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ORIGINAL PAPER

Second-order optimality conditions for inequality constrained problems with locally Lipschitz data

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Abstract In this paper we obtain second-order optimality conditions of Fritz John and Karush–Kuhn–Tucker types for the problem with inequality constraints in nonsmooth settings using a new second-order directional derivative of Hadamard type. We derive necessary and sufficient conditions for a point $\bar{x}$ to be a local minimizer and an isolated local one of order two. In the primal necessary conditions we suppose that all functions are locally Lipschitz, but in all other conditions the data are locally Lipschitz, regular in the sense of Clarke, Gâteaux differentiable at $\bar{x}$, and the constraint functions are second-order Hadamard differentiable at $\bar{x}$ in every direction. It is shown by an example that regularity and Gâteaux differentiability cannot be removed from the sufficient conditions.

Keywords Nonsmooth optimization · Local minimizer · Second-order isolated local minimizer · Second-order conditions for optimality · Lagrange multipliers

1 Introduction

Classical second-order optimality conditions for constrained problems were generalized in several directions. One of them is to weaken the regularity assumptions, which are a priori supposed (see, for example [1,2]). Another modern direction is determined by the main task of the nonsmooth optimization, that is to extend the optimality conditions to problems, which are not necessarily twice continuously differentiable. There are a lot of literature concerning second-order conditions of Fritz John type or Karush–Kunh–Tucker one for twice continuously differentiable problems, some papers where

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$C^{1,1}$ problems are studied [9,11,12,16,17,30,31], and several works continuously differentiable problems [8,14,19,21,23]. The papers, in which more general classes of functions are the object of investigations are rather limited. In [18,28] are obtained conditions for problems with locally Lipschitz data: in [18] in terms of G-functions, and in [28] necessary conditions under some constraint qualifications. Our aim is to obtain as weaker conditions, as possible.

Recently appeared the papers Ginchev and Ivanov [14], Ivanov [19], where second-order optimality conditions were derived for inequality constrained problems with C^1 data. The functions from [14,19] are Fréchet and twice directionally differentiable. The sufficient conditions for an isolated local minimum of order two there hold for $C^{1,1}$ problems. It is shown by an example that these conditions are not satisfied when the functions belong to the class C^1. What can we do? One way to overcome this barrier is to change the type of the local minimizer [14, Section 5]. Another way is to choose another directional derivative. In this paper we apply the second possibility.

In the present paper we introduce a new second-order directional derivative of Hadamard type for nonsmooth functions. We consider the nonlinear programming problem with inequality constraints and a set constraint

$$\text{Minimize } f_0(x) \text{ subject to } x \in X,\ \ f_i(x) \leq 0, \quad i = 1, 2, \ldots, m, \tag{P}$$

where X is an open set in some Banach space $\mathbf{E}$ and $f_i : X \to \mathbf{R}$, $i = 0, 1, \ldots, m$ are real functions, defined on X. We derive optimality conditions of Lagrange multipliers type for the problem (P) in terms of this derivative. If it is not written otherwise, we suppose that the Banach space $\mathbf{E}$ is infinite-dimensional. We extend and improve the necessary and sufficient conditions in Ben-Tal [5] for problems with inequality constraints and twice continuously differentiable data to locally Lipschitz or Gâteaux differentiable ones. We obtain necessary and sufficient conditions for an isolated minimum of order two, a notion introduced in Auslender [4]. We extend some conditions from [14,19] to problems with locally Lipschitz and Gâteaux differentiable objective and constraint functions.

We begin with some preliminary definitions. Denote by $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty\} \cup \{+\infty\}$ the extended real line. Besides the usual operations with infinities we accept that $0.(\pm\infty) = (\pm\infty).0 = 0$. Denote by $\mathbf{E}^*$ the topological dual space of $\mathbf{E}$, by $\langle \cdot, \cdot \rangle$ the duality pairing between $\mathbf{E}^*$ and $\mathbf{E}$, by $\| \cdot \|_*$ the norm in weak-* topology, that is

$$\|\xi\|_* = \sup \{\langle \xi, d\rangle \mid d \in \mathbf{E},\ \ \|d\| \leq 1\}.$$

