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ORIGINAL PAPER

A variational inequality formulation of the environmental pollution control problem

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Abstract In this paper we develop the time-dependent pollution control problem in which different countries aim to determine the optimal investment allocation in environmental projects and the tolerable pollutant emissions, so as to maximize their welfare. We provide the equilibrium conditions governing the model and derive the evolutionary variational inequality formulation. The existence of solutions is investigated and a numerical example is also presented.

Keywords Nash equilibrium · Evolutionary variational inequality · Kyoto Protocol

1 Introduction

The 1997 Kyoto Protocol, the international agreement linked to the United Nations Framework Convention on Climate Change, prescribes that 37 industrialized countries and the European Community, labeled as "Annex I Parties", must reduce their greenhouse gas emissions at least 5% below the 1990 levels for the 2008–2012 period. Under the Treaty countries must meet their targets primarily through national measures. However, some other market-based mechanisms are offered: the emissions trading, known as "the carbon market"; the clean development mechanism and the joint implementation (JI). In this paper we focus on the JI which, in the Kyoto Protocol, is described as follows (see [35]): "for the purpose of meeting its commitments under Article 3, any Party included in Annex I may transfer to, or acquire from, any other such Party emission reduction units (ERUs) resulting from projects aimed at reducing anthropogenic emissions by sources or enhancing anthropogenic removals by sinks of

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greenhouse gases in any sector of the economy...”. In other words, it allows countries with emission reduction or limitation commitments to collect rewards in the form of ERUs from an emission-reduction or emission removal project in another Annex I Party, where the abatement costs are lower. Therefore, joint implementation offers Parties a flexible and cost-efficient means of fulfilling a part of their Kyoto commitments, while the host country benefits from foreign investment and technology transfer. In addition, JI mechanism provides additional incentives for research into environmental projects and promotes sustainable economic growth in countries in transition to a market economy.

The larger number of papers concerning the environmental defence issues suggest a game-theoretic approach. The static game theoretic literature is too vast to list here all the results achieved up to now, hence we just refer to those that mainly inspired this paper and address the reader to the references therein for further discussions. In [5] investment strategies in the context of the JI mechanism are investigated. In particular, a two-player game is considered and solved under three possible scenarios: the autarky model, the non-cooperative joint implementation and the cooperative joint implementation. The underlying equilibrium concepts are the Nash equilibrium and the generalized Nash equilibrium (see [12,13,16,24,25,28] for both an overview on Nash equilibria and some applications). The equilibrium investment strategies are then characterized by means of the Lagrangian approach. In [13] the environmental pollution control model as in [5] is generalized to the case of N players. In addition, a formulation in terms of a quasi-variational inequality is suggested (see [1] for a survey on theory and applications on quasi-variational inequalities). Several papers have been devoted to environmental problems under a dynamic game perspective, however much less has been done about the Kyoto mechanisms. In [3] a differential game is proposed in order to analyze the hot-air effect. In [4,6] a two-player finite-horizon differential game model of JI is analyzed using nonlinear damage cost function and different possible scenarios are described.

The purpose of this paper is to present a new approach in the study of investment strategies in Kyoto’s mechanism, based on the variational inequality theory. In fact, to-date, no variational inequality framework has been applied to such a model. In addition, we attempt to refine models in the literature, taking into account the evolution of the system with respect to time, and hence assuming that all data are time-specific. It is indeed well-recognized (see for instance [2,8,11,14,22,23,29–33]) the importance of studying dynamic problems in order to represent model adjustment processes and equilibrium with lags. The mathematical framework chosen for the study of the model is that of infinite dimensional variational inequalities (see the fundamental manuscripts [17,34] and the books [9,15,21] for both theory and applications on finite and infinite dimensional variational inequalities). We emphasize that we will not develop novel mathematical techniques, but we will show how existing abstract results can be concretely applied in order to analyze and tackle the pollution control problem. In particular, we study the case in which the JI mechanism collapses to the autarky model, namely when countries seek to fulfill the Kyoto commitments investing only in local environmental projects. We suppose that different countries simultaneously aim to determine the optimal investment allocation in environmental projects and the tolerable pollutant emissions so as to maximize their welfare. Countries follow multicriteria decision-making processes since they seek to maximize their revenue and minimize both the investments in environmental projects and the damage from pollution. We show how the optimal solution of the multiobjective problem of each country solves an evolutionary variational inequality, for which we are able to ensure the existence of solutions. Moreover, we state the equilibrium conditions governing the model, which are proved to identify a solution to the evolutionary variational inequality.

The paper is organized as follows. In Sect. 2 the multicriteria decision-making behaviors of countries are investigated and the equilibrium conditions are stated. Section 3 presents the variational inequality formulation, while Sect. 4 contains a discussion on existence results. In Sect. 5 a numerical example is provided and, finally, Sect. 6 summarizes our findings and presents some further research issues.

