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ORIGINAL PAPER

Cooperative games in facility location situations with regional fixed costs

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Abstract A continuous single-facility location problem, where the fixed cost (or installation cost) depends on the region where the new facility is located, is studied by mean of cooperative Game Theory tools. Core solutions are proposed for the total cost allocation problem. Sufficient conditions in order to have a nonempty core are given, then the Weber problem with regional fixed costs is studied.

Keywords Facility location problems · Cost sharing · Core solutions · Weber problem

1 Introduction

The facility location problem has been intensively studied. It concerns the question of locating some facilities in a continuous or discrete space in order to minimize the total cost of opening sites and transporting goods or services to customers (see, for example, [4,8,11], etc.). Several papers deal with single or multiple facility location, competitive location or dynamic location, and so on. The location problem has been studied also in a Game Theory context: the Hotelling model [7] is a competitive location problem where the location of two duopolists, whose decision variables are locations and prices, is chosen. References for spatial competition can be found in [1]. A multi-facility location problem where each competitor controls one or more facilities trying to optimize the transportation costs has been studied in [9], the proposed solution of the location problem is the Nash equilibrium of the location game.

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An additional question related to a facility location problem is whether the total cost can be allocated to the different customers in a fair way (allocation problem). Some cost allocation games arising from location problems in graphs have been investigated in [5,6]. The cost allocation game associated to a continuous single-facility location game is studied in [14], where some sufficient conditions in order to have a non empty core are proved. In [14], the total cost is the sum of the variable costs (transportation costs) and the fixed cost (installation cost). In line with previous papers that study from a computational point of view a single-facility Weber location problem in presence of zone varying fixed costs [2], we consider in this paper the location–allocation problem for a single-facility in a continuous space with different fixed costs depending on the chosen location. In fact, the opening cost of a facility may depend on the cost of land, on the taxation of different regions, on zone-restrictions, and so on.

In this paper the cooperative cost TU-game corresponding to a continuous single-facility location problem is defined in the case of fixed regional costs. In Sect. 2, the facility location situation is presented and some properties are given. In Sect. 3, the cost TU-game is defined and the existence of core solutions is discussed, then the Weber case is studied.

2 The location problem

A set of n users of a certain facility ($n \in \mathcal{N}, n > 1$) is located in n points $a_1, \dots, a_n$ in the plane $\mathbb{R}^2$. Let us denote by N the set of points and let Ω be of compact rectangle containing N. The total transportation cost is measured by a monotone function[1] $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ and n non-decreasing functions $f_i : \mathbb{R} \rightarrow \mathbb{R}$ such that $\Phi(f_1(\|x - a_1\|), \dots, f_n(\|x - a_n\|))$ is the cost of connecting each point in N with facility x and $\| \cdot \|$ is any norm on $\mathbb{R}^2$. Moreover, we consider an additional opening cost of the new facility $x \in \Omega$ depending on x, namely a non-negative function $K : \mathbb{R}^2 \rightarrow \mathbb{R}_+$.

As in [2], we divide the plane into mutually exclusive zones and assign a constant value to all points in the same zone (for example, regional taxation satisfies this assumption). We assume that $\Omega = \bigcup_{j=1}^{h} P_j$ with P_j non-overlapping closed convex polygons (for example, rectangles) and for any $j = 1, \dots, h$ we let

$$K(x) = k_j \quad \forall x \in P'_j$$

for a non-negative constant $k_j \in \mathbb{R}_+$, where P'_j represents the set of the interiors points of P_j and the set of its free edges, i.e. the set of boundary points that belong uniquely to it. Moreover we assume that on a common edge B_l shared by two adjacent zones P_i and P_j the minimum of the two fixed costs is applied

$$K(x) = \min\{k_i, k_j\}, \quad \forall x \in B_l = P_i \cap P_j.$$

[1] Monotonicity with respect to the partial order relation: $x \leq y \Leftrightarrow x_i \leq y_i \ \forall i$, for $x, y \in \mathbb{R}^n$.