Definition 1 [7] Let be given a locally Lipschitz function $f : X \to \mathbf{R}$ with an open domain $X \subset \mathbf{E}$. The Clarke's generalized derivative of f at the point x in direction d is defined as follows:

$$f^0(x, d) = \limsup_{(t,y)\to(+0,x)} (f(y + td) - f(y))/t.$$

We accept here that t approaches to 0 with positive values and $y \in X$ is such that $\|y - x\| \to 0$. The set $\partial f(x) := \{\xi \in \mathbf{E}^* \mid \langle \xi, d\rangle \leq f^0(x, d)\}$ for all $d \in \mathbf{E}$ is called the Clarke's subdifferential (the Clarke's generalized gradient) at x [7]. Each its element is called a subgradient. We have

$$f^0(x, d) = \max\{\langle \xi, d\rangle \mid \xi \in \partial f(x)\}.$$

Lemma 1 [7] *Let the function* $f : \mathbf{E} \to \mathbf{R}$ *satisfy the Lipschitz condition with a constant* L *in some neighbourhood of the point* $x \in \mathbf{E}$. *Then* $\|\xi\|_* \leq L$ *for all* $\xi \in \partial f(x)$.

Definition 2 The directional derivative $f'(x, d)$ of the function f at the point $x \in X$ in direction $d \in \mathbf{E}$, if it exists, is defined by

$$f'(x, d) = \lim_{t \to +0} (f(x + td) - f(x))/t. \tag{1}$$

The function f is called directionally differentiable on X, if the derivative $f'(x, d)$ exists for all $x \in X$ and any direction $d \in \mathbf{E}$. If the limit (1) exists for all $t \in \mathbf{R}$, not necessarily positive, and there is a linear continuous operator $\nabla f(x)$ such that $f'(x, d) = \nabla f(x)(d)$ for all $d \in \mathbf{E}$, then f is called Gâteaux differentiable at $x \in X$. We usually denote $\nabla f(x)(d)$ by $\langle \nabla f(x), d\rangle$.

Definition 3 A locally Lipschitz function f, defined on the open set X, is called regular [7] at the point $x \in X$ if there exists the directional derivative $f'(x, d)$ in every direction $d \in \mathbf{E}$ and $f'(x, d) = f^0(x, d)$ for each $d \in \mathbf{E}$.

Lebourg's mean-value theorem [22, Theorem 2.3.7] *Let the function* f *be Lipschitz on some open set from the Banach space* $\mathbf{E}$ *containing the closed segment* $[x, y] = \{z \mid z = x + t(y - x),\ t \in [0, 1]\}$ *with endpoints* $x \in \mathbf{E}$, $y \in \mathbf{E}$. *Then there exists a point* u *from the open segment* (x, y) *and* $\xi \in \partial f(u)$ *such that*

$$f(y) - f(x) = \langle \xi, y - x\rangle .$$

Banach–Alaoglu theorem [26, Section 3.15] *Let* $\mathbf{E}$ *be a normed vector space, and let* $\mathbf{E}^*$ *be its dual. Then the closed unit ball* B *of* $\mathbf{E}^*$, *that is*

$$B = \{\xi \in \mathbf{E}^* \mid \|\xi\|_* \leq 1\},$$

is compact in the weak- topology.*

The following claim is well known.

Lemma 2 *Let* K *be a compact set in the topological space* $\mathbf{E}$. *Suppose that* $\{x_k\}$ *is an infinite sequence of points from* K. *Then* $\{x_k\}$ *has at least one accumulation point from* K.

We introduce the following definition:

Definition 4 Let be given a locally Lipschitz function $f : X \to \mathbf{R}$ with an open domain $X \subset \mathbf{E}$. The lower second-order directional derivative $f''_-(x, u)$ of f at the point $x \in X$ in direction $d \in \mathbf{E}$ is defined as element of $\overline{\mathbf{R}}$ by

$$f''_-(x, d) = \liminf_{(t,d') \to (+0,d)} 2t^{-2}(f(x + td') - f(x) - tf^0(x, d)),$$

where $f^0(x,d)$ is the Clarke's generalized derivative at the point x in direction d. We accept here that t converges to 0 with positive values, and $d' \in \mathbf{E}$ is such that $\|d'-d\| \to 0$. This is a derivative of Hadamard type because of the convergence $(t,d') \to (+0,d)$. It is well defined, because the Clarke's derivative is finite when the function is locally Lipschitz. The type of the second-order derivative differs from the type of the first-order one.

We call the function $f : X \to \mathbf{R}$ second-order Hadamard directionally differentiable at $x \in X$ in direction $d \in \mathbf{E}$ if there exists the limit

$$f''(x,d) = \lim_{(t,d')\to(+0,d)} 2t^{-2}(f(x+td') - f(x) - tf^0(x,d)).$$

It is customary that the second-order derivative corresponds to the first-order one (see, for example, Ginchev [10] where a second-order Hadamard-directional derivative is defined using the first-order Hadamard derivative). Really this is not a very strict rule. For example, in the epi-derivative of Rockafellar [25] is employed convergence of Hadamard type and the convergence in the derivative of Chaney [6] is similar. Other second-order derivatives of Hadamard type were introduced in Studniarski [27], Aubin and Frankowska [3], Ginchev and Ivanov [15]. Among these derivatives optimality conditions are derived in Chaney [6] for semismooth problems, Rockafellar [25], Ward [29], Ivanov [20] for $C^{1,1}$ problems.

