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Vector variational-like inequalities and vector optimization problems in Asplund spaces

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Abstract In this paper, the Minty vector variational-like inequality, the Stampacchia vector variational-like inequality, and the weak formulations of these two inequalities defined by means of Mordukhovich limiting subdifferentials are introduced and studied in Asplund spaces. Some relations between the vector variational-like inequalities and vector optimization problems are established by using the properties of Mordukhovich limiting subdifferentials. An existence theorem of solutions for the weak Minty vector variational-like inequality is also given.

Keywords Mordukhovich limiting subdifferential · Vector variational-like inequality · Vector optimization problem · Asplund space

1 Introduction

The vector variational inequality was first introduced and studied by Giannessi [11] in the setting of finite-dimensional Euclidean spaces. Since then, several applications have been shown to a wide range of problems in various disciplines in the natural and social sciences. Consequently, vector variational inequalities have been generalized in various directions, in particular, vector variational-like inequality problems, see [1,2,4,6,7,10,14,16,18,19,25,26,31] and the references therein. The vector varia-

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tional-like inequalities are closely related to the concept of the invex and pre-invex functions which generalize the notion of the convexity of functions. As a matter of fact, the concept of invexity plays the same role in vector variational-like inequalities as the classical convexity plays in vector variational inequalities. The concept of the invexity was first introduced by Hanson [15]. More recently, the characterizations and applications for generalized invexity were studied by many authors, see [13,17,23,27,29,30,32,33,35] and the references therein.

The relation between the vector variational inequality and the smooth vector optimization problem has been studied by many authors (see, for example, [12,31,34] and the references therein). Giannessi [12] showed the relation between the Minty vector variational inequality and the differentiable, convex optimization problem. Yang et al. [34] extended the results of Giannessi [12] for differentiable but pseudoconvex functions. In addition, Yang and Yang [31] gave some relations between the Minty variational-like inequalities and the vector optimization problems for differentiable but pseudoinvex vector-valued functions. Vector variational-like inequalities and nonsmooth vector optimization problems have also been studied by many authors (see, for example, [1,3,18,19,25] and the references therein). Very recently, Rezaie and Zafarani [25] showed some relations between the vector variational-like inequalities and vector optimization problems for nondifferential functions under generalized monotonicity. Al-Homidan and Ansari [1] studied the relation among the generalized Minty vector variational-like inequality, generalized Stampacchia vector variational-like inequality, and vector optimization problem for nondifferential and nonconvex functions. The main results in [1] and [25] were obtained in the setting of Clarke subdifferential. Since the class of Clarke differential is larger than the class of Mordukhovich subdifferential, it is necessary to study the vector variational-like inequalities and vector optimizations problem in the setting of Mordukhovich subdifferential (see [8,9,21,22]).

Motivated and inspired by the work mentioned above, in this paper, we introduce the Minty vector variational-like inequality, Stampacchia vector variational-like inequality, and the weak formulations of these two inequalities defined by means of Mordukhovich limiting subdifferentials in Asplund spaces. Some relations between the vector variational-like inequalities and vector optimization problems are established by using the properties of Mordukhovich limiting subdifferentials. We also present an existence result for the solutions of the weak Minty vector variational-like inequality. The results presented in this paper generalize some main results in Al-Homidan and Ansari [1] and Yang and Yang [31].

2 Preliminary results

Throughout this paper X is an Asplund space. A Banach space X is Asplund, or it has the Asplund property, if every convex continuous function $g : U \to R$ defined on an open convex subset U of X is Fréchet differentiable on a dense subset of U. This class includes all Banach spaces having Fréchet smooth bump functions (in particular, spaces with Fréchet smooth renorms, hence, every reflexive space) and spaces with separable duals, etc. Let X^* denote the topological dual of X and $\langle \cdot, \cdot \rangle$ be the duality pairing between X and X^*. A mapping $g : X \to Y$ is Fréchet differentiable at $\bar{x}$ if there exists a linear continuous operator $\nabla g(\bar{x}) : X \to Y$, called the Fréchet derivative of g at $\bar{x}$, such that

$$\lim_{x \to \bar{x}} \frac{g(x) - g(\bar{x}) - \nabla g(\bar{x})(x - \bar{x})}{\|x - \bar{x}\|} = 0.$$

Let Ω be a nonempty subset of X. Given $x \in \Omega$ and $\varepsilon \geq 0$, the set of $\varepsilon-$normals to Ω at x is defined by

$$\hat{N}_{\varepsilon}(x; \Omega) = \left\{ x^* \in X^* : \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x \rangle}{\|u - x\|} \leq \varepsilon \right\}.$$

If $x \notin \Omega$, we put $\hat{N}_{\varepsilon}(x; \Omega) = \emptyset$ for all $\varepsilon \geq 0$.

