Optimization Letters (2008) 2:239–249
DOI 10.1007/s11590-007-0054-5

ORIGINAL PAPER

On global optimizations with polynomials

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Received: 3 May 2006 / Accepted: 7 May 2007 / Published online: 4 July 2007

Abstract We consider the problem of finding the (unconstrained) global minimum of a real valued polynomial $p(x) : R^n \longrightarrow R$. We study the problem of finding the bounds of global minimizers. It is shown that the unconstrained optimization reduces to some constrained optimizations which can be approximated by solving some convex linear matrix inequality (LMI) problems.

Keywords Global optimizations · Polynomials · Positive semi-definite programs

1 Introduction

Given a real-valued polynomial $p(x) : R^n \longrightarrow R$, we are interested in solving the problem

$$P : p^* := \min \{p(x) \mid x \in R^n\}, \tag{1.1}$$

that is, finding the global minimum p^* of $p(x)$ and, if possible, a global minimizer x^*. In the one-dimensional case, that is, when $n = 1$, Shor [2](see also [3,4]) first showed that (1.1) reduces to a convex problem. In [1], J.Lasserre showed that the unconstrained optimization (1.1) can be approximated as closely as desired (and often can be obtained exactly) by solving a finite sequence of convex LMI optimization problems(or positive semi-definite programs). The main theorem in Lasserre [1] says

This research was partly supported by the National Science Foundation of China under grant No. 10671145.

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that, if a $2m$-degree polynomial $p(x)$, with global minimum p^*, has a global minimizer x^* satisfying $\|x^*\| \leq a$ for some $a > 0$, then p^* can be approximated as closely as desired by solving a finite sequence of convex LMI optimization problems. But there is no information on how to obtain such a bounds a. In this paper, we study the problem of finding the bounds a such that there is a global minimizer x^* staying in $\|x\| \leq a$. We will obtain the desired positive bounds a by solving some constrained optimizations. We also show that the global unconstrained minimization (1.1) of a polynomial reduces to some constrained optimizations which can be approximated by some convex linear matrix inequality (LMI) problems.

The paper is organized as follows. In Sect. 2, we briefly introduce the main result in [1] on solving (1.1) by a sequence of convex LMI optimization problems when there is a positive real a such that a global minimizer x^* stays in $\|x\| \leq a$. In Sect. 3, we study the problem of finding the desired bounds a by solving some constrained optimizations. We show that solving (1.1) amounts to solving some constrained auxiliary problems. In Sect. 4, we give some examples. In the last section, we present an approach of finding the bounds of global minimizers of (1.1) by some convex LMI optimization problems.

2 Preliminary results

Let

$$1, x_1, x_2, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_1x_n, x_2x_3, \ldots, x_n^2, \ldots, x_1^m, \ldots, x_n^m \tag{2.1}$$

be a base for the m-degree real-valued polynomial $p(x) : R^n \longrightarrow R$, and let $s(m)$ be its dimension. We adopt the notation in [1]. If $p(x) : R^n \longrightarrow R$ is an m-degree polynomial, write

$$p(x) = \sum_{\alpha} p_\alpha x^\alpha \text{ with } x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \quad \text{and} \sum_i \alpha_i \leq m, \tag{2.2}$$

where $p = \{p_\alpha\} \in R^{s(m)}$ is the coefficient vector of $p(x)$ in the basis (2.1). In [1], the theory of moments(see also Berg [6], Curto and Fialkow [7,8], Jacobi [9], Putinar [10], Putinar and Vasilescu [11], Simon [12], Schmudgen [13]) is applied to the study of solving (1.1). Let $y = \{y_\alpha\}$(with $y_{\{0,\ldots,0\}} = 1$) be the vector of moments up to order m of some probability measure μ_y and $M_m(y)$ be the moment matrix $M_m(y)$ of dimension $s(m)$, with its (i, j) entry being $y_{\beta(i,j)}$. For a real-valued polynomial $q(x)$ of degree w with coefficient vector $q \in R^{s(w)}$, the matrix $M_m(qy)$ is defined by

$$M_m(qy)(i, j) = \sum_\alpha q_\alpha y_{\{\beta(i,j)+\alpha\}}. \tag{2.3}$$

Then Lasserre [1] discussed the situation in which one knows in advance that a global minimizer x^* of $p(x)$ has norm less than a for some $a > 0$, that is, $p(x^*) = p^* = \min P$ and $\|x^*\| \leq a$. With $x \longrightarrow q(x) = a^2 - \|x\|^2$, for every $N \geq m$, Lasserre [1] introduced the following convex LMI problem Q_a^N:

$$\begin{cases} \inf_y \ \sum_\alpha p_\alpha y_\alpha \\ M_N(y) \geq 0 \\ M_{N-1}(qy) \geq 0, \end{cases} \tag{2.4}$$

and proved the following main result (see Theorem 3.4(a) in [1]):