Consider the problem (P). Denote by S the feasible set, that is

$$S = \{x \in X \mid f_i(x) \le 0,\ i = 1,2,\dots,m\}.$$

For every point $x \in S$ denote by $I(x)$ the set of active constraints

$$I(x) = \{i \in \{1,2,\dots,m\} \mid f_i(x) = 0\}.$$

Definition 5 A direction d is called critical at the feasible point x if

$$(f_i)^0(x,d) \le 0 \quad \text{for all } i \in \{0\} \cup I(x).$$

For a given feasible point x and a critical direction d denote

$$J(x,d) = \{i \in \{0\} \cup I(x) \mid (f_i)^0(x,d) = 0\}.$$

2 Conditions for local and global optimality

Theorem 1 (Necessary conditions for a local minimum, Primal form) *Let the set $X \subseteq \mathbf{E}$ be open, $\bar{x}$ be a local minimizer of the problem (P). Suppose that the functions f_i, $i \in \{0\} \cup I(\bar{x})$ are locally Lipschitz, f_i, $i \in I(\bar{x})$ are second-order Hadamard directionally differentiable at $\bar{x}$ in every critical direction, and the functions f_i, $i \notin I(\bar{x})$*

are continuous at $\bar{x}$. Then, for every critical direction $d \in \mathbf{E}$ there is no $z \in \mathbf{E}$, which solves the system

$$\begin{aligned} &(f_0)^0(\bar{x}, z) + (f_0)''_-(\bar{x}, d) < 0, \quad \text{if } 0 \in J(\bar{x}, d) \\ &(f_i)^0(\bar{x}, z) + f_i''(\bar{x}, d) < 0, \quad \text{if } i \in J(\bar{x}, d) \setminus \{0\}. \end{aligned} \tag{2}$$

Proof Let $J(\bar{x}, d) \neq \emptyset$. Suppose the contrary that there exists a critical direction d such that the system (2) has a solution $z \in \mathbf{E}$.

Consider the following cases:

1.1^0) Suppose that $0 \in J(\bar{x}, d)$. There exist infinite sequences $\{t_k\}$ and $\{d_k\}$ such that $t_k > 0$, $t_k \to +0$, $d_k \to d$ and

$$(f_0)''_-(\bar{x}, d) = \lim_{k\to+\infty} 2t_k^{-2}\left(f_0(\bar{x} + t_k d_k) - f_0(\bar{x}) - t_k (f_0)^0(\bar{x}, d)\right).$$

We prove that the points $x_k = \bar{x} + t_k d_k + 0.5t_k^2 z$ are feasible for all sufficiently large positive integers k and $f_0(x_k) < f_0(\bar{x})$.

According to the Lebourg's mean-value theorem [22] there exist $\theta_k \in (0, 1)$ and $\eta'_k \in \partial f_0(\bar{x} + t_k d_k + 0.5t_k^2 \theta_k z)$ such that

$$f_0(x_k) - f_0(\bar{x} + t_k d_k) = 0.5t_k^2 \left\langle \eta'_k, z \right\rangle.$$

Since f_0 satisfies the Lipschitz condition in a neighbourhood of $\bar{x}$, then, by Lemm 1, there exists $L_0 > 0$ such that $\|\eta'_k\|_* \leq L_0$ for all sufficiently large numbers k. By Banach-Alaoglu theorem and Lemma 2 we suppose without loss of generality that the sequence $\{\eta'_k\}$ is weakly-* convergent. Let $\eta'_k \to \eta_0$ in the weak-* topology, that is η_0 be the weak-* limit. Since the set valued map $\partial f_0(\cdot)$ is weakly-* closed (see [7, Proposition 2.1.5]), then $\eta_0 \in \partial f_0(\bar{x})$.

We have

$$\begin{aligned} &\lim_{k\to+\infty} 2t_k^{-2}\left(f_0(x_k) - f_0(\bar{x})\right) \\ &= \lim_{k\to+\infty} \Big(2t_k^{-2}(f_0(x_k) - f_0(\bar{x} + t_k d_k)) + 2t_k^{-2}(f_0(\bar{x} + t_k d_k) - f_0(\bar{x}) - t(f_0)^0(\bar{x}, d))\Big) \\ &= \langle \eta_0, z\rangle + (f_0)''_-(\bar{x}, d) \leq (f_0)^0(\bar{x}, z) + (f_0)''_-(\bar{x}, d) < 0. \end{aligned}$$

Hence $f_0(x_k) < f_0(\bar{x})$ for all sufficiently large k.