Let $\bar{x} \in \Omega$. Then $x^* \in X^*$ is a limiting normal to Ω at $\bar{x}$ if there are sequences $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\Omega} \bar{x}$, and $x_k^* \xrightarrow{w^*} x^*$ such that $x_k^* \in \hat{N}_{\varepsilon_k}(x_k; \Omega)$ for all $k \in N$. The collection of such normals

$$N(\bar{x}; \Omega) = \limsup_{\substack{x \xrightarrow{\Omega} \bar{x} \\ \varepsilon \downarrow 0}} \hat{N}_{\varepsilon}(x; \Omega)$$

is the limiting normal cone to Ω at $\bar{x}$. Put $N(\bar{x}; \Omega) = \emptyset$ for $\bar{x} \notin \Omega$. Note that the symbol $u \xrightarrow{\Omega} x$ means that $u \to x$ with $u \in \Omega$. The symbol $\xrightarrow{w^*}$ stands for convergence in weak star topology, and N denotes the set of all natural numbers.

Considering the extended-real-valued function $g : X \to \bar{R} = [-\infty, +\infty]$, we say that g is proper if $g(x) > -\infty$ for all $x \in X$ and its domain, dom $g = \{x \in X : g(x) < +\infty\}$ is nonempty. The epigraph of g is defined as

$$epi\ g = \{(x, a) \in X \times R : g(x) \leq a\}.$$

Considering a point $\bar{x} \in X$ with $|g(\bar{x})| < \infty$, the set

$$\partial g(\bar{x}) = \{x^* \in X^* : (x^*, -1) \in N((\bar{x}, g(\bar{x})); epi g)\}$$

is the limiting subdifferential of g at $\bar{x}$ and its elements are limiting subdifferentials of g at this point. We put $\partial g(\bar{x}) = \emptyset$ if $|g(\bar{x})| = \infty$.

The Fréchet subdifferential of g at $\bar{x}$ is defined by

$$\hat{\partial} g(\bar{x}) = \left\{ x^* \in X^* : \liminf_{x \to \bar{x}} \frac{g(x) - g(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}.$$

g is said to be lower regular at $\bar{x}$ if $\hat{\partial} g(\bar{x}) = \partial g(\bar{x})$.

For more details about the properties of the above-mentioned results, see Mordukhovich [21].

Hereafter, unless otherwise specified, we assume that $S \subseteq X$ is a nonempty open invex set with respect to mapping $\eta : S \times S \to X$. S is said to be an invex set wrt the mapping $\eta : S \times S \to X$ if $x + \lambda\eta(y, x) \in S$ for all $x, y \in S$ and $\lambda \in [0, 1]$. Suppose that $g : S \to \bar{R}$ is a Lipschitz continuous function.

Definition 2.1 g is said to be invex wrt η on S if for any $x, y \in S$ and $\xi \in \partial g(x)$,

$$\langle \xi, \eta(y, x)\rangle \leq g(y) - g(x).$$

Definition 2.2 g is said to be preinvex wrt η on S if for any $x, y \in S$ and $\lambda \in [0, 1]$,

$$g(x + \lambda\eta(y, x)) \leq \lambda g(y) + (1 - \lambda)g(x).$$

Definition 2.3 ∂g is said to be monotone wrt η on S if, for any $x, y \in S, \xi \in \partial g(x)$ and $\zeta \in \partial g(y)$,

$$\langle \xi, \eta(y, x)\rangle + \langle \zeta, \eta(x, y)\rangle \leq 0.$$

Remark 2.1 It is easy to see that if g is said to invex wrt η on S, then ∂g is monotone wrt η on S.

Definition 2.4 A mapping $\eta : S \times S \to X$ is said to be skew if, for any $x, y \in S$,

$$\eta(x, y) + \eta(y, x) = 0.$$

Definition 2.5 ([5]) Let x, y be two arbitrary points of S. A set P_{xz} is said to be a closed η-path joining the points x and $z = x + \eta(y, x)$ (contained in S) if

$$P_{xz} = \{u = x + \lambda\eta(y, x) : \lambda \in [0, 1]\}.$$

Analogously, an open η-path joining the points x and $z = x + \eta(y, x)$ (contained in S) is

$$P^o_{xz} = \{u = x + \lambda\eta(y, x) : \lambda \in (0, 1)\}.$$

Yang et.al [33] introduced the following Condition A.