Note that some point may belong to the intersection of two or more common edges; in this case we consider the minimum between the fixed costs of all the involved regions. The fixed cost $K(x)$ is very large (infinite) for $x \notin \Omega$. We assume the following:

i) $\Phi, f_1, \ldots, f_n$ are lower semicontinuous, increasing and convex functions in their respective domains, such that $\Phi(0) = 0$, $f_1(0) = 0, \ldots, f_n(0) = 0$.

We denote by (N, Φ, f, K), with $f = (f_1, \ldots, f_n)$, the *continuous single-facility location situation*. The problem is to find a point $x^* \in \Omega$ solution to

$$\min_{x \in \Omega} \Phi(f_1(\|x - a_1\|), \ldots, f_n(\|x - a_n\|)) + K(x) \qquad (P1)$$

Let us note that the problem ($P1$) can be defined also in the whole space $\mathbb{R}^2$ since the fixed cost $K(x)$ is very high for $x \notin \Omega$. Moreover, the problem ($P1$) under the assumption i) admits always at least an optimal solution. In fact, the function $\delta(x) = \Phi(f_1(\|x - a_1\|), \ldots, f_n(\|x - a_n\|)) + K(x)$ turns out to be lower semicontinuous at any $x \in \Omega$ and Ω is a compact set.

The case $\Phi(z_1, \ldots, z_n) = \sum_{i=1}^{n} z_i$ and $f_i(\|x - a_i\|) = w_i \|x - a_i\|_2$, with w_i positive constants for each $i = 1, \ldots, n$ and $\| \|_2$ the Euclidean norm, corresponds to the problem considered in [2] and with the additional requirement $K = 0$ is nothing but the classical single-facility minisum or Weber problem. The case $\Phi(z_1, \ldots, z_n) = \max\{z_1, \ldots, z_n\}$, $f_i(\|x - a_i\|) = w_i \|x - a_i\|_2$ and $K = 0$ is nothing but the classical single-facility minimax problem. All the mentioned location problems can be solved by using standard procedure as the Weiszfeld algorithm with the assumption of no fixed cost $K = 0$ [8], with the assumption of a constant fixed cost $K(x) = k \ \forall x$ [13] and with the assumption of regional fixed costs [2].

Let us denote by x_M^* the solution to ($P1$) with $K = 0$, i.e. without fixed costs:

$$x_M^* \in \operatorname*{argmin}_{x \in \Omega} \Phi(f_1(\|x - a_1\|), \ldots, f_n(\|x - a_n\|)).$$

Since the function $\Phi(f_1(\|x - a_1\|), \ldots, f_n(\|x - a_n\|))$ turns out to be convex under the assumption i), we have the following result that generalizes Property 1 of [2] proved for the Weber problem.

Theorem 2.1 *Under assumption i), an optimal solution x^* to the problem (P1) may be found such that $x^* = x_M^*$ or x^* is a point on an edge B_l where the fixed cost $K(x^*) < K(x_M^*)$ for any $x^* \in B_l$.*

Example 2.1 Let $N = \{a_1, a_2, a_3\}$ be 3 demand points in the rectangle $\Omega = [0, 1] \times [0, \sqrt{3}/2]$ located in $(0, 0)$, $(1/2, \sqrt{3}/2)$, $(1, 0)$, respectively. Let us consider the problem ($P1$) with $\Phi(z_1, \ldots, z_n) = \sum_{i=1}^{n} z_i$, $f_i(\|x - a_i\|) = \|x - a_i\|_2^2$ for each $i = 1, \ldots, n$ and $K(x) = k \ \forall x$, $(k > 0)$. In this case a new facility x will be located in the barycenter of the triangle with vertices a_1, a_2, a_3, i.e. $x_M^* = (1/2, \sqrt{3}/6)$. This example [14] can be interpreted as a continuous single facility location situation where 3 towns (each located in $a_i \in \Omega$) want to build an airport jointly and the building cost is k euros. Moreover, each town has to create infrastructures communicating the towns and the airport directly proportional to the square distance between the town and the facilities.