2 The multicriteria decision-making problem and the equilibrium conditions

In this section we present the evolutionary environmental pollution control problem and give the corresponding equilibrium conditions. We study the system in the finite time horizon $[0, \bar{t}]$, with $\bar{t} > 0$. Let N be the number of countries involved in the Treaty. Let $e^i(t)$ denote the gross emissions resulting from the industrial production of country i at time $t \in [0, \bar{t}]$ and let $e(t) = (e^1(t), \dots, e^N(t))^T$ be the total gross emission vector. We assume that the emissions of each country are proportional to the industrial output, thus we can define the revenue R_i as follows

$$R_i(t, e^i(t)) : [0, \bar{t}] \times \mathbb{R}_+ \to \mathbb{R}_+.$$

Emissions can be reduced by investing in environmental projects. Let $I^i(t)$ be the amount of environmental investments held by country i in local projects at time $t \in [0, \bar{t}]$ and let $I(t) = (I^1(t), \dots, I^N(t))^T$. The benefit of this investment lies in the acquisition of emission reduction units, which are assumed to be proportional to the investment, namely, $\gamma_i(t) I^i(t)$, where $\gamma_i(t)$ is a positive technological efficiency parameter that depends on the technologies and laws of the country. The net emission in country i is given by the difference between the gross emissions and the reduction resulting from local investments, namely

$$e^i(t) - \gamma_i(t) I^i(t) \geq 0 \text{ a.e. in } [0, \bar{t}].$$

Moreover, the net emissions are assumed to be equal to a prescribed threshold $E_i(t)$, with $E_i(t) > 0$ a.e.in $[0, \bar{t}]$, in other words the following environmental constraint must hold

$$e^i(t) - \gamma_i(t) I^i(t) = E_i(t) \text{ a.e. in } [0, \bar{t}]. \tag{1}$$

The above constraint describes an instantaneous relationship, in the sense that a tolerable level $E_i(t)$ is requested at time t and is satisfied at the same time. Clearly, a delayed reaction, namely requested at time t and verified at time $t + \delta(t)$, where $\delta(t)$ is a non negative delay factor, would be more realistic. However, the instantaneous approach appears as necessary in order to capture basic features of the model, especially in view of further improvements.

Let the local investment cost of country i be given by

$$C_i(t, I^i(t)) : [0, \bar{t}] \times \mathbb{R}_+ \to \mathbb{R}_+.$$

We also assume that pollution in one country can affect also other countries, hence damages from pollution in one country depend on the net emissions of all countries according to the function

$$D_i(t, e(t), I(t)) = D_i\left(t, \sum_{i=1}^{N}(e^i(t) - \gamma_i(t)I^i(t))\right) : [0, \bar{t}] \times \mathbb{R}_+^{2N} \to \mathbb{R}_+.$$

We choose as our functional setting the Hilbert space $L^2([0, \bar{t}], \mathbb{R}^2)$ of square-integrable functions defined in the closed interval $[0, \bar{t}]$, endowed with the scalar product $\langle \cdot, \cdot \rangle_{L^2} = \int_0^{\bar{t}} \langle \cdot, \cdot \rangle dt$ and the usual associated norm $\| \cdot \|_{L^2}$.

We assume that $R_i(t, e^i(t))$ is measurable in t $\forall e^i \in \mathbb{R}_+$ and continuous in e^i; $C_i(t, I^i(t))$ is measurable in t $\forall I^i \in \mathbb{R}_+$ and continuous in I^i; $D_i(t, e(t), I(t))$ is measurable in t $\forall e, I \in \mathbb{R}_+^N$ and continuous with respect to e and I. Moreover, we assume that there exist $\frac{\partial R_i(t,e^i)}{\partial e^i}$ measurable in t $\forall e^i \in \mathbb{R}_+$ and continuous in e^i; $\frac{\partial C_i(t,I^i)}{\partial I^i}$ measurable in t $\forall I^i \in \mathbb{R}_+$ and continuous in I^i; $\frac{\partial D_i(t,e,I)}{\partial e^i(t)}$ and $\frac{\partial D_i(t,e,I)}{\partial I^i(t)}$ measurable in t and continuous with respect to e and I. In addition, we require the following conditions:

$$\exists \delta_1^i \in L^2([0, \bar{t}]) : \left| \frac{\partial R_i(t, e^i)}{\partial e^i} \right| \leq \delta_1^i(t) + |e^i|, \tag{2}$$

$$\exists \delta_2^i \in L^2([0, \bar{t}]) : \left| \frac{\partial C_i(t, I^i)}{\partial I^i} \right| \leq \delta_2^i(t) + |I^i|, \tag{3}$$

$$\exists \delta_3^i \in L^2([0, \bar{t}]) : \left| \frac{\partial D_i(t, e, I)}{\partial e^i(t)} \right| \leq \delta_3^i(t) + |e|, \tag{4}$$

$$\exists \delta_4^i \in L^2([0, \bar{t}]) : \left| \frac{\partial D_i(t, e, I)}{\partial I^i(t)} \right| \leq \delta_4^i(t) + |I|. \tag{5}$$

We denote by $I^{-i}(t)$ the vector of all the investments held by all the countries except for i. Analogously $e^{-i}(t)$ is the vector of the gross emissions of countries different from i. Sometimes we will write $e(t) = (e^i(t), e^{-i}(t))$ and $I(t) = (I^i(t), I^{-i}(t))$. The goal of country i consists in maximizing the welfare function, namely maximizing the revenue, minimizing the investments in emission reduction as well as the damage from pollution. Therefore, the corresponding optimization problem with $(e^{-i}(t), I^{-i}(t))$ fixed at $(e^{*-i}(t), I^{*-i}(t))$ is given by

$$\max_{(e^i(t), I^i(t)) \in K_i} \int_0^{\bar{t}} W_i(t, e^i(t), I^i(t), e^{*-i}(t), I^{*-i}(t))dt, \tag{6}$$

where

$$W_i(t, e^i(t), I^i(t), e^{*-i}(t), I^{*-i}(t)) = R_i(t, e^i(t)) - C_i(t, I^i(t))$$
$$-D_i\left(t, e^i(t) - \gamma_i(t)I^i(t) + \sum_{j \neq i}(e^{*j}(t) - \gamma_j(t)I^{*j}(t))\right)$$

and

$$K_i = \Big\{(e^i(t), I^i(t)) \in L^2([0, \bar{t}], \mathbb{R}^2) : e^i(t) \geq 0, \quad I^i(t) \geq 0 \text{ a.e. in } [0, \bar{t}]\ ,$$
$$0 \leq e^i(t) - \gamma_i(t)I^i(t) = E_i(t) \text{ a.e. in } [0, \bar{t}]\Big\}. \tag{7}$$