Condition A: Let $S \subseteq X$ be an invex set wrt η and let $g : S \to R$ be a function. Then,

$$g(y + \eta(x, y)) \leq g(x), \forall x, y \in S.$$

Mohan and Neogy [20] introduced the following Condition C. Condition C: for any $x, y \in S, \lambda \in [0, 1]$,

$$\eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y),$$
$$\eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y).$$

Remark 2.2 Yang et al. [35] showed that, if $\eta : S \times S \to X$ satisfies Condition C, then

$$\eta(y + \lambda\eta(x, y), y) = \lambda\eta(x, y), \lambda \in [0, 1].$$

Lemma 2.1 ([27]) *Let g and η satisfy Condition A and Condition C. If g is invex wrt η on S, then g is preinvex wrt η on S.*

Lemma 2.2 ([28]) *Let x and y be two arbitrary points of S. $z = x + \eta(y, x)$, g is lower regular on P^o_{xz}, then there exists $c \in P^o_{xz}$, $\xi \in \partial g(c)$,*

$$\langle \xi, \eta(y, x)\rangle = g(z) - g(x).$$

In other words, there exists $\lambda \in (0, 1)$, $c = x + \lambda\eta(y, x)$, $\xi \in \partial g(c)$,

$$\langle \xi, \eta(y, x)\rangle = g(x + \eta(y, x)) - g(x).$$

Lemma 2.3 ([24]) *Let S be a nonempty convex subset of a Housdorff topological vector space X, let K be nonempty compact subset of S, suppose that $A, B : S \rightrightarrows S$ are setvalued mappings satisfying the following conditions:*

- (A1) *$Ax \subset Bx$ for all $x \in S$;*
- (A2) *Bx is a convex set for all $x \in S$;*
- (A3) *$Ax \neq \emptyset$ for all $x \in S$;*
- (A4) *$A^{-1}y = \{x \in S, y \in Ax\}$ is an open set for each $x \in S$;*
- (A5) *for each finite subset N of S, there exists a compact,convex and nonempty subset L_N of S, such that $L_N \supset N$, and $Ax \cap L_N \neq \emptyset$ for all $x \in L_N \backslash K$.*

Then there exists a $\bar{x} \in B\bar{x}$.

Let $f = (f_1, \ldots, f_l) : X \to R^l$ be a vector-valued function. In this paper, we consider the following vector optimization problem:

$$\text{(VOP)} \qquad \text{Minimize}\, f(x) = (f_1(x), \ldots, f_l(x)) \quad \text{subject to} \quad x \in S,$$

where $f_i : S \to R (i = 1, \ldots, l)$ are given functions.

A point $\bar{x} \in S$ is said to be an efficient (or Pareto) solution (respectively, weak efficient solution) of (VOP) if for all $y \in S$,

$$f(y) - f(\bar{x}) = (f_1(y) - f_1(\bar{x}), \ldots, f_l(y) - f_l(\bar{x})) \notin -R^l_+ \backslash \{0\},$$
$$(\text{respectively, } f(y) - f(\bar{x}) = (f_1(y) - f_1(\bar{x}), \ldots, f_l(y) - f_l(\bar{x})) \notin -\text{int} R^l_+)$$

where R^l_+ is the nonnegative orthant of R^l and 0 is the origin of the nonnegative orthant.

3 Minty vector variational-like inequalities

In this section, we consider the following Minty vector variational-like inequality problem and Stampacchia vector variational-like inequality problem:

(MVVLIP) Find $\bar{x} \in S$ such that, for all $y \in S$ and all $\xi_i \in \partial f_i(y), i \in I = \{1, \ldots, l\}$,

$$\langle \xi, \eta(y, \bar{x}) \rangle_l = (\langle \xi_1, \eta(y, \bar{x}) \rangle, \ldots, \langle \xi_l, \eta(y, \bar{x}) \rangle) \notin -R^l_+ \backslash \{0\}.$$

(SVVLIP) Find $\bar{x} \in S$ such that, for all $y \in S$ there exists $\xi_i \in \partial f_i(\bar{x}), i \in I = \{1, \ldots, l\}$,

$$\langle \xi, \eta(y, \bar{x}) \rangle_l = (\langle \xi_1, \eta(y, \bar{x}) \rangle, \ldots, \langle \xi_l, \eta(y, \bar{x}) \rangle) \notin -R^l_+ \backslash \{0\}.$$