Theorem A *Let $p(x) : R^n \longrightarrow R$ be a $2m$-degree polynomial as in (2.2) with global minimum $p^* = \min P$ and such that $\|x^*\| \leq a$ for some $a > 0$ at some global minimizer x^*. Then as $N \longrightarrow \infty$, the sequence $\{\inf Q_a^N\}$ is monotonously increasing and*

$$\inf Q_a^N \to p^*. \tag{2.5}$$

In the constrained case, we consider the optimization with a real-valued polynomial $p(x) : R^n \to R$, that is,

$$P_K \to p_K^* := \min_{x \in K} p(x), \tag{2.6}$$

where K is a compact set defined by polynomial inequalities $g_i(x) \geq 0$ with $g_i(x) : R^n \to R$ being a real-valued polynomial of degree w_i, $i = 1, 2, \ldots, l$. We assume that K satisfies the following assumption.

Assumption 2.1 The set K is compact and there exists a real-valued polynomial $u(x) : R^n \to R$ such that $\{u(x) \geq 0\}$ is compact, and

$$u(x) = u_0(x) + \sum_{k=1}^{l} g_i(x) u_i(x) \quad \text{for all } x \in R^n, \tag{2.7}$$

where the polynomials $u_i(x)$ are all sums of squares, $i = 1, 2, \ldots, l$.

Let $\tilde{w}_i := [w_i/2]$ be the smallest integer larger than $w_i/2$, and with $N \geq [m/2]$ and $N \geq \max_i \tilde{w}_i$, Lasserre [1] introduced the following convex LMI problem Q_K^N:

$$\begin{cases} \inf_y \ \sum_\alpha p_\alpha y_\alpha \\ M_N(y) \geq 0 \\ M_{N-\tilde{w}_i}(g_i y) \geq 0, \quad i = 1, 2, \ldots, i. \end{cases} \tag{2.8}$$

and proved the following result (see Theorem 4.2(a) in [1]):

Theorem B *Let $p(x) : R^n \longrightarrow R$ be an m-degree polynomial and K be compact set $\{g_i \geq 0, \ i = 1, 2, \ldots, l\}$. Let Assumption 2.1 hold, and let $p_K^* := \min_{x \in K} p(x)$. Then as $N \longrightarrow \infty$, the sequence $\{\inf Q_K^N\}$ is monotonously increasing and*

$$\inf Q_K^N \to p_K^*. \tag{2.9}$$

3 The estimation of the bounds of global minimizers of (1.1)

Let $p(x)$ be a m-degree polynomial on R^n. We have the following expression by Taylor series

$$\begin{cases} p(x) - p(0) = \sum_{k=1}^{n} \frac{\partial p(0)}{\partial x_k} x_k + \frac{1}{2!} \sum_{\{1\le k_1,k_2\le n\}} \frac{\partial^2 p(0)}{\partial x_{k_1} \partial x_{k_2}} x_{k_1} x_{k_2} + \cdots \\ \quad + \frac{1}{l!} \sum_{\{1\le k_1,k_2,\ldots,k_l\le n\}} \frac{\partial^l p(0)}{\partial x_{k_1} \partial x_{k_2} \cdots \partial x_{k_l}} x_{k_1} x_{k_2} \cdots x_{k_l} + \cdots \\ \quad + \frac{1}{m!} \sum_{\{1\le k_1,k_2,\ldots,k_m\le n\}} \frac{\partial^m p(0)}{\partial x_{k_1} \partial x_{k_2} \cdots \partial x_{k_l}} x_{k_1} x_{k_2} \cdots x_{k_m}. \end{cases} \tag{3.1}$$

where in a sum when $k_\mu \neq k_\nu$ the term is counted two times.

For every positive r on $\{x \in R^n \mid \|x\| = r\}$, with a change of variables $s_k = \frac{x_k}{r}$, $k = 1, 2, \ldots, n$, by (3.1), we have

$$\begin{cases} p(x) - p(0) = r \sum_{k=1}^{n} \frac{\partial p(0)}{\partial x_k} s_k + \frac{r^2}{2!} \sum_{\{1\le k_1,k_2\le n\}} \frac{\partial^2 p(0)}{\partial x_{k_1} \partial x_{k_2}} s_{k_1} s_{k_2} + \cdots \\ \quad + \frac{r^l}{l!} \sum_{\{1\le k_1,k_2,\ldots,k_l\le n\}} \frac{\partial^l p(0)}{\partial x_{k_1} \partial x_{k_2} \cdots \partial x_{k_l}} s_{k_1} s_{k_2} \cdots s_{k_l} + \cdots \\ \quad + \frac{r^m}{m!} \sum_{\{1\le k_1,k_2,\ldots,k_m\le n\}} \frac{\partial^m p(0)}{\partial x_{k_1} \partial x_{k_2} \cdots \partial x_{k_l}} s_{k_1} s_{k_2} \cdots s_{k_m}. \end{cases} \tag{3.2}$$