1.2^0) Using similar arguments we can prove that for all $i \in J(\bar{x}, d) \setminus \{0\}$ there exist $\eta_i \in \partial f_i(\bar{x})$ with

$$\lim_{k\to+\infty} 2t_k^{-2}\left(f_i(x_k) - f_i(\bar{x})\right) = \langle \eta_i, z\rangle + f_i''(\bar{x}, d) \leq (f_i)^0(\bar{x}, z) + f_i''(\bar{x}, d) < 0.$$

Therefore $f_i(x_k) < f_i(\bar{x}) = 0$ for all sufficiently large integers k.

1.3^0) For every $i \in I(\bar{x})$ such that $i \notin J(\bar{x}, d)$ we have

$$\limsup_{t \to +0} t^{-1}(f_i(\bar{x} + td) - f_i(\bar{x})) \le (f_i)^0(\bar{x}, d) = -2\varepsilon_i < 0.$$

Therefore $f_i(\bar{x} + t_k d) - f_i(\bar{x}) < -t_k \varepsilon_i$ for all sufficiently large k. By f_i is locally Lipschitz we obtain that there exist a constant L_i and neighbourhood of $\bar{x}$ such that f_i satisfies the Lipschitz condition with a constant L_i on this neighbourhood. Therefore

$$|f_i(x_k) - f_i(\bar{x} + t_k d)| \le L_i t_k \|d_k - d + 0.5 t_k z\| < t_k \varepsilon_i$$

for all sufficiently large k. It follows from here that

$$f_i(x_k) = (f_i(x_k) - f_i(\bar{x} + t_k d)) + (f_i(\bar{x} + t_k d) - f_i(\bar{x})) < t_k \varepsilon_i - t_k \varepsilon_i = 0$$

for all sufficiently large k.

1.4^0) For every $i \in \{1, 2, \ldots, m\} \setminus I(\bar{x})$ we have $f_i(\bar{x}) < 0$. Hence, by continuity, $f_i(x_k) < 0$ for all sufficiently large k.

1.5^0) Thus, for all sufficiently large positive integers k the point x_k is feasible and $f_0(x_k) < f_0(\bar{x})$, which contradicts our assumption that $\bar{x}$ is a local minimizer.

2^0) Consider the case $0 \notin J(\bar{x}, d)$ that is $(f_0)^0(\bar{x}, d) < 0$. Using the arguments of part 1.3^0 we can prove that $f_0(x_k) < f_0(\bar{x})$ for all sufficiently large integers k. By the arguments of parts 1.2^0, 1.3^0 and 1.4^0 we obtain a contradiction to the hypothesis $\bar{x}$ is a local minimizer.

3^0) At last, we prove that $J(\bar{x}, d) \neq \emptyset$. If $J(\bar{x}, d) = \emptyset$, then $(f_i)^0(\bar{x}, d) < 0$ for all i. Taking into account the arguments of parts 1.3^0 and 1.4^0 our assumption contradicts the statement $\bar{x}$ is a local minimizer. □

Theorem 2 (Necessary conditions for s local minimum, dual form) *Let the set X be open and the feasible point $\bar{x}$ be a local minimizer of the problem (P). Suppose that the functions f_i, $i \in \{0\} \cup I(\bar{x})\}$ are locally Lipschitz, regular and Gâteaux differentiable at $\bar{x}$, the functions f_i, $i \in I(\bar{x})$ are second-order Hadamard directionally differentiable at $\bar{x}$ in every critical direction, f_i, $i \notin I(\bar{x})$ are continuous at $\bar{x}$. Then, corresponding to any critical direction d such that $(f_0)''_-(\bar{x}, d) < +\infty$ if $0 \in J(\bar{x}, d)$ and $f''_i(\bar{x}, d) < +\infty$, $i \in J(\bar{x}, d) \setminus \{0\}$, there exist nonnegative multipliers $\lambda_0, \lambda_1, \ldots, \lambda_m$, $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_m) \neq 0$ with*

$$\lambda_i f_i(\bar{x}) = 0,\ i = 1, \ldots, m, \qquad \sum_{i \in \{0\} \cup I(\bar{x})} \lambda_i \nabla f_i(\bar{x}) = 0,$$
$$\lambda_i \langle \nabla(f_i)(\bar{x}), d \rangle = 0, \quad i \in \{0\} \cup I(\bar{x}),$$

and

$$\lambda_0 (f_0)''_-(\bar{x}, d) + \sum_{i \in I(\bar{x})} \lambda_i f''_i(\bar{x}, d) \ge 0.$$