Theorem 3.1 *Let S be an open invex set wrt $\eta : S \times S \to X$ such that any η−path is contained in S and η is skew and satisfies Condition C. For each $i \in I = \{1, \ldots, l\}$, assume that $f_i : S \to R (i = 1, \ldots, l)$ is Lipschitz continuous, invex wrt η, lower regular on S and satisfy Condition A. Then, $\bar{x} \in S$ is a solution of (MVVLIP) if and only if it is an efficient solution of (VOP).*

Proof Let $\bar{x} \in S$ be a solution of (MVVLIP) but not an efficient solution of (VOP). Then, there exists $z \in S$ such that

$$f(z) - f(\bar{x}) = (f_1(z) - f_1(\bar{x}), \ldots, f_l(z) - f_l(\bar{x})) \in -R^l_+ \backslash \{0\}. \tag{3.1}$$

Set $z(\lambda) := \bar{x} + \lambda \eta(z, \bar{x})$, for all $\lambda \in [0, 1]$. Since S is invex, $z(\lambda) \in S$, for all $\lambda \in [0, 1]$. By Lemma 2.1, each f_i is preinvex wrt η. Thus, for each $i = 1, \ldots, l$,

$$f_i(z(\lambda)) = f_i(\bar{x} + \lambda \eta(z, \bar{x})) \leq \lambda f_i(z) + (1 - \lambda) f_i(\bar{x})$$

and so

$$f_i(\bar{x} + \lambda \eta(z, \bar{x})) - f_i(\bar{x}) \leq \lambda [f_i(z) - f_i(\bar{x})]$$

for all $\lambda \in [0, 1]$. In particular, for $\lambda = 1$ and for each $i = 1, \ldots, l$, we have

$$f_i(\bar{x} + \eta(z, \bar{x})) - f_i(\bar{x}) \leq f_i(z) - f_i(\bar{x}) \tag{3.2}$$

By Lemma 2.2, there exist $\lambda_i \in (0, 1)$ and $\xi_i \in \partial f_i(z(\lambda_i))$ with $z(\lambda_i) = \bar{x} + \lambda_i \eta(z, \bar{x})$ such that, for each $i = 1, \ldots, l$

$$\langle \xi_i, \eta(z, \bar{x}) \rangle = f_i(\bar{x} + \eta(z, \bar{x})) - f_i(\bar{x}), \tag{3.3}$$

It follows from (3.2) and (3.3) that, for each $i = 1, \ldots, l$,

$$\langle \xi_i, \eta(z, \bar{x}) \rangle \leq f_i(z) - f_i(\bar{x}), \tag{3.4}$$

Suppose that $\lambda_1, \dots, \lambda_l$ are all equal. Then by Condition C and Remark 2.2, we have for all $\lambda_i \in (0,1)$, and for each $i = 1, \dots, l$,

$$\langle \xi_i, \eta(z(\lambda_i), \bar{x}) \rangle = \lambda_i \langle \xi_i, \eta(z, \bar{x}) \rangle \leq \lambda_i (f_i(z) - f_i(\bar{x})) \tag{3.5}$$

By (3.1) and (3.5), we know that it contradicts the fact that the $\bar{x}$ is a solution of (MVVLIP).

Consider the case when $\lambda_1, \dots \lambda_l$ are not all equal. Let $\lambda_1 \neq \lambda_2$. Then by (3.4), we have

$$\langle \xi_1, \eta(z, \bar{x}) \rangle \leq f_1(z) - f_1(\bar{x}) \tag{3.6}$$

and

$$\langle \xi_2, \eta(z, \bar{x}) \rangle \leq f_2(z) - f_2(\bar{x}) \tag{3.7}$$

Since f_1 and f_2 are invex wrt η, by Remark 2.1, ∂f_1 and ∂f_2 are monotone wrt η, then for all

$$\langle \xi_1 - \zeta_1, \eta(z(\lambda_1), z(\lambda_2)) \rangle \geq 0, \quad \text{for all } \zeta_1 \in \partial f_1(z(\lambda_2)). \tag{3.8}$$

and

$$\langle \zeta_2 - \xi_2, \eta(z(\lambda_1), z(\lambda_2)) \rangle \geq 0, \quad \text{for all } \zeta_2 \in \partial f_2(z(\lambda_1)). \tag{3.9}$$