Remark 1 Conditions (2)–(5) ensure that $\int_0^{\bar{t}} W_i(t, e^i(t), I^i(t), e^{*-i}(t), I^{*-i}(t))dt$ is well-defined. In fact, from the Lagrange Theorem, given $(t_0, e^i(t_0)) \in [0, \bar{t}] \times \mathbb{R}_+$ it follows that

$$R_i(t, e^i(t)) = \int_0^1 \frac{\partial R_i(t, \tau e^i(t) + (1-\tau)e^i(t_0))}{\partial e^i}(e^i(t) - e^i(t_0))d\tau + R_i(t_0, e^i(t_0)),$$

hence

$$\int_0^{\bar{t}} |R_i(t, e^i(t))|^2 dt \leq \int_0^T \left(\int_0^1 \left|\frac{\partial R_i(t, \tau e^i(t) + (1-\tau)e^i(t_0))}{\partial e^i}\right| |e^i(t) - e^i(t_0)| d\tau\right)^2 dt$$
$$+\bar{t}|R_i(t_0, e^i(t_0))|^2$$
$$\leq \int_0^{\bar{t}} \left(\int_0^1 (\delta_1(t) + |\tau e^i(t) + (1 - \tau e^i(t_0))\Big|)||e^i(t) - e^i(t_0)|d\tau\right)^2 dt$$
$$+\bar{t}|R_i(t_0, e^i(t_0))|^2.$$

The last terms can be immediately estimated by means of constants. In other words, the integral $\int_0^{\bar{t}} |R_i(t, e^i(t))|^2 dt$ is finite and, as a consequence, $R_i(t, e^i(t))$ is square-integrable. With a similar reasoning we are able to prove that also $C_i(t, I^i(t))$ and $D_i\left(t, e^i(t) - \gamma_i(t)I^i(t) + \sum_{j \neq i}(e^{j*}(t) - \gamma_j(t)I^{j*}(t))\right)$ are square-integrable, hence W_i belongs to L^2. In addition, W_i belongs to the class of Nemytskii operators (see Example 2.5.5 p. 159 and Example 4.7.3 p. 341 in [27] for definitions and main properties), therefore W_i is continuous with respect to the strong topology.

The equilibrium concept underlying the model is that of Nash equilibrium, which we recall as follows.

Definition 1 A vector of emissions and investments $(e^*(t), I^*(t)) \in \prod_i^N K_i$ is a Nash equilibrium if, for each $i = 1, \ldots, N$, $(e^{*i}(t), I^{*i}(t))$ is an optimal solution of problem (6) in the variables $(e^i(t), I^i(t))$, with $(e^{-i}(t), I^{-i}(t))$ fixed at $(e^{*-i}(t), I^{*-i}(t))$.

We are also able to give equivalent equilibrium conditions in an implicit manner, namely in terms of marginal welfare and marginal gap from the optimal solution.

Definition 2 A vector of emissions and investments $(e^*(t), I^*(t)) \in \prod_i^N K_i$ is an equilibrium of the evolutionary environmental pollution control problem if and only if for each $i = 1, \ldots, N$ and a.e. in $[0, \bar{t}]$ it satisfies the system of inequalities

$$-\frac{\partial R_i(t, e^{*i}(t))}{\partial e^i} + \frac{\partial D_i(t, \sum_{j=1}^N e^{*j}(t) - \gamma_j(t) I^{*j}(t))}{\partial e^i} + \overline{\nu_i}(t) - \overline{\tau_i}(t) \geq 0, \tag{8}$$

$$\frac{\partial C_i(t, I^{*i}(t))}{\partial I^i} + \frac{\partial D_i(t, \sum_{j=1}^N e^{*j}(t) - \gamma_j(t) I^{*j}(t))}{\partial I^i} - \overline{\nu_i}(t) + \overline{\tau_i}(t) \geq 0, \tag{9}$$

and equalities

$$\left(-\frac{\partial R_i(t, e^{*i}(t))}{\partial e^i(t)} + \frac{\partial D_i(t, \sum_{j=1}^N e^{*j}(t) - \gamma_j(t) I^{*j}(t))}{\partial e^i} + \overline{\nu_i}(t) - \overline{\tau_i}(t)\right) \times e^{*i}(t) = 0, \tag{10}$$

$$\left(\frac{\partial C_i(t, I^{*i}(t))}{\partial I^i} + \frac{\partial D_i(t, \sum_{j=1}^N e^{*j}(t) - \gamma_j(t) I^{*j}(t))}{\partial I^i} - \overline{\nu_i}(t) + \overline{\tau_i}(t)\right) \times \gamma_i(t) I^{*i}(t) = 0, \tag{11}$$

simultaneously, where $\overline{\nu_i}(t) \in L^2([0, \bar{t}])$ is the Lagrange multiplier attached to the environmental constraint and can be viewed as the marginal abatement cost to be borne by country i; whereas $\overline{\tau_i}(t) \in L^2([0, \bar{t}])$ is the Lagrange multiplier attached to the non-negativity constraint of net emissions and can be viewed as the shadow price associated with this constraint.