Let us now consider two different regions in the rectangle, say $\Omega = A_1 \cup A_2$ where $A_1 = \{x = (x_1, x_2) \in \Omega : 0 \leq x_1 \leq 1/4\}$. The regional fixed costs are $K(x) = 1$ for $x \in A_1$ and $K(x) = 1 + \varepsilon$ for $x \in A_2$ ($\varepsilon > 0$). In this case for ε sufficiently large ($\varepsilon > 3/16$) the new facility x will be located in $x^* = (1/4, \sqrt{3}/6)$. In this case we have to solve the problem

$$\min_{x\in\Omega} \delta(x) = \min_{(x_1,x_2)\in\Omega} x_1^2 + x_2^2 + (x_1 - 1)^2 + x_2^2 + (x_1 - 1/2)^2 + (x_2 - \sqrt{3}/2)^2 + K(x_1, x_2)$$

and $K(x^*) = 1 < K(x_M^*) = 1 + \varepsilon$ as claimed in Theorem 2.1, where $x_M^* = (1/2, \sqrt{3}/6)$ is the solution of the problem without fixed costs.

3 The allocation problem

3.1 The cost allocation game and the core

A very natural request of the agents in a continuous single-facility location situation is to find an optimal location of the facility as well as to share the corresponding total costs, i.e. the cost allocation problem (see, for example, [3]). More precisely, if the optimal location of the situation (N, Φ, f, K) is $x^* \in \Omega$ solution of $(P1)$, the agents want to divide in a suitable way the total cost

$$\delta(x^*) = \Phi(f_1(\|x^* - a_1\|), \ldots, f_n(\|x^* - a_n\|)) + K(x^*).$$

Let us consider the following *cost allocation game*, i.e. a cost TU-game (N, c) associated to (N, Φ, f, K) with characteristic function c defined by

$$c(S) = \min_{x\in\Omega}\{\Phi(d^S(x)) + K(x)\}, \quad \forall S \subseteq N$$

and $c(\emptyset) = 0$ [10]; here, for each coalition $S \subseteq N$, $d^S(x)$ is the vector in $\mathbb{R}^n$ whose ith component is $f_i(\|x - a_i\|)$ if $a_i \in S$ and zero otherwise. Then $c(N) = \Phi(d^N(x^*)) + K(x^*)$. Sometimes we need to stress that a solution to $\min_{x\in\Omega}\{\Phi(d^S(x)) + K(x)\}$ depends on S, so we denote it by $x^*(S)$.

A property that immediately follows from the definition of the cost allocation game is that, under assumption i), the characteristic function c is monotonic, i.e. $c(S) \leq c(T)$ for all $S, T \subseteq N$ with $S \subseteq T$. In fact, for any $S, T \subseteq N$ with $S \subseteq T$ and any $x \in \Omega$, we have $d^S(x) \leq d^T(x)$ because f_i are increasing functions. Since Φ is an increasing function we have $\Phi(d^S(x)) \leq \Phi(d^T(x))$ and then $\Phi(d^S(x)) + K(x) \leq \Phi(d^T(x)) + K(x)$ for any $x \in \Omega$.

The cost allocation game is not in general subadditive, i.e. $c(S \cup T) \leq c(S) + c(T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$.

Example 3.1 Let us consider $\Omega = [0, 1] \times \{0\}$ and let $N = \{a_1, a_2\}$ be 2 demand points located in (0, 0), (1, 0), respectively. Let us consider the problem $(P1)$ with

$\Phi(z_1, z_2) = z_1 + z_2$, $f_i(\|x - a_i\|) = \|x - a_i\|_2^2$ for each $i = 1, 2$ and $K(x)$ defined by

$$K(x) = \begin{cases} 1/10 & x \leq 1/2, \\ 3/10 & \text{otherwise.} \end{cases}$$

In this case a new facility x will be located in the barycenter of the segment, i.e. $x^* = 1/2$ and the cost allocation game is a 2-player TU-game with characteristic function $c(\{1\}) = 1/10, c(\{2\}) = 3/10, c(\{1, 2\}) = 3/5$. It is not subadditive since $c(\{1, 2\}) = 3/5 > c(\{1\}) + c(\{2\}) = 2/5$.