Proof The case $J(\bar{x}, d) = \emptyset$ is impossible, since $\bar{x}$ is a local minimum. Let d be an arbitrary fixed critical direction. It follows from Theorem 1 that the system

$$\begin{aligned} &\langle \nabla f_0(\bar{x}), z\rangle + (f_0)''_-(\bar{x}, d) < 0, \quad \text{if } 0 \in J(\bar{x}, d) \\ &\langle \nabla f_i(\bar{x}), z\rangle + f''_i(\bar{x}, d) < 0, \quad \text{if } i \in J(\bar{x}, d) \setminus \{0\} \end{aligned}$$

is inconsistent. We remove all inequalities from this system such that the respective second-order derivative is $-\infty$.

Consider the matrix $\mathbf{A}$, whose rows are $\{\nabla f_i(\bar{x}) \mid i \in J(\bar{x}, d)\}$ and the vector $\mathbf{b}$, whose components are $\{-f''_i(\bar{x}, d) \mid i \in J(\bar{x}, d)\}$ if $0 \notin J(\bar{x}, d)$ and

$$\{-(f_0)''_-(\bar{x}, d), -f''_i(\bar{x}, d), i \in J(\bar{x}, d) \setminus \{0\}\} \quad \text{if } 0 \in J(\bar{x}, d).$$

With these notations Theorem 1 states that the linear system $\mathbf{A}z < \mathbf{b}$ has no solution. This is equivalent to say that the linear program $\max\{y \mid \mathbf{A}z + \hat{\mathbf{y}} \le \mathbf{b}\}$ has an optimal value $\bar{y} \le 0$. Here $\hat{\mathbf{y}}$ is the vector with all components equal y. Thus, the dual program

$$\min \left\{\mathbf{b}^T\lambda \mid \mathbf{A}^T\lambda = 0, \sum \lambda_i = 1, \lambda_i \ge 0\right\}$$

has a non positive optimal value, i.e. the system

$$\mathbf{A}^T\lambda = 0, \quad \mathbf{b}^T\lambda \le 0, \ \lambda \ge 0, \ \lambda \ne 0 \tag{3}$$

has a solution. If we put $\lambda_i = 0$ for $i \in (\{0\} \cup I(\bar{x}))\setminus J(\bar{x}, d)$ or the respective second-order derivative is $-\infty$, then we see from (3) that $\lambda_0, \ \lambda_1, \ldots, \lambda_m$ satisfy the statement of the theorem. □

Definition 6 Recall that a function $f : X \to \mathbf{R}$ is said to be quasiconvex at the point $x \in X$ (with respect to X) [24] if the conditions

$$y \in X, \ f(y) \le f(x), \ t \in [0, 1], \ (1-t)x+ty \in X \quad \text{imply} \quad f((1-t)x+ty) \le f(x).$$

If the set X is convex, then the function f is called quasiconvex on X when for all $x, \ y \in X$ and $t \in [0, 1]$ it holds $f((1-t)x + ty) \le \max(f(x), f(y))$.

We introduce the following notion which is based on Definition 1 in Ginchev and Ivanov [13].

Definition 7 We call a locally Lipschitz function f second-order pseudoconvex at $x \in X$ if for all $y \in X$ the following implications hold:

$$\begin{aligned} &f(y) < f(x) \quad \text{implies} \quad f^0(x, y - x) \le 0; \\ &f(y) < f(x), \ f^0(x, y - x) = 0 \quad \text{imply} \quad f''_-(x, y - x) < 0. \end{aligned}$$

We call f second-order pseudoconvex on X if it is second-order pseudoconvex at every $x \in X$.

Theorem 3 (Sufficient conditions for a global minimum) *Let the constraint set X be open and convex, $\bar{x}$ be a feasible point. Suppose that the functions f_i, $i \in \{0\} \cup I(\bar{x})$ are locally Lipschitz, regular and Gâteaux differentiable at $\bar{x}$, f_0 is second-order pseudoconvex at $\bar{x}$, f_i, $i \in I(\bar{x})$ are quasiconvex at $\bar{x}$ and second-order Hadamard differentiable at $\bar{x}$ in every critical direction $d \neq 0$. If for each critical direction $d \neq 0$ there exist Lagrange nonnegative multipliers $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that*

$$\lambda_i f_i(\bar{x}) = 0, \; i = 1, \ldots, m, \quad \nabla f_0(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla f_i(\bar{x}) = 0$$

$$(f_0)''_-(\bar{x}, d) + \sum_{i \in I(\bar{x})} \lambda_i f_i''(\bar{x}, d) \geq 0,$$

then $\bar{x}$ is a global minimizer of (P).