Systems (8) and (10) have the following interpretation. A typical country will only emit if the marginal damage cost of emitting minus the marginal revenue of emitting equals the shadow price of net emissions minus the marginal abatement cost. In other words, the intuitive rule so that marginal costs equal marginal revenues holds, where the marginal cost is given by the marginal damage cost of emitting plus the marginal abatement cost, whereas the marginal revenue is represented by the marginal revenue of emitting plus the shadow price of emitting. If the marginal damage cost of emitting minus the marginal revenue of emitting is greater than the shadow price of net emissions minus the marginal abatement cost, then it will be unfeasible for the country to emit.

Systems (9) and (11) have the following interpretation. If the marginal damage cost of investing plus the marginal cost of investing equals the marginal abatement cost minus the shadow price of net emissions, then the investments are positive. Also in this situation marginal costs equal marginal revenues. In fact, we consider as the marginal cost the marginal damage cost of investing plus the marginal cost of investing plus the marginal abatement cost, and as marginal revenue the shadow price of emitting. If the marginal damage cost of investing plus the marginal cost of investing is greater than the marginal abatement cost minus the shadow price of net emissions, it will be unfeasible for the country to invest.

Finally, we observe that $\overline{\tau_i}(t) - \overline{\nu_i}(t)$ can also be regarded as the marginal gap between the optimal marginal damage cost and the optimal marginal revenue.

We note that in Definition 2 unknown multipliers $\overline{\nu_i}(t)$ and $\overline{\tau_i}(t)$ appear, however we will show that the equilibrium conditions are equivalent to a variational inequality in which they do not appear.

3 The variational inequality formulation

In this section we prove that the optimal solution of problem (6) is also a solution to an evolutionary variational inequality problem, which in turn is equivalent to equilibrium conditions (8)–(11).

Theorem 1 *Let us assume that, for $i = 1, \ldots, N$, $W_i(t, e^i(t), I^i(t), e^{*-i}(t), I^{*-i}(t))$ is concave and weakly upper semicontinuous with respect to $(e^i(t), I^i(t))$. Let us further suppose that $\int_0^{\bar{t}} W_i(t, e^i(t), I^i(t), e^{*-i}(t), I^{*-i}(t))dt$ is Fréchet differentiable for all i with respect to (e^i, I^i) (it suffices that the $W_i(t, e^i(t), I^i(t), e^{*-i}(t), I^{*-i}(t))$ is continuously differentiable with respect to $(e^i(t), I^i(t))$). Then a vector $(e^{*i}(t), I^{*i}(t)) \in K_i$ is an optimal solution of problem* (6) *if and only if it is a solution of the evolutionary variational inequality*:

$$\int_0^{\bar{t}} \Bigg\{ \left(-\frac{\partial R_i(t, e^{*i}(t))}{\partial e^i} + \frac{\partial D_i(t, \sum_{j=1}^N e^{*j}(t) - \gamma_j(t) I^{*j}(t))}{\partial e^i} \right)(e^i(t) - e^{*i}(t))$$
$$+ \left(\frac{\partial C_i(t, I^{*i}(t))}{\partial I^i} + \frac{\partial D_i(t, \sum_{j=1}^N e^{*j}(t) - \gamma_j(t) I^{*j}(t))}{\partial I^i} \right)$$
$$\times \gamma_i(t)(I^i(t) - I^{*i}(t)) \Bigg\} dt \geq 0, \forall (e^i(t), I^i(t)) \in K_i. \tag{12}$$

Proof Let us assume that $(e^{*i}(t), I^{*i}(t))$ is a solution to problem (6). Then for all $(e^i(t), I^i(t)) \in K_i$ the function

$$G(\lambda) = \int_0^{\bar{t}} W_i(t, \lambda e^{*i}(t) + (1-\lambda)e^i(t), \lambda I^{*i}(t) + (1-\lambda) I^i(t), e^{*-i}(t), I^{*-i}(t))\, dt,$$
$$\lambda \in [0, 1],$$

admits a maximal solution at $\lambda = 1$ and $G'(1) \geq 0$. We now consider the derivative with respect to λ

$$G'(\lambda) = \frac{\partial}{\partial \lambda} \int_0^{\bar{t}} \Bigg\{ R_i(t, \lambda e^{*i}(t) + (1-\lambda)e^i(t)) - C_i(t, \lambda I^{*i}(t) + (1-\lambda)I^i(t)) - D_i\Bigg(t, \sum_{j \neq i}(e^{*j}(t) - \gamma_j(t)I^{*j}(t)) + \lambda e^{*i}(t) + (1-\lambda)e^i(t) - \lambda\gamma_i(t)I^{*i}(t) - (1-\lambda)\gamma_i(t)I^i(t)\Bigg)\Bigg\} dt$$

$$= \int_0^{\bar{t}} \Bigg\{ \Bigg(\frac{\partial R_i(t, \lambda e^{*i}(t) + (1-\lambda)e^i(t))}{\partial e^i} - \frac{\partial}{\partial e^i} D_i \Bigg(t, \sum_{j \neq i}(e^{*j}(t) - \gamma_j(t)I^{*j}(t)) + \lambda e^{*i}(t) + (1-\lambda)e^i(t) - \gamma_i(t)(\lambda I^{*i}(t) - (1-\lambda)I^i(t))\Bigg)\Bigg)(e^{*i}(t) - e^i(t)) + \Bigg(-\frac{\partial C_i(t, I^{*i}(t))}{\partial I^i} - \frac{\partial}{\partial e^i} D_i \Bigg(t, \sum_{j \neq i}(e^{*j}(t) - \gamma_j(t)I^{*j}(t)) + \lambda e^{*i}(t) + (1-\lambda)e^i(t) - \gamma_i(t)(\lambda I^{*i}(t) - (1-\lambda)I^i(t))\Bigg)\Bigg) \gamma_i(t)(I^{*i}(t) - I^i(t))\Bigg\} dt \geq 0.$$