By Condition C and the Remark 2.2, we have

$$\begin{aligned}
\eta(z(\lambda_1), z(\lambda_2)) &= \eta(\bar{x} + \lambda_1 \eta(z, \bar{x}), \bar{x} + \lambda_2 \eta(z, \bar{x})) \\
&= \eta\,(\bar{x} + \lambda_2 \eta(z, \bar{x}) + (\lambda_1 - \lambda_2)\eta(z, \bar{x}), \bar{x} + \lambda_2 \eta(z, \bar{x})) \\
&= \eta(\bar{x} + \lambda_2 \eta(z, \bar{x}) + \tfrac{(\lambda_1 - \lambda_2)}{1 - \lambda_1} \eta(z, \bar{x} + \lambda_1 \eta(z, \bar{x})), \bar{x} + \lambda_2 \eta(z, \bar{x})) \\
&= \tfrac{(\lambda_1 - \lambda_2)}{1 - \lambda_1} \eta(z, \bar{x} + \lambda_1 \eta(z, \bar{x})) \\
&= (\lambda_1 - \lambda_2) \eta(z, \bar{x})
\end{aligned} \tag{3.10}$$

If $\lambda_1 > \lambda_2$, then by (3.8) and (3.10), we obtain

$$\langle \xi_1 - \zeta_1, \eta(z(\lambda_1), z(\lambda_2)) \rangle = (\lambda_1 - \lambda_2) \langle \xi_1 - \zeta_1, \eta(z, \bar{x}) \rangle \geq 0,$$

and so

$$\langle \xi_1, \eta(z, \bar{x}) \rangle \geq \langle \zeta_1, \eta(z, \bar{x}) \rangle.$$

From (3.6), we have

$$\langle \zeta_1, \eta(z, \bar{x}) \rangle \leq f_1(z) - f_1(\bar{x}), \quad \text{for all } \zeta_1 \in \partial f_1(z(\lambda_2)). \tag{3.11}$$

If $\lambda_1 < \lambda_2$, then by (3.9) and (3.10), we have

$$\langle \zeta_2 - \xi_2, \eta(z(\lambda_1), z(\lambda_2)) \rangle = (\lambda_1 - \lambda_2) \langle \zeta_2 - \xi_2, \eta(z, \bar{x}) \rangle \geq 0$$

and so

$$\langle \xi_2, \eta(z, \bar{x}) \rangle \geq \langle \zeta_2, \eta(z, \bar{x}) \rangle.$$

From (3.7), we have

$$\langle \zeta_2, \eta(z, \bar{x}) \rangle \leq f_2(z) - f_2(\bar{x}), \quad \text{for any } \zeta_2 \in \partial f_2(z(\lambda_1)). \tag{3.12}$$

Letting $\bar{\lambda} = \min(\lambda_1, \lambda_2)$, by (3.11), (3.12), we can find $\bar{\xi}_i \in \partial f_i(z(\lambda))$ such that

$$\langle \bar{\xi}_i, \eta(z, \bar{x}) \rangle \leq f_i(z) - f_i(\bar{x}), \quad \text{for } i = 1, 2.$$

Hence, we can find $\lambda^* \in (0, 1)$ and $\xi_i^* \in \partial f_i(z(\lambda^*))$ such that $\lambda^* = \min(\lambda_1, \cdots, \lambda_l)$ and

$$\langle \xi_i^*, \eta(z, \bar{x}) \rangle \leq f_i(z) - f_i(\bar{x}), \quad \text{for each } i = 1, \ldots l. \tag{3.13}$$

By (3.1) and (3.13), we have

$$(\langle \xi_1^*, \eta(z, \bar{x}) \rangle, \ldots, \langle \xi_l^*, \eta(z, \bar{x}) \rangle) \in -R_+^l \backslash \{0\}.$$

It follows from Remark 2.2 that

$$(\langle \xi_1^*, \eta(z(\lambda^*), \bar{x}) \rangle, \ldots, \langle \xi_l^*, \eta(z(\lambda^*), \bar{x}) \rangle) \in -R_+^l \backslash \{0\},$$

which contradicts the fact that $\bar{x}$ is a solution of (MVVLIP).