Proof Assume the contrary that there exists $x \in S$ with $f_0(x) < f_0(\bar{x})$. We prove that $x - \bar{x}$ is a critical direction. By the second-order pseudoconvexity $\nabla f_0(\bar{x})(x - \bar{x}) \leq 0$. Due to the quasiconvexity and $f_i(x) \leq f_i(\bar{x})$, $i \in I(\bar{x})$, we have $\nabla f_i(\bar{x})(x - \bar{x}) \leq 0$ for all $i \in I(\bar{x})$, which implies that $x - \bar{x}$ is critical.

Using the assumptions of the theorem we obtain that there exist nonnegative multipliers $\lambda_1, \lambda_2, \ldots, \lambda_m$ with $\lambda_i f_i(\bar{x}) = 0$, $i = 1, \ldots, m$ and

$$\nabla f_0(\bar{x})(x - \bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla f_i(\bar{x})(x - \bar{x}) = 0$$

such that $(f_0)''_-(\bar{x}, x - \bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i f_i''(\bar{x}, x - \bar{x}) \geq 0$. Therefore $\lambda_i = 0$ when $i \notin I(\bar{x})$. Using that $x - \bar{x}$ is critical we obtain that $\nabla f_0(\bar{x})(x - \bar{x}) = 0$ and $\lambda_i \nabla f_i(\bar{x})(x - \bar{x}) = 0$ for all $i \in I(\bar{x})$. Then $\nabla f_i(\bar{x})(x - \bar{x}) = 0$ when $\lambda_i > 0$. It follows from the second-order pseudoconvexity that $(f_0)''_-(\bar{x}, x - \bar{x}) < 0$. By the quasiconvexity $f_i''(\bar{x}, x - \bar{x}) \leq 0$ for all $i \in I(\bar{x})$. We conclude from here that

$$(f_0)''_-(\bar{x}, x - \bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i f_i''(\bar{x}, x - \bar{x}) < 0,$$

which contradicts the hypothesis. □

3 Conditions for a second-order isolated local minimum

Definition 8 A feasible point $\bar{x}$ is called an isolated local minimizer of second-order for the problem (P) if there exist a neighborhood N of $\bar{x}$ and a constant $C > 0$ with $f_0(x) \geq f_0(\bar{x}) + C\|x - \bar{x}\|^2$ for all $x \in N \cap S$, where S is the set of feasible points.

Theorem 4 (Sufficient conditions, dual form) *Let* **E** *be a finite-dimensional space. Suppose that X is an open set, the functions f_i, $i \in \{0\} \cup I(\bar{x})$ are locally Lipschitz, regular and Gâteaux differentiable at the feasible point $\bar{x}$, and the functions f_i, $i \in I(\bar{x})$ are second-order Hadamard differentiable at $\bar{x}$ in every critical direction*

$d \neq 0$. *If for every nonzero critical direction d there exist Lagrange multipliers $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$ such that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \neq 0$ and*

$$\sum_{i \in \{0\} \cup I(\bar{x})} \lambda_i \nabla f_i(\bar{x}) = 0; \tag{4}$$

$$\lambda_i f_i(\bar{x}) = 0, \quad i = 1, 2, \ldots, m; \tag{5}$$

$$\lambda_0 (f_0)''_-(\bar{x}, d) + \sum_{i \in I(\bar{x})} \lambda_i f_i''(\bar{x}, d) > 0, \tag{6}$$

then $\bar{x}$ is an isolated local minimizer of second-order.

Proof Suppose the contrary that $\bar{x}$ is not an isolated local minimizer of order two. Therefore for every sequence of positive numbers $\{\varepsilon_k\}$, converging to 0, there exists a sequence of feasible points $\{x_k\}$ such that $\|x_k - \bar{x}\| \leq \varepsilon_k$ and

$$f_0(x_k) < f_0(\bar{x}) + 0.5\varepsilon_k \|x_k - \bar{x}\|^2.$$

Denote $t_k = \|x_k - \bar{x}\|$, $d_k = (x_k - \bar{x})/t_k$. Without loss of generality we may assume that $d_k \to d$, because the space is finite-dimensional. Hence

$$f_0(\bar{x} + t_k d_k) < f_0(\bar{x}) + 0.5\, \varepsilon_k t_k^2, \quad f_i(\bar{x} + t_k d_k) \leq 0,\ i = 1, 2, \ldots m. \tag{7}$$