Therefore, we have

$$G'(1) = \int_0^{\bar{t}} \Bigg\{ \Bigg(\frac{\partial R_i(t, e^{*i}(t))}{\partial e^i} - \frac{\partial D_i(t, \sum_{j=1}^N e^{*j}(t) - \gamma_j(t)I^{*j}(t))}{\partial e^i} \Bigg) \times (e^{*i}(t) - e^i(t)) + \Bigg(-\frac{\partial C_i(t, I^{*i}(t))}{\partial I^i} - \frac{\partial D_i(t, \sum_{j=1}^N e^{*j}(t) - \gamma_j(t)I^{*j}(t))}{\partial I^i} \Bigg) \times \gamma_i(t)(I^{*i}(t) - I^i(t)) \Bigg\} dt \geq 0,$$

which is nothing but the variational inequality.

Conversely, let us assume that $(e^{*i}(t), I^{*i}(t), e^{*-i}(t), I^{*-i}(t))$ satisfies variational inequality (12). Since $W_i(t, e^i(t), I^i(t), e^{*-i}(t), I^{*-i}(t))$ is concave a.e. in $[0, \bar{t}]$, the functional

$$\widetilde{W}_i(e^i, I^i, e^{*-i}, I^{*-i}) = \int_0^{\bar{t}} W_i(t, e^i(t), I^i(t), e^{*-i}(t), I^{*-i}(t))dt,$$

is concave. Hence, by the characterization of concave function via the Fréchet derivative we have

$$\begin{aligned} &\widetilde{W}_i(e^i, I^i, e^{*-i}, I^{*-i}) - \widetilde{W}_i(e^{*i}, I^{*i}, e^{*-i}, I^{*-i}) \\ &\quad \leq \frac{\partial \widetilde{W}_i(e^{*i}, I^{*i}, e^{*-i}, I^{*-i})}{\partial e^i}(e^i - e^{*i}) \\ &\qquad + \frac{\partial \widetilde{W}_i(e^{*i}, I^{*i}, e^{*-i}, I^{*-i})}{\partial I^i}\gamma_i(I^i - I^{*i}) \leq 0, \end{aligned}$$

and hence (e^{*i}, I^{*i}) is a maximal solution.

Remark 2 Since the feasible sets K_i, $i = 1, \dots, N$, are separate, it is possible to express the problem as the following single variational inequality

$$\begin{aligned} \int_0^T \sum_{i=1}^N \Bigg\{ &\left(-\frac{\partial R_i(t, e^{*i}(t))}{\partial e^i} + \frac{\partial D_i(\sum_{j=1}^N e^{*j}(t) - \gamma_j(t) I^{*j}(t))}{\partial e^i} \right)(e^i(t) - e^{*i}(t)) \\ &+ \left(\frac{\partial C_i(t, I^{*i}(t))}{\partial I^i} + \frac{\partial D_i(\sum_{j=1}^N e^{*j}(t) - \gamma_j(t) I^{*j}(t))}{\partial I^i} \right) \\ &\times \gamma_i(t)(I^i(t) - I^{*i}(t)) \Bigg\} dt \geq 0, \end{aligned} \tag{13}$$

$\forall (e(t), I(t)) = (e^i(t), I^i(t))_{i=1,\dots,N} \in K$, where $K = \prod_{i=1}^N K_i$.

We now prove that the pollution control equilibrium vector satisfying conditions (8)–(11) is also a solution to variational inequality problem (13).

Theorem 2 *A vector of emissions and investments $(e^*(t), I^*(t)) \in K$ is an equilibrium of the evolutionary environmental pollution control problem if and only if it is solution to variational inequality problem (13).*

Proof Let $(e^*(t), I^*(t)) \in K$ be an equilibrium pattern according to Definition 2. Therefore, from (8) and (10), for all $(e(t), I(t)) \in K$, for each $i = 1, \dots N$ and a.e. in $[0, \bar{t}]$ we have

$$\begin{aligned} &\left(-\frac{\partial R_i(t, e^{*i}(t))}{\partial e^i} + \frac{\partial D_i(t, \sum_{j=1}^N e^{*j}(t) - \gamma_j(t) I^{*j}(t))}{\partial e^i} + \overline{\nu_i}(t) - \overline{\tau_i}(t) \right) \\ &\times (e^i(t) - e^{*i}(t)) \geq 0. \end{aligned}$$

Analogously, from (9) and (11) we obtain

$$\begin{aligned} &\left(\frac{\partial C_i(t, I^{*i}(t))}{\partial I^i} + \frac{\partial D_i(\sum_{j=1}^N e^{*j}(t) - \gamma_j(t) I^{*j}(t))}{\partial I^i} - \overline{\nu_i}(t) + \overline{\tau_i}(t) \right) \\ &\times \gamma_i(t)(I^i(t) - I^{*i}(t)) \geq 0. \end{aligned}$$

Summing up the above inequalities and taking into account (1), we find a.e. in $[0, \bar{t}]$

$$\sum_{i=1}^{N}\Bigg\{\Bigg(-\frac{\partial R_i(t, e^{*i}(t))}{\partial e^i}+\frac{\partial D_i(\sum_{j=1}^{N} e^{*j}(t)-\gamma_j(t)I^{*j}(t))}{\partial e^i}\Bigg)(e^i(t)-e^{*i}(t))$$
$$+\Bigg(\frac{\partial C_i(t, I^{*i}(t))}{\partial I^i}+\frac{\partial D_i(\sum_{j=1}^{N} e^{*j}(t)-\gamma_j(t)I^{*j}(t))}{\partial I^i}\Bigg)\gamma_i(t)(I^i(t)-I^{*i}(t))\Bigg\}\geq 0,$$

and hence (13).