Then for each positive integer $l : 1 \le l \le m$, we pose a constrained optimization as follows

$$\begin{aligned} \min \quad & J_l(s) = \sum_{\{1\le k_1,k_2,\ldots,k_l\le n\}} \frac{\partial^l p(0)}{\partial x_{k_1} \partial x_{k_2} \cdots \partial x_{k_l}} s_{k_1} s_{k_2} \cdots s_{k_l} \\ s.t. \quad & \|s\|^2 = s_1^2 + s_2^2 + \cdots + s_n^2 = 1. \end{aligned} \tag{3.3}$$

For each positive integer $l : 1 \le l \le m$, let I_l denote the minimum of (3.4).

First of all, if there is a positive integer $j : 1 \le j \le m$ such that $I_i \ge 0$ when $j \le i \le m$, we conclude from (3.2) that, for every x on $\{\|x\| = r\}$,

$$\begin{cases} p(x) - p(0) \ge r \sum_{k=1}^{n} \frac{\partial p(0)}{\partial x_k} s_k + \frac{r^2}{2!} \sum_{\{1\le k_1,k_2\le n\}} \frac{\partial^2 p(0)}{\partial x_{k_1} \partial x_{k_2}} s_{k_1} s_{k_2} + \cdots \\ \quad + \frac{r^{j-1}}{(j-1)!} \sum_{\{1\le k_1,k_2,\ldots,k_{j-1}\le n\}} \frac{\partial^{j-1} p(0)}{\partial x_{k_1} \partial x_{k_2} \cdots \partial x_{k_{j-1}}} s_{k_1} s_{k_2} \cdots s_{k_{j-1}}. \end{cases} \tag{3.4}$$

From (3.4), we immediately see that when $I_1 = I_2 = \cdots = I_m = 0$, the optimum of (1.1) $p^* = p(0)$. On the other hand, if there is a positive integer $j : 1 \le j \le m$ such that $I_i = 0$ when $j + 1 \le i \le m$ and $I_j > 0$, then by (3.4), we have, for $r \ge 1$ and every x on $\{\|x\| = r\}$,

$$\begin{cases} p(x) - p(0) \ge r \sum_{k=1}^{n} \frac{\partial p(0)}{\partial x_k} s_k + \frac{r^2}{2!} \sum_{\{1\le k_1,k_2\le n\}} \frac{\partial^2 p(0)}{\partial x_{k_1} \partial x_{k_2}} s_{k_1} s_{k_2} + \cdots \\ \qquad + \frac{r^j}{j!} \sum_{\{1\le k_1,k_2,\ldots,k_j\le n\}} \frac{\partial^j p(0)}{\partial x_{k_1} \partial x_{k_2} \cdots \partial x_{k_j}} s_{k_1} s_{k_2} \cdots s_{k_j} \\ \qquad \ge I_j \frac{r^j}{j!} - (n + \frac{n^2}{2!} + \cdots + \frac{n^{j-1}}{(j-1)!}) M_{j-1} r^{j-1}, \end{cases} \tag{3.5}$$

where

$$M_{j-1} = \max\left\{|\frac{\partial^l p(0)}{\partial x_{k_1}\partial x_{k_2}\cdots\partial x_{k_l}}|:\ 0\le l\le j-1;\, 1\le k_1,\ldots,k_l\le n\right\}, \quad (3.6)$$

noting that $M_0 = |p(0)|$. We define, for $r > 1$,

$$\Phi(r) = I_j\frac{r^j}{j!} - \left(n + \frac{n^2}{2!} + \cdots + \frac{n^{j-1}}{(j-1)!}\right)M_{j-1}r^{j-1}. \quad (3.7)$$

If $j = 1$, we have

$$\Phi(r) = I_1 r - |p(0)|. \quad (3.8)$$

Writing $\sigma_1 = \max\{1, \frac{|p(0)|}{I_1}\}$, by (3.5), when $\|x\| > \sigma_1$, we have

$$p(x) \ge p(0). \quad (3.9)$$

It follows that, in this case, i.e., $j = 1$, the global minimizers of (1.1) should stay in $\|x\| \le \sigma_1$. If $j \ge 2$, we have

$$\frac{d\Phi(r)}{dr} = r^{j-2}\left[I_j\frac{jr}{j!} - (n + \frac{n^2}{2!} + \cdots + \frac{n^{j-1}}{(j-1)!})(j-1)M_{j-1}\right]. \quad (3.10)$$