Conversely, suppose that $\bar{x} \in S$ is an efficient solution of (VOP), then

$$(f_1(y) - f_1(\bar{x}), \ldots, f_l(y) - f_l(\bar{x})) \notin -R_+^l \backslash \{0\}, \quad \text{for all } y \in S. \tag{3.14}$$

Since f_i is invex wrt η, we then have

$$\langle \xi_i, \eta(\bar{x}, y) \rangle \leq f_i(\bar{x}) - f_i(y), \quad \text{for all } y \in S, \xi_i \in \partial f_i(y).$$

Because η is skew, we get

$$\langle \xi_i, \eta(y, \bar{x}) \rangle \geq f_i(y) - f_i(\bar{x}), \quad \text{for all } y \in S, \xi_i \in \partial f_i(y). \tag{3.15}$$

By (3.14) and (3.15), we know that $\bar{x} \in S$ is a solution of (MVVLIP). This completes the proof. □

Remark 3.1 (i) Theorem 3.1 improves Theorem 3.1 of [1] in the following two aspects: (a) the space R^n is extended to the Asplund space; (b) the Conditions C(c) of Theorem 3.1 in [1] is removed.

(ii) Theorem 3.1 also improves Theorem 3.1 of [31] in the following aspects: (a) the space R^n is extended to the Asplund space; (b) the differential functions f_i are extended to the nondifferential ones.

Theorem 3.2 *Let S be an invex set wrt $\eta : S \times S \to X$. For each $i \in I = \{1, \ldots, l\}$, $f_i : S \to R\ (i = 1, \ldots, l)$ is Lipschitz continuous and invex wrt η on S. If $\bar{x} \in S$ is a solution of (SVVLIP), then it is an efficient solution of (VOP); Furthermore, if η is skew, then it is a solution of (MVVLIP).*

Proof Suppose that $\bar{x} \in S$ is a solution of (SVVLIP) but not an efficient solution of (VOP). Then there exists $y \in S$ such that

$$(f_1(y) - f_1(\bar{x}), \ldots, f_l(y) - f_l(\bar{x})) \in -R^l_+ \setminus \{0\}.$$

As each f_i is invex wrt η, we have

$$\langle \xi_i, \eta(y, \bar{x}) \rangle \leq f_i(y) - f_i(\bar{x}), \quad \text{for all } \xi_i \in \partial f_i(\bar{x})$$

and so

$$(\langle \xi_1, \eta(y, \bar{x}) \rangle, \ldots, \langle \xi_l, \eta(y, \bar{x}) \rangle) \in -R^l_+ \setminus \{0\} \text{ for all } \xi_i \in \partial f_i(\bar{x}),$$

which contradicts the fact that $\bar{x}$ is a solution of (SVVLIP). Furthermore, if η is skew, by the proof of Theorem 3.1, we know that it is a solution of (MVVLIP). This completes the proof. □

4 Weak Minty vector variational-like inequalities

In this section, we consider the following weak Minty vector variational-like inequalities problem and weak Stampacchia vector variational-like inequalities problem.

(WMVVLIP) Find $\bar{x} \in S$ such that, for all $y \in S$ and all $\xi_i \in \partial f_i(y), i \in I = \{1, \ldots, l\}$,

$$\langle \xi, \eta(y, \bar{x}) \rangle_l = (\langle \xi_1, \eta(y, \bar{x}) \rangle, \ldots, \langle \xi_l, \eta(y, \bar{x}) \rangle) \notin -\mathrm{int} R^l_+.$$

(WSVVLIP) Find $\bar{x} \in S$ such that, for all $y \in S$ there exists $\xi_i \in \partial f_i(\bar{x}), i \in I = \{1, \ldots, l\}$,

$$\langle \xi, \eta(y, \bar{x}) \rangle_l = (\langle \xi_1, \eta(y, \bar{x}) \rangle, \ldots, \langle \xi_l, \eta(y, \bar{x}) \rangle) \notin -\mathrm{int} R^l_+.$$

Now we show the relation among the sets of solutions of (WMVVLIP) and (WSVVLIP) and that of the weak efficient solutions of (VOP).