Conversely, let $(e^*(t), I^*(t)) \in K$ be a solution to the variational inequality problem. By contradiction we assume that the equilibrium conditions are not verified. Without loss of generality we can suppose that conditions (8)–(10) are not satisfied (the other possible situations can be proved with a similar reasoning). Hence, there exist an index $s \in \{1, \ldots, N\}$ and a set $U \subset [0, \bar{t}]$ with positive measure, such that

$$-\frac{\partial R_s(t, e^{*s}(t))}{\partial e^s}+\frac{\partial D_s(t, \sum_{j=1}^{N} e^{*j}(t)-\gamma_j(t)I^{*j}(t))}{\partial e^s}<\overline{\tau_s}(t)-\overline{\nu_s}(t), \quad \text{on } U. \tag{14}$$

Taking into account constraint (1), variational inequality (13) may be written as

$$0\leq\int_0^T\sum_{i=1}^{N}\Bigg\{\Bigg(-\frac{\partial R_i(t, e^{*i}(t))}{\partial e^i}+\frac{\partial D_i(\sum_{j=1}^{N} e^{*j}(t)-\gamma_j(t)I^{*j}(t))}{\partial e^i}\Bigg)(e^i(t)-e^{*i}(t))$$
$$+\Bigg(\frac{\partial C_i(t, I^{*i}(t))}{\partial I^i}+\frac{\partial D_i(\sum_{j=1}^{N} e^{*j}(t)-\gamma_j(t)I^{*j}(t))}{\partial I^i}\Bigg)\gamma_i(t)(I^i(t)-I^{*i}(t))\Bigg\}\,dt$$
$$=\int_0^T\sum_{i=1}^{N}\Bigg\{\Bigg(-\frac{\partial R_i(t, e^{*i}(t))}{\partial e^i}+\frac{\partial D_i(\sum_{j=1}^{N} e^{*j}(t)-\gamma_j(t)I^{*j}(t))}{\partial e^i}+\overline{\nu_i}(t)-\overline{\tau_i}(t)\Bigg)$$
$$\times(e^i(t)-e^{*i}(t))$$
$$+\Bigg(\frac{\partial C_i(t, I^{*i}(t))}{\partial I^i}+\frac{\partial D_i(\sum_{j=1}^{N} e^{*j}(t)-\gamma_j(t)I^{*j}(t))}{\partial I^i}-\overline{\nu_i}(t)+\overline{\tau_i}(t)\Bigg)$$
$$\times\gamma_i(t)(I^i(t)-I^{*i}(t))\Bigg\}\,dt.$$

Let us choose $I^i(t) = I^{*i}(t)$, for $i = 1, \ldots, N$, $e^i(t) = e^{*i}(t)$ for $i \neq s$ in $[0, \bar{t}]$ and

$$e^s(t)\begin{cases} = e^{*s}(t) & \text{if } t \in [0, \bar{t}] \setminus U, \\ > e^{*s}(t) & \text{if } t \in U. \end{cases}$$

In virtue of the choices of $e(t)$ and $I(t)$ and due to condition (14), the above variational inequality reduces to

$$\int_U \left(-\frac{\partial R_s(t, e^{*s}(t))}{\partial e^s} + \frac{\partial D_s(\sum_{j=1}^N e^{*j}(t) - \gamma_j(t) I^{*j}(t))}{\partial e^s} + \overline{\nu_s}(t) - \overline{\tau_s}(t) \right)$$
$$\times (e^s(t) - e^{*s}(t))dt < 0,$$

which is an absurd assertion.

4 Existence of solutions

The existence of solutions can be ensured under suitable assumptions as in [19], where a comprehensive study on necessary conditions for the solvability of variational inequality problems is presented. We now recall some useful definitions adapted to our case. In order to simplifying notation, we set for $i = 1, \ldots, N$

$$-\nabla W_i(t, e^i, I^i, e^{*-i}, I^{*-i}) = \begin{pmatrix} \frac{\partial R_i(t, e^{*i}(t))}{\partial e^i} - \frac{\partial D_i(t, \sum_{j=1}^N e^{*j}(t) - \gamma_j(t) I^{*j}(t))}{\partial e^i} \\ -\frac{\partial C_i(t, I^{*i}(t))}{\partial I^i} - \frac{\partial D_i(t, \sum_{j=1}^N e^{*j}(t) - \gamma_j(t) I^{*j}(t))}{\partial I^i} \end{pmatrix}^T.$$

Definition 3 $-\nabla W_i(t, e^i, I^i, e^{*-i}, I^{*-i})$ is said to be Fan-hemicontinuous iff for all $(\eta^i, \Gamma^i) \in K_i$ the function

$$(e^i, I^i) \mapsto \langle -\nabla W_i(t, e^i, I^i, e^{*-i}, I^{*-i}), (\eta^i, \Gamma^i) - (e^i, I^i) \rangle_{L^2}$$

is weakly lower semicontinuous on K.