Remark 3.1 In the lucky case where everybody prefers the same location $\bar{x} \in \Omega$,

$$\bar{x} = \underset{x \in \Omega}{\operatorname{argmin}}\{\Phi(d^S(x)) + K(x)\}, \quad \forall S \subseteq N$$

and $\Phi(z_1, \ldots, z_n) = \sum_{i=1}^n z_i$, the game turns out to be subadditive. In fact

$$\begin{aligned} c(S \cup T) &= \sum_{i:a_i \in S \cup T} f_i(\|\bar{x} - a_i\|) + K(\bar{x}) \\ &= \sum_{i:a_i \in S} f_i(\|\bar{x} - a_i\|) + \sum_{i:a_i \in T} f_i(\|\bar{x} - a_i\|) + K(\bar{x}) \leq c(S) + c(T) \end{aligned}$$

for all $S, T \subseteq N$ with $S \cap T = \emptyset$.

The situation where any group of agents prefers the same location may happen if there exists a unique point with a very low fixed cost. This situation is no longer possible if the fixed cost is constant in the whole set Ω: for example, any singleton $S_i = \{a_i\}$ chooses $\bar{x} = a_i$ for any $i = 1, \ldots, n$.

The question arises of how to allocate the total costs $c(N)$. We will use the concept of core to analyze the situation (see [12]). A *core allocation* is a vector $(x_1, \ldots, x_n)$ of real numbers satisfying the following conditions:

1. Collective rationality (efficiency): $\sum_{i=1}^n x_i = c(N)$
2. Coalitions rationality (core property): $\sum_{i:a_i \in S} x_i \leq c(S)$ for any coalition $S \subseteq N$.

The core of a cost allocation game can be empty even though it is subadditive. We consider in the following example constant fixed cost.

Example 3.2 Let $N = \{a_1, a_2, a_3\}$ be three demand points in the rectangle $\Omega = [0, 1] \times [0, \sqrt{3}/2]$ located in $(0, 0)$, $(1/2, \sqrt{3}/2)$, $(1, 0)$, respectively. Let us consider the problem ($P1$) with $\Phi(z_1, \ldots, z_n) = \sum_{i=1}^n z_i$, $f_i(\|x - a_i\|) = \|x - a_i\|_2^4$ for each $i = 1, \ldots, n$ and $K(x) = 5/24 \ \forall x$. In this case a new facility x will be located in the barycenter of the triangle with vertices a_1, a_2, a_3, i.e. $x_M^* = (1/2, \sqrt{3}/6)$. In this case $c(\{1\}) = c(\{2\}) = c(\{3\}) = 5/24, c(\{1, 2\}) = c(\{1, 3\}) = c(\{2, 3\}) = 1/3, c(\{1, 2, 3\}) = 13/24$ and the core is empty.

A natural approach to allocation games having empty core as in Example 3.2 is to deal with approximate core solutions. Other cost allocation methods could be used when the cost allocation game (N, c) has an empty core, for example the Shapley value. Let us remark that the *egalitarian solution* $(c(N)/3, c(N)/3, c(N)/3)$ is nothing but the Shapley value of the TU-game in Example 3.2.

3.2 Existence of core allocations

In line with previous results in [14], we give now a sufficient condition in order to have a nonempty core. We denote by $k_{min} = \min_{i=1,\dots,h} k_i$ and by $D(S) = \max_{x\in\Omega} \Phi(d^S(x))$ for any $S \subseteq N$.

Theorem 3.1 *Assume that*

$$D(N) \leq \frac{k_{min}}{n-1} \tag{1}$$

then the egalitarian solution $(c(N)/n, \dots, c(N)/n)$ *is in the core.*

Proof Let $S \subset N$ be a coalition with cardinality $|S| \leq n-1$. For any $x \in \Omega$, we have

$$\begin{aligned} \frac{|S|}{n}\left[\Phi(d^N(x)) + K(x)\right] &\leq \frac{|S|}{n}\left[D(N) + K(x)\right] \\ &\leq \frac{(n-1)}{n}\left[\frac{K(x)}{n-1} + K(x)\right] \leq \Phi(d^S(x)) + K(x) \end{aligned}$$

then $|S|c(N)/n \leq c(S)$. □

Condition (1) means that $\Phi(d^N(x)) \leq D(N) \leq k_{min}/(n-1) \leq K(x)/(n-1)$ for any $x \in \Omega$, then for any $i = 1, \dots, h$

$$\Phi(d^N(x)) \leq \frac{k_i}{n-1} \quad \forall x \in A_i$$

This last condition represents regional constraints and is fulfilled if k_{min} is large enough. If we consider the constant fixed cost case $k_1 = \cdots = k_h$ in this last inequality and the minimum with respect to $x \in \Omega$, we have the sufficient condition proved in [14].