Theorem 4.1 *Let S be an invex set wrt $\eta : S \times S \to X, \eta$ is skew and satisfies Condition C. For each $i \in I = \{1, \ldots, l\}$, $f_i : S \to R\,(i = 1, \cdots, l)$ is Lipschitz continuous and invex wrt η on S. Then, $\bar{x} \in S$ is a solution of (WMVVLIP) if and only if it is a solution of (WSVVLIP).*

Proof Let $\bar{x} \in S$ is a solution of (WMVVLIP). For any $y \in S$ and $t \in (0, 1]$, since S is invex wrt η, we have

$$x(t) := \bar{x} + t\eta(y, \bar{x}) \in S.$$

As $\bar{x} \in S$ is a solution of (WMVVLIP), there exist $\xi_i^t \in \partial f_i(x(t))$ such that

$$\left(\left\langle \xi_1^t, \eta(x(t), \bar{x})\right\rangle, \ldots, \left\langle \xi_l^t, \eta(x(t), \bar{x})\right\rangle\right) \notin -\mathrm{int}R_+^l.$$

Hence, by Remark 2.2 and $t \in (0, 1]$, we have

$$\left(\left\langle \xi_1^t, \eta(y, \bar{x})\right\rangle, \ldots, \left\langle \xi_l^t, \eta(y, \bar{x})\right\rangle\right) \notin -\mathrm{int}R_+^l.$$

Since $f_i : S \to R\,(i = 1, \ldots, l)$ is Lipschitz continuous on S, we know that $\left\{\xi_i^t\right\}$ is bounded due to Proposition 1.85 in Mordukhovich [21]. Because $\lim_{t\to 0^+} x(t) = \bar{x}$, recalling X is Asplund, $\left\{\xi_i^t\right\}$ has a subsequence that converges weak* to some $\xi_i \in \partial f_i(\bar{x})$ for $i = 1, \ldots, l$. Hence, for any $y \in S$, there exist $\xi_i \in \partial f_i(\bar{x}), i = 1, \cdots, l$, such that

$$(\langle \xi_1, \eta(y, \bar{x})\rangle, \ldots, \langle \xi_l, \eta(y, \bar{x})\rangle) \notin -\mathrm{int}R_+^l.$$

This shows that $\bar{x} \in S$ is a solution of (WSVVLIP).

Conversely, suppose that $\bar{x} \in S$ is a solution of (WSVVLIP). Then for any $y \in S$, there exists $\xi_i \in \partial f_i(\bar{x}), i = 1, \ldots, l$, such that

$$(\langle \xi_1, \eta(y, \bar{x})\rangle, \ldots, \langle \xi_l, \eta(y, \bar{x})\rangle) \notin -\mathrm{int}R_+^l.$$

Since each f_i is invex wrt η, by Remark 2.1, we have

$$\langle \zeta_i - \xi_i, \eta(y, \bar{x})\rangle \geq 0, \text{ for any } y \in S, \zeta_i \in \partial f_i(y).$$

Hence, for any $y \in S$ and $\zeta_i \in \partial f_i(y), i = 1, \ldots, l$,

$$(\langle \zeta_1, \eta(y, \bar{x})\rangle, \ldots, \langle \zeta_l, \eta(y, \bar{x})\rangle) \notin -\mathrm{int}R_+^l.$$

Thus, $\bar{x} \in S$ is a solution of (WMVVLIP). This completes the proof. □

By Theorems 3.2 and 4.1, we can easily obtain the following theorem which present the equivalence among the sets of solutions of (WMVVLIP) and (WSVVLIP) and that of the weak efficient solutions of (VOP).

Theorem 4.2 *Let S be an invex set wrt $\eta : S \times S \to X$, η is skew and satisfies Condition C. For each $i \in I = \{1, \cdots, l\}$, assume that $f_i : S \to R\,(i = 1, \cdots, l)$ is Lipschitz continuous and invex wrt η on S. Then, the sets of solutions of (WMVVLIP) and (WSVVLIP) and that of the weak efficient solutions of (VOP) are equal.*

Remark 4.1 Theorem 4.2 extends Theorem 4.1 in [1] from R^n to the Asplund space.

Next we present an existence result for the solutions of (WMVVLIP).

Theorem 4.3 *Let S be an convex set of X, for each $f_i : S \to R\,(i = 1, \cdots, l)$ is Lipschitz continuous and invex wrt η on S, K be a nonempty compact subset of S. η is affine, skew and continuous about the first variable. Moreover, suppose that for each finite subset N of S, there exists a compact, convex and nonempty set L_N of S, such that $N \subset L_N$ and for all $x \in L_N \backslash K$, there is $y \in L_N$ such that there exists $\xi_i \in \partial f_i\,(y)\,, i = 1, \cdots, l$ satisfying*

$$(\langle \xi_1, \eta\,(y, x)\rangle\,, \ldots, \langle \xi_l, \eta\,(y, x)\rangle) \in -intR^l_+.$$

Then (WMVVLIP) has a solution.

