha(t), \frac{\partial h}{\partial x_\gamma}(\pi_{x^\circ}(t)) >> \Bigg)dt^\alpha$$

$$+\int_{\gamma_{t_0,t_1}} D_\theta < D_\sigma\eta(\pi_x(t),\pi_{x^\circ}(t)), < \lambda, \frac{\partial f_\alpha}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) >$$

$$+ < \mu_\alpha(t), \frac{\partial g}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) > + < \nu_\alpha(t), \frac{\partial h}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) >> dt^\alpha$$

$$- \int_{\gamma_{t_0,t_1}} D_\sigma < \eta(\pi_x(t), \pi_{x^\circ}(t)), D_\theta\Bigg(< \lambda, \frac{\partial f_\alpha}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) >$$

$$+ < \mu_\alpha(t), \frac{\partial g}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) > + < \nu_\alpha(t), \frac{\partial h}{\partial x_{\theta\sigma}}(\pi_{x^\circ}(t)) > \Bigg) > dt^\alpha$$

$$< -\left(\lambda_\ell \rho_1^\ell + \rho_2 + \rho_3\right) \|\theta(x(\cdot), x^\circ(\cdot))\|^2.$$

According to [13], [16], a total divergence is equal to a total derivative. Therefore, the left hand side of the previous inequality is null, and we obtain

$$0 < -\left(\lambda_\ell \rho_1^\ell + \rho_2 + \rho_3\right) \|\theta(x(\cdot), x^\circ(\cdot))\|^2.$$

Since $\|\theta(x(\cdot), x^\circ(\cdot))\|$ is positive, it follows a contradiction. Thus, the point $x^\circ(\cdot)$ is an efficient solution for problem (MP), and this completes the proof.

By replacing the integrals from hypotheses b), c) of Theorem 2 by the integral

$$\int_{\gamma_{t_0,t_1}} \left[< \mu_\alpha(t), g(\pi_x(t)) > + < \nu_\alpha(t), h(\pi_x(t)) > \right] dt^\alpha,$$

the following statement is obtained.

Corollary 1 *Let us consider the vector λ, a feasible solution $x^\circ(\cdot)$ of problem* (MP) *and the functions $\mu(\cdot)$, $\nu(\cdot)$ which satisfy the relations in Theorem 1. Suppose that the following conditions are fulfilled*:

(a) *for each $\ell = \overline{1,r}$, $F^\ell(x(\cdot)) = \int_{\gamma_{t_0,t_1}} f_\alpha^\ell(\pi_x(t))dt^\alpha$ is (ρ_1^ℓ, b)-quasiinvex at the point $x^\circ(\cdot)$ with respect to η and θ*;

(b) *the functional $\int_{\gamma_{t_0,t_1}} \left[< \mu_\alpha(t), g(\pi_x(t)) > + < \nu_\alpha(t), h(\pi_x(t)) > \right] dt^\alpha$ is (ρ_2, b)-quasiinvex at the point $x^\circ(\cdot)$ with respect to η and θ*;

(c) *one of the integrals of* a) *or* b) *is strictly-quasiinvex at the point $x^\circ(\cdot)$ with respect to η and θ*;

(d) $\lambda_\ell \rho_1^\ell + \rho_2 \geq 0$.

Then the point $x^\circ(\cdot)$ is an efficient solution of problem (MP).

4 Conclusion and further development

In our previous work [10], we initiated an optimization theory for the second order jet bundle. We considered the problem of minimization of vectors of curvilinear functionals (well known as mechanical work), thought as multi-time multi-objective variational problem, subject to PDE and/or PDI constraints (limited resources). Within this framework, we introduced necessary conditions. As natural continuation of our results in [10], and strongly motivated by its possible applications in Mechanics, the present work introduced a study of sufficient efficiency conditions for (MP).

Since ratio programming problems with objective function of our type arise from applied areas as decision problems in management, game theory, engineering studies and design, we will orient our future research to these problems.

Acknowledgments The authors are deeply indebted to the anonymous referee for the careful reading and valuable suggestions. These ones helped us to greatly improve the presentation of the paper.

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