Definition 4 The map $-\nabla W_i(t, e^i, I^i, e^{*-i}, I^{*-i})$ is said to be pseudomonotone in the sense of Karamardian (K-pseudomonotone) iff for all $(e^i, I^i), (\overline{e}^i, \overline{I}^i) \in K_i$

$$\langle -\nabla W_i(t, \overline{e}^i, \overline{I}^i, e^{*-i}, I^{*-i}), (\overline{e}^i, \overline{I}^i) - (e^i, I^i) \rangle_{L^2} \geq 0$$
$$\to \langle -\nabla W_i(t, e^i, I^i, e^{*-i}, I^{*-i}), (\overline{e}^i, \overline{I}^i) - (e^i, I^i) \rangle_{L^2} \geq 0.$$

Definition 5 The map $-\nabla W_i(t, e^i, I^i, e^{*-i}, I^{*-i})$ is said to be lower hemicontinuous along line segments, iff the function:

$$(\eta^i, \Gamma^i) \mapsto \langle -\nabla W_i(t, \eta^i, \Gamma^i, e^{*-i}, I^{*-i}), (e^i, I^i) - (\overline{e}^i, \overline{I}^i) \rangle_{L^2}$$

is lower semicontinuous for all $(e^i, I^i), (\overline{e}^i, \overline{I}^i) \in K_i$ on the line segments $[(e^i, I^i), (\overline{e}^i, \overline{I}^i)]$.

Theorem 3 *Let us assume that for every player i and every $(e^i, I^i) \in K_i$ function $-\nabla W_i(t, e^i, I^i, e^{*-i}, I^{*-i})$ is Fan-hemicontinuous and there exist $(\overline{e}^i, \overline{I}^i) \in K_i$ and $R > \|(\overline{e}^i, \overline{I}^i)\|_{L^2}$ such that*

$$\begin{aligned} &\langle -\nabla W_i(t, e^i, I^i, e^{*-i}, I^{*-i}), (\overline{e}^i, \overline{I}^i) - (e^i, I^i)\rangle_{L^2} \geq 0, \\ &\forall (e^i, I^i) \in K_i \cap \{(e^i, I^i) \in L^2([0, T], \mathbb{R}^2) : \|(e^i, I^i)\|_{L^2} = R\}. \end{aligned} \tag{15}$$

Then the variational inequality problem (12) admits solutions.

Remark 3 As proved in [7], relationship (15) is ensured under condition that there exists $(\widetilde{e}^i, \widetilde{I}^i) \in K_i$

$$\lim_{\|(e^i, I^i)\|_{L^2} \to \infty, (e^i, I^i) \in K_i} \frac{\langle -\nabla W_i(t, e^i, I^i, e^{*-i}, I^{*-i}), (e^i, I^i) - (\widetilde{e}^i, \widetilde{I}^i)\rangle_{L^2}}{\|(e^i, I^i)\|_{L^2}} = +\infty.$$

Other existence results can be given assuming a certain kind of monotonicity. In particular, we have the following outcome

Theorem 4 *Let us assume that for every player i and every $(e^i, I^i) \in K_i$ function $-\nabla W_i(t, e^i, I^i, e^{*-i}, I^{*-i})$ is K-pseudomonotone and lower hemicontinuous along line segments. Let us further suppose that (15) is satisfied. Then the variational inequality problem (12) admits solutions.*

We note that the advantage of using Theorem 4 lies in the fact that the lower hemicontinuity is ensured by assumptions (2)–(5).

Finally, we note that, in virtue of Remark 2, the existence of solutions to problem (13) is also established.

5 A numerical example

In this section, for illustrative purposes, we present a numerical example. In the time interval $[0, \bar{t}] = [0, 1]$, we consider three countries (labeled as 1, 2 and 3, respectively) characterized by the functions:

$$\begin{aligned} R_i(t, e^i(t)) &= -\frac{1}{2}(e^i(t))^2 + 300e^i(t), \\ C_i(t, I^i(t)) &= \frac{150}{2}(I^i(t))^2, \\ D_i(t, e(t), I(t)) &= (e^1(t) - \gamma_1(t)I^1(t) + e^2(t) - \gamma_2(t)I^2(t) \\ &\quad + e^3(t) - \gamma_3(t)I^3(t))^2, i = 1, 2, 3, \end{aligned}$$

and the quantities: $\gamma_1(t) = \frac{1}{2}t$, $\gamma_2(t) = 2t$, $\gamma_3(t) = t$, $E_1(t) = 50t + 1$, $E_2(t) = 30t$ and $E_3(t) = 25t + 2$. After some calculations, it is easy to verify that the pollution control problem with the above data is described by the evolutionary variational inequality:

$$
\begin{aligned}
&\int_0^1 \Big\{\Big(3e^1(t) + 2e^2(t) + 2e^3(t) - 2\gamma_1(t)I^1(t) - 2\gamma_2(t)I^2(t) - 2\gamma_3(t)I^3(t) \\
&\quad -300\Big)\,(e^1(t) - e^{*1}(t)) \\
&\quad + \Big(-2\gamma_1(t)e^1(t) - 2\gamma_1(t)e^2(t) - 2\gamma_1(t)e^3(t) + (150 + 2(\gamma_1(t))^2)I^1(t) \\
&\quad +2\gamma_1(t)\gamma_2(t)I^2(t) + 2\gamma_1(t)\gamma_3(t)I^3(t)\Big)\,\gamma_1(t)(I^1(t) - I^{*1}(t)) \\
&\quad + \Big(2e^1(t) + 3e^2(t) + 2e^3(t) - 2\gamma_1(t)I^1(t) - 2\gamma_2(t)I^2(t) - 2\gamma_3(t)I^3(t) \\
&\quad -300\Big)\,(e^2(t) - e^{*2}(t)) \\
&\quad + \Big(-2\gamma_2(t)e^1(t) - 2\gamma_2(t)e^2(t) - 2\gamma_2(t)e^3(t) + (150 + 2(\gamma_2(t))^2)I^2(t) \\
&\quad + 2\gamma_1(t)\gamma_2(t)I^1(t) + 2\gamma_2(t)\gamma_3(t)I^3(t)\Big)\,\gamma_2(t)(I^2(t) - I^{*2}(t)) \\
&\quad + \Big(2e^1(t) + 2e^2(t) + 3e^3(t) - 2\gamma_1(t)I^1(t) - 2\gamma_2(t)I^2(t) - 2\gamma_3(t)I^3(t) \\
&\quad -300\Big)\,(e^3(t) - e^{*3}(t)) \\
&\quad + \Big(-2\gamma_3(t)e^1(t) - 2\gamma_3(t)e^2(t) - 2\gamma_3(t)e^3(t) + (150 + 2(\gamma_3(t))^2)I^3(t) \\
&\quad + \ 2\gamma_1(t)\gamma_3(t)I^1(t) + 2\gamma_2(t)\gamma_3(t)I^2(t)\Big)\,\gamma_3(t)(I^3(t) - I^{*3}(t))\Big\}\,dt \geq 0, \\
&\quad \forall (e(t), I(t)) \in K_1 \times K_2 \times K_3,
\end{aligned}
$$

where

$$
\begin{aligned}
K_i = \Big\{&(e^i(t), I^i(t)) \in L^2([0,1], \mathbb{R}^2) : e^i(t) \geq 0, \quad I^i(t) \geq 0, \text{ a.e. in } [0,1] \\
&0 \leq e^i(t) - \gamma_i(t)I^i(t) = E_i(t), \text{ a.e. in } [0,1]\Big\}, \quad i = 1, 2, 3.
\end{aligned}
$$

Under regularity assumptions, see [20] and references therein, solutions are continuous, hence we are led to solve a sequence of finite-dimensional variational inequality problems. We omit the statements of the example, implemented as an M-script file of MatLab and solved by applying the extragradient method with constant steplength as in [18] (see also [26] for further discussions on efficient computational procedures), and directly provide numerical results in graphical form.

Figure 1 shows the time evolution of emissions and investments and the different investment policies adopted by the countries. We observe that investments are much higher in country 2 where the efficiency coefficient is higher. This means that country 2 is able to bear a higher investment effort in order to control the emission levels. Moreover, for all the countries both emissions and investments increase until a maximum value is reached. In other words, the investment curve reaches its maximum when emissions reach their higher level. After that time, both emissions and

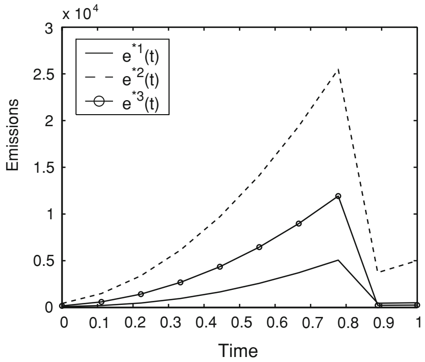

Fig. 1 Curves of equilibria

investments reduce with time along the equilibrium trajectories. The results also confirm the behaviors of emissions and investments at equilibrium (see Definition 2). In fact, the emissions $e^{*1}(t)$, $e^{*2}(t)$, $e^{*3}(t)$ are positive almost everywhere, so that, for all the countries, the marginal damage cost of emitting minus the marginal revenue of emitting equals the shadow price of net emissions minus the marginal abatement cost (see (8) and (10)). In addition, $I^{*2}(t)$ is positive almost everywhere, so that, for country 2, the marginal damage cost of investing plus the marginal cost of investing equals the marginal abatement cost minus the shadow price of net emissions (see (9) and (11)). Finally, since after a certain time investments $I^{*1}(t)$ and $I^{*3}(t)$ vanish, inequality (9) tends to be strictly satisfied for countries 2 and 3. It is worth noting that different scenarios are possible, depending on the values of the model parameters $\gamma_i(t)$, $E_i(t)$, $i = 1, 2, 3$.

6 Conclusions

In this paper we presented an evolutionary variational inequality framework for the study of the environmental pollution control problem, with particular regard to Kyoto Protocol's commitments. We examined multicriteria decision-making problems of countries involved in the Treaty, aiming at maximizing their revenue and minimizing both the investments in environmental projects and the damage from pollution. We showed how the optimal solution of the optimization problem of each country satisfies an evolutionary variational inequality, for which we were able to ensure the existence of solutions. Moreover, we stated the equilibrium conditions that we characterized in terms of the evolutionary variational inequality. To the best of our knowledge for the first time, an evolutionary variational inequality approach was applied to the study of investment strategies in Kyoto's mechanisms.

Future extensions of the work may include the following issues. First, the model could be refined by introducing delay effects in the environmental constraint $e^i(t) - \gamma_i(t)I^i(t) \leq E_i(t)$, so that real reactions to environmental projects could be studied. Second, a memory term which shows how the current equilibrium situation is affected by the equilibrium distributions of the previous observation times (see [20]) could be introduced. Third, the infinite dimensional duality theory (see [10]) could be applied to our problem in order to shed light on intrinsic properties and economic interpretation of the model.

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