Proof Let $A, B : S \rightrightarrows S$ be two set-valued mappings defined as follows:

$$Ax = \left\{y \in S : \exists \xi_i \in \partial f_i\,(y)\,, (\langle \xi_1, \eta\,(y, x)\rangle\,, \ldots, \langle \xi_l, \eta\,(y, x)\rangle) \in -\operatorname{int}R^l_+\right\}.$$
$$Bx = \left\{y \in S : \forall \xi_i \in \partial f_i\,(x)\,, (\langle \xi_1, \eta\,(y, x)\rangle\,, \ldots, \langle \xi_l, \eta\,(y, x)\rangle) \in -\operatorname{int}R^l_+\right\}.$$

We will show that the set-valued mappings satisfy all the conditions (A1), (A2), (A4), (A5) of Lemma 2.3, but B does not have a fixed point. Thus, by the Lemma 2.3, we know there exists $x \in S$ such that $Ax = \emptyset$ and so x is a solution of (WMVVLIP).

First, we show the condition (A1) of Lemma 2.3 holds. Taking $x \in S$, and $y \in Ax$, then there exists $\xi_i \in \partial f_i\,(y)$, such that

$$(\langle \xi_1, \eta\,(y, x)\rangle\,, \ldots, \langle \xi_l, \eta\,(y, x)\rangle) \in -\operatorname{int}R^l_+.$$

As each f_i is invex wrt η, for any $\zeta_i \in \partial f_i\,(x)$, we have

$$\langle \xi_i - \zeta_i, \eta\,(y, x)\rangle \geq 0$$

and so

$$(\langle \zeta_1, \eta\,(y, x)\rangle\,, \ldots, \langle \zeta_l, \eta\,(y, x)\rangle) \in -\operatorname{int}R^l_+.$$

Thus, $y \in Bx$.

Now, we show the condition (A2) of Lemma 2.3 holds. Let $x \in S$, $y_1, y_2 \in Bx$ and $\lambda \in [0, 1]$. For each $\xi_i \in \partial f_i\,(x)$,

$$(\langle \xi_1, \eta\,(y_1, x)\rangle\,, \ldots, \langle \zeta_l, \eta\,(y_1, x)\rangle) \in -\operatorname{int}R^l_+,$$
$$(\langle \xi_1, \eta\,(y_2, x)\rangle\,, \ldots, \langle \zeta_l, \eta\,(y_2, x)\rangle) \in -\operatorname{int}R^l_+.$$

Since η is affine about the first variable, for each $i = 1, \ldots, l$, we have

$$\langle \xi_i, \eta(\lambda y_1 + (1-\lambda) y_2, x)\rangle = \lambda \langle \xi_i, \eta(y_1, x)\rangle + (1-\lambda)\langle \xi_i, \eta(y_2, x)\rangle < 0$$

and so

$$(\langle \xi_1, \eta(\lambda y_1 + (1-\lambda) y_2, x)\rangle, \ldots, \langle \xi_l, \eta(\lambda y_1 + (1-\lambda) y_2, x)\rangle) \in -\operatorname{int} R^l_+.$$

It follows that $\lambda y_1 + (1-\lambda) y_2 \in Bx$.

Next, we show the condition (A4) of Lemma 2.3 holds. Let $y \in S$, $x_n \in \left(A^{-1}y\right)^c$ such that $x_n \to x$. Then $x_n \notin A^{-1}y$. Let $\xi_i \in \partial f_i(y)$,

$$(\langle \xi_1, \eta(y, x_n)\rangle, \ldots, \langle \xi_l, \eta(y, x_n)\rangle) \notin -\operatorname{int} R^l_+.$$

Since η is skew and continuous about the first variable, we obtain

$$(\langle \xi_1, \eta(y, x)\rangle, \ldots, \langle \xi_l, \eta(y, x)\rangle) \notin -\operatorname{int} R^l_+.$$

By the hypotheses, condition (A5) of Lemma 2.3 also holds. It follows that B does not have a fixed point, because otherwise it would exists some $x \in S$ such that $\xi_i \in \partial f_i(x)$,

$$(\langle \xi_1, \eta(x, x)\rangle, \ldots, \langle \zeta_l, \eta(x, x)\rangle) \in -\operatorname{int} R^l_+.$$

Since η is skew, we know that $\eta(x, x) = 0$, a contradiction. Thus by Lemma 2.3, (WMVVLIP) has a solution. This completes the proof. □

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