The sufficient condition in Theorem 3.1 is in line with the usual situation where high fixed costs make profitable the located facility. Let us point out that this paper is concentrated on a short term analysis of the location–allocation problem. In a long term analysis, the transportation costs may become very high and a different model is needed.

3.3 The Weber facility location problem

Let us consider now the Weber problem $\Phi(z_1, \dots, z_n) = \sum_{i=1}^n z_i$ and $f_i(\|x - a_i\|) = \|x - a_i\|_2$ for each $i = 1, \dots, n$, and the corresponding continuous single facility location situation (N, Φ, f, K), $K : \mathbb{R}^2 \to \mathbb{R}_+$ being the fixed cost function. The problem is to find a point $x^*(N) \in \Omega$ solution to

$$\min_{x\in\Omega} \sum_{i=1}^{n} \|x - a_i\|_2 + K(x)$$

So for any coalition $S \subseteq N$, $c(S) = \sum_{i:a_i\in S} \|x^*(S) - a_i\|_2 + K(x^*(S))$.

The *proportional rule* [10] considers a vector

$$x_i^P = \|x^*(N) - a_i\|_2 + \alpha_i K(x^*(N)), \quad i = 1, \dots, n$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a vector of positive real proportionality coefficients such that $\sum_{i=1}^n \alpha_i = 1$. For example, if the players are towns these coefficient can be proportional to the number of inhabitants or to the population density. We use the notation $\alpha_S = \sum_{i:a_i\in S} \alpha_i$ for any coalition $S \subseteq N$.

Let us remark that the egalitarian solution $(c(N)/n, \dots, c(N)/n)$ studied in Sect. 3.2 leads to share the total transportation cost among players, while the proportional rule $(x_1^P, \dots, x_n^P)$ assigns to each player his transportation cost plus a quota of the fixed cost.

The following proposition gives a sufficient condition for the Weber problem in order to have the proportional rule $(x_1^P, \dots, x_n^P)$ in the core.

Theorem 3.2 *If for any coalition $S \subseteq N$ we have*

$$|S|\|x^*(N) - x^*(S)\|_2 \le K(x^*(S)) - \alpha_S K(x^*(N)) \tag{2}$$

then $(x_1^P, \dots, x_n^P)$ is in the core.

Proof Since $\sum_{i=1}^n x_i^P = \sum_{i=1}^n \|x^*(N) - a_i\|_2 + K(x^*(N)) = c(N)$, we must prove that $\sum_{i:a_i\in S} x_i^P \le c(S)$ for each S coalition with $|S| \le n - 1$. One has

$$\begin{aligned}
\sum_{i:a_i\in S} x_i^P &= \sum_{i:a_i\in S} \|x^*(N) - a_i\|_2 + \alpha_S K(x^*(N)) \\
&\le \sum_{i:a_i\in S} (\|x^*(S) - a_i\|_2 + \|x^*(N) - x^*(S)\|_2) + \alpha_S K(x^*(N)) \\
&= |S|\|x^*(N) - x^*(S)\|_2 + \sum_{i:a_i\in S} \|x^*(S) - a_i\|_2 + \alpha_S K(x^*(N)) \\
&\le \sum_{i:a_i\in S} \|x^*(S) - a_i\|_2 + K(x^*(S)) = c(S)
\end{aligned}$$

then $(x_1^P, \dots, x_n^P)$ is in the core. □

Condition (2) describes a compensation effect between the transportation costs and the fixed costs in order to have the vector $(x_1^P, \dots, x_n^P)$ in the core: it means that the

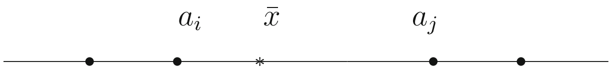

Fig. 1 The facility location problem on a line

extra transportation cost that users in S have in the case where the grand coalition will form [i.e. the left-hand side of inequality (2)] has to be lower than the additional fixed cost supported by agents of S when they belong to coalition S with respect to the fixed cost they would pay in case of grand coalition.