On the other hand we have

$$\frac{f_0(\bar{x} + t_k d) - f_0(\bar{x})}{t_k} = \frac{f_0(\bar{x} + t_k d) - f_0(\bar{x} + t_k d_k)}{t_k} + \frac{f_0(\bar{x} + t_k d_k) - f_0(\bar{x})}{t_k}. \tag{8}$$

Since f_0 is locally Lipschitz, there exists a constant L_0 such that

$$|f_0(\bar{x} + t_k d) - f_0(\bar{x} + t_k d_k)| \leq L_0\, t_k \|d_k - d\|$$

for all sufficiently large k. Using that d_k approaches d it follows from (7) and (8) that $\langle \nabla f_0(\bar{x}), d \rangle \leq 0$. Due to similar arguments we can prove that $\langle \nabla f_i(\bar{x}), d \rangle \leq 0$, $i \in I(\bar{x})$. Therefore d is a critical direction. It follows from $\|d_k\| = 1$ that $d \neq 0$. Then we obtain from (4) and (7) that

$$\begin{aligned} &\lambda_0 (f_0)''_-(\bar{x}, d) + \sum_{i \in I(\bar{x})} \lambda_i f_i''(\bar{x}, d) \\ &\leq \liminf_{k \to +\infty} \Bigg(\lambda_0 f_0(\bar{x} + t_k d_k) - \lambda_0 f_0(\bar{x}) - \lambda_0 \langle \nabla f_0(\bar{x}), d \rangle \\ &\quad + \sum_{i \in I(\bar{x})} (\lambda_i f_i(\bar{x} + t_k d_k) - \lambda_i \langle \nabla f_i(\bar{x}), d \rangle) \Bigg) \Big/ \left(0.5 t_k^2\right) \leq \liminf_{k \to +\infty} \lambda_0 \varepsilon_k = 0 \end{aligned}$$

which is a contradiction. □

The following example shows that the assumption f_i, $i \in \{0\} \cup I(\bar{x})$ are regular and differentiable is essential in Theorems 3 and 4.

Example 1 Consider the problem

$$\text{Minimize } f_0(x) \text{ subject to } f_1(x) \le 0,$$

where the functions $f_0 : \mathbf{R} \to \mathbf{R}$ and $f_1 : \mathbf{R} \to \mathbf{R}$ are defined as follows:

$$f_0(x) = \begin{cases} x^2, & \text{if } x \le 0 \\ -x^2, & \text{if } x > 0 \end{cases}, \quad f_1(x) = \begin{cases} x, & \text{if } x \le 0 \\ 0, & \text{if } x > 0. \end{cases}$$

The objective function f_0 is continuously differentiable and second-order pseudoconvex. The constraint function f_1 is locally Lipschitz and quasiconvex. Let $\bar{x} = 0$. The set of the critical directions consists of the directions $d \in \mathbf{R}$ such that $d \le 0$, because $f_0^0(\bar{x}, d) = 0$ for every $d \in \mathbf{R}$, and $f_1^0(\bar{x}, d) = d$ if $d \ge 0$, $f_1^0(\bar{x}, d) = 0$ if $d < 0$. Therefore $\partial f_1(0) = [0, 1]$. We have $(f_0)''_-(\bar{x}, d) = 2d^2$ and $f_1''(\bar{x}, d) = -\infty$ for every critical direction d such that $d < 0$. The following inequalities are satisfied for every critical direction $d \ne 0$ with Lagrange multiplier $\lambda_1 = 0$:

$$f_0^0(\bar{x}, d) + \lambda_1 f_1^0(\bar{x}, d) \ge 0, \quad (f_0)''_-(\bar{x}, d) + \lambda_1 f_1''(\bar{x}, d) > 0,$$

because we accept that $0.(-\infty) = 0$. On the other hand $\bar{x} = 0$ is not a local and global minimizer.