From condition (2) we have an upper bound for the coefficients $\alpha_i,\ i \in S$ for any coalition $S \subset N$:

$$\alpha_S \leq \frac{K(x^*(S))}{K(x^*(N))} - \frac{|S| \|x^*(N) - x^*(S)\|_2}{K(x^*(N))}. \tag{3}$$

By considering the core property for the coalition S we have a greater upper bound

$$\alpha_S \leq \frac{K(x^*(S))}{K(x^*(N))} - \frac{\sum_{i \in S} \|x^*(N) - a_i\|_2 - \|x^*(S) - a_i\|_2}{K(x^*(N))}. \tag{4}$$

In fact, $\|x^*(N) - a_i\|_2 \leq \|x^*(N) - x^*(S)\|_2 + \|x^*(S) - a_i\|_2$ for any $i \in S$ and any $x^*(S)$. Then, for some vector α, it is possible that the proportional value $x_1^P, \ldots, x_n^P$ belongs to the core even if condition (2) is not satisfied. Such a situation may happen, for example, when all facilities are located on a line in the plane. This occurs when we have a railway line and the problem is to decide where to build a railway station to improve the service to the inhabitants of the region.

For simplicity let us consider $\Omega = [0, L](L > 0)$ and the set of facilities $N = \{a_1, \ldots, a_n\}$ located in Ω with coordinates $x_{a_1}, \ldots, x_{a_n}$. We suppose also that the fixed cost defined in Ω is

$$K(x) = \begin{cases} k_1 & x \leq \bar{x}, \\ k_2 & \text{otherwise.} \end{cases}$$

with $0 < k_1 < k_2$ and the solution of the corresponding Weber problem is $x^*(N) = \bar{x}$. Let us focus the analysis on two consecutive facilities $a_i, a_j (x_{a_i} < x_{a_j})$ as in Fig. 1.

In this case if $k_1 + (x_{a_j} - \bar{x}) < k_2$, we have that $c(\{a_i\}) = k_1, c(\{a_j\}) = k_1 + (x_{a_j} - \bar{x}), c(\{a_i, a_j\}) = k_1 + (x_{a_j} - x_{a_i})$. If the proportional value $x_i^P = \|x^*(N) - a_i\|_2 + \alpha_i k_1$ belongs to the core, by inequality (4) for $S = \{a_i\}$ we have the bound $\alpha_i \leq 1 - \frac{\|x^*(N) - a_i\|_2}{k_1}$ and for $S = \{a_j\}$ the trivial inequality $\alpha_j \leq 1$. On the other hand, if player a_i and player a_j decide to locate *their* facility in $x^*(\{a_i, a_j\}) = a_i$, inequality (3) gives $\alpha_i + \alpha_j \leq 1 - \frac{2\|x^*(N) - a_i\|_2}{k_1}$, that is not satisfied with $\alpha_i > 1 - \frac{2\|x^*(N) - a_i\|_2}{k_1}$. If this is the case, we do not have the compensation effect given by condition (2). Note that for player a_i and player a_j it is indifferent to locate $x^*(\{a_i, a_j\})$ in any point of the interval $[x_{a_i}, \bar{x}]$.

Finally, let us remark that in the constant fixed cost case, condition (2) is similar to condition (5) of [14].

4 Concluding remarks

The paper considers a continuous single-facility location problem: the goal is to select a location that minimizes the total transportation costs from the new location to the demand points, plus a fixed cost (location problem). The corresponding total cost has then to be shared between agents (allocation problem). The location–allocation problem is studied in the case where the fixed cost is no longer constant with respect to locations. In the literature the paper by Brimberg and Salhi [2] explores a Weber single facility location problem with zone-dependent fixed cost determining the optimal solution and its properties. Here we extend this result for a general class of single facility location problem with zone-dependent fixed cost. Moreover, the allocation problem is modelled by using a TU-game and the core solution concept. Sufficient conditions are given to guarantee the existence of allocations that belong to the core of the game. Similar results concerning the allocation game have been obtained by Puerto and Garcia-Jurado [14] in the simpler case of constant fixed cost.

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