Theorem 5 (Sufficient conditions, Primal form) *Let X be an open set in the finite-dimensional space $\mathbf{E}$. Suppose that the functions f_i, $i \in \{0\} \cup I(\bar{x})$ are locally Lipschitz, regular, and Gâteaux differentiable at the feasible point $\bar{x}$. f_i, $i \in I(\bar{x})$ are second-order Hadamard differentiable at $\bar{x}$ in every critical direction $d \ne 0$. If for every nonzero critical direction d such that $(f_0)''_-(\bar{x}, d) < +\infty$ when $0 \in J(\bar{x}, d)$ and $f_i''(\bar{x}, d) < +\infty$, $i \in J(\bar{x}, d) \setminus \{0\}$ there is no $z \in \mathbf{E}$, which solves the system*

$$\begin{aligned} &\langle \nabla f_0(\bar{x}), z \rangle + (f_0)''_-(\bar{x}, d) \le 0, \quad \text{if } 0 \in J(\bar{x}, d) \\ &\langle \nabla f_i(\bar{x}), z \rangle + f_i''(\bar{x}, d) \le 0, \quad \text{if } i \in J(\bar{x}, d) \setminus \{0\}, \end{aligned}$$

then $\bar{x}$ is an isolated local minimizer of second-order.

Proof Let d be an arbitrary fixed nonzero critical direction. We prove that there exist Lagrange multipliers $\lambda_i \ge 0$ such that conditions (4), (5), and (6) are satisfied. Consider the matrix $\mathbf{A}$, whose rows are $\{\nabla f_i(\bar{x}) \mid i \in \{0\} \cup I(\bar{x}, d)\}$, and the vector $\mathbf{b}$, whose components are $\{-f_i''(\bar{x}, d) \mid i \in J(\bar{x}, d)\}$ if $0 \notin J(\bar{x}, d)$ and

$$\{-(f_0)''_-(\bar{x}, d), -f_i''(\bar{x}, d), i \in J(\bar{x}, d) \setminus \{0\}\} \quad \text{if } 0 \in J(\bar{x}, d).$$

With these notations it follows from the hypothesis that the linear system $\mathbf{A}z \le \mathbf{b}$ has no solutions. We remove all inequalities from the system such that the respective components of the vector $\mathbf{b}$ are not finite. Therefore the linear programming problem

$\max\{y \mid \mathbf{A}z + \hat{\mathbf{y}} \leq \mathbf{b}\}$ has strictly negative optimal value. Here $\hat{\mathbf{y}}$ is the vector with all components equal y. Thus, the dual program

$$\min\left\{\mathbf{b}^T\lambda \mid \mathbf{A}^T\lambda = 0,\ \sum \lambda_i = 1, \lambda_i \geq 0\right\}$$

is solvable and it has negative optimal value. Therefore (4), (5), (6) are satisfied. It follows from Theorem 4 that $\bar{x}$ is an isolated local minimizer of second-order. □

Theorem 6 (Necessary conditions, primal form) *Let the set $X \subseteq \mathbf{E}$ be open, $\bar{x}$ be an isolated local minimizer of order two of the problem (P). Suppose that the functions f_i, $i \in \{0\} \cup I(\bar{x})$ are locally Lipschitz, regular, and Gâteaux differentiable at $\bar{x}$. Suppose that f_i, $i \in I(\bar{x})$ are second-order Hadamard differentiable at $\bar{x}$ in every critical direction, f_i, $i \notin I(\bar{x})$ are continuous at $\bar{x}$. Then, for every critical direction $d \in \mathbf{E}$ there is no $z \in \mathbf{E}$, which solves the system*

$$\begin{aligned} &(f_0)^0(\bar{x}, z) + (f_0)''_-(\bar{x}, d) \leq 0, \quad \text{if } 0 \in J(\bar{x}, d) \\ &(f_0)^0(\bar{x}, z) + f_i''(\bar{x}, d) < 0, \quad \text{if } i \in J(\bar{x}, d) \setminus \{0\}. \end{aligned} \tag{9}$$

Proof Suppose the contrary. Then there exists a critical direction d so that the system (9) has a solution $z \in \mathbf{E}$. Let ε be an arbitrary positive number. Therefore, the system

$$\begin{aligned} &(f_0)^0(\bar{x}, z) + (f_0)''_-(\bar{x}, d) < \varepsilon, \quad \text{if } 0 \in J(\bar{x}, d) \\ &(f_i)^0(\bar{x}, z) + f_i''(\bar{x}, d) < 0, \quad \text{if } i \in J(\bar{x}, d) \setminus \{0\}. \end{aligned}$$

has a solution $z \in \mathbf{E}$. Using the arguments of Theorem 1 we obtain that there exist infinite sequences $\{t_k\}$ and $\{d_k\}$ such that $t_k > 0$, $t_k \to 0$, $d_k \to d$,

$$f_0\left(\bar{x} + t_k d_k + 0.5 t_k^2 z\right) < f_0(\bar{x}) + 0.5\varepsilon t_k^2$$

and $\bar{x} + t_k d_k + 0.5 t_k^2 z$ is a feasible point for all sufficiently large k. This is impossible, because $\bar{x}$ is an isolated local minimizer of order two. □

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