When

$$r > \sigma_j := \max\left\{1, \frac{j!(n+\frac{n^2}{2!}+\cdots+\frac{n^{j-1}}{(j-1)!})(j-1)M_{j-1}}{jI_j}, \frac{j!(n+\frac{n^2}{2!}+\cdots+\frac{n^{j-1}}{(j-1)!})M_{j-1}}{I_j}\right\}$$
$$= \max\left\{1, \frac{j!(n+\frac{n^2}{2!}+\cdots+\frac{n^{j-1}}{(j-1)!})M_{j-1}}{I_j}\right\}, \quad (3.11)$$

we have

$$\frac{d\Phi(r)}{dr} > 0, \quad \Phi(r) > 0. \quad (3.12)$$

Therefore, when $r > \sigma_j$, $\Phi(r)$ monotonously increases. By (3.5), (3.12), we have, when $r > \sigma_j$,

$$p(x) - p(0) \ge \Phi(r) > 0. \quad (3.13)$$

It follows that, in this case, i.e., $j \ge 2$, the minimizers of (1.1) stays in $\|x\| \le \sigma_j$.

As a conclusion, we have reached the following results.

Theorem 3.1 *If there is a positive integer $j : 1 \leq j \leq m$ such that $I_i = 0$ when $j+1 \leq i \leq m$ and $I_j > 0$, then all of global minimizers of* (1.1) *stay in $\|x\| \leq \sigma_j$, where* $\sigma_j := \max\left\{1, \frac{j!(n+\frac{n^2}{2!}+\cdots+\frac{n^{j-1}}{(j-1)!})M_{j-1}}{I_j}\right\}$

Corollary 3.2 *If $I_m > 0$, then there is a positive real a such that all of global minimizers of* (1.1) *stay in $\|x\| \leq a = \sigma_m$.*

It is very complicated in the case that a positive integer $j : 1 \leq j \leq m$ exists such that $I_i = 0$ when $j+1 \leq i \leq m$ and $I_j < 0$. We give several examples in this case as follows:

$$\begin{aligned} p_1(x) &= x_1^2 - 2x_1x_2 + x_2^2 - x_1, \\ p_2(x) &= x_1^4 - 2x_1^2x_2^2 + x_2^4 + x_1x_2, \\ p_3(x) &= x_1^2 - 2x_1x_2 + x_2^2 + 2(x_1 - x_2), \\ p_4(x) &= x_1^2x_2^2(x_1^2 + x_2^2 - 1). \end{aligned} \tag{3.14}$$

It is easy to see that there are global minimizers for $p_3(x)$, $p_4(x)$ and there are no global minimizers for $p_1(x)$, $p_2(x)$. But we have following results to deal with these examples.

Theorem 3.3 *If, for a $(s_1^*, s_2^*, \ldots, s_n^*)^{\mathrm{T}}$ with $(s_1^*)^2 + (s_2^*)^2 + \cdots + (s_n^*)^2 = 1$, there is a $j : 1 \leq j \leq m$ such that $J_{j+1}(s^*) = \cdots = J_m(s^*) = 0$ and $J_j(s^*) < 0$, then there is no global minimizer for $p(x)$.*

Proof By (3.2), the assumption of this theorem implies

$$p(rs^*) - p(0) = rJ_1(s^*) + \frac{r^2}{2!}J_2(s^*) + \cdots + \frac{r^j}{j!}J_j(s^*). \tag{3.15}$$

Consequently, $\lim_{r\to\infty} p(rs^*) = -\infty$. Hence there is no global minimizer for the $p(x)$.

Corollary 3.4 *If $I_m < 0$, then there is no global minimizer for $p(x)$.*

Proof Clearly there is a $(s_1^*, s_2^*, \ldots, s_n^*)^{\mathrm{T}}$ with $(s_1^*)^2 + (s_2^*)^2 + \cdots + (s_n^*)^2 = 1$ such that $J_m(s^*) = \min\{J_m(s) : \|s\| = 1\} = I_m$. Since $J_m(s^*) = I_m < 0$, we see that there is no global minimizer for the $p(x)$ by Theorem 3.3.

Remark 3.1 We see that $p_1(x)$, $p_2(x)$ in (3.15) meet the conditions of Theorem 3.3, therefore there are no global minimizers for $p_1(x)$, $p_2(x)$.

Example 3.1 Let's have a look at the polynomial

$$p(x) = x_1^4 x_2^2 + x_1^2 x_2^4 + x_1^2 x_2 + x_1 x_2 + x_2. \tag{3.16}$$

Concerning $p(x)$, it's easy to see that, for $s^* = (1, 0)^{\mathrm{T}}$, $J_6(s^*) = \cdots = J_1(s^*) = 0$ and for $\hat{s} = (0, 1)^{\mathrm{T}}$, $J_6(\hat{s}) = \cdots = J_2(\hat{s}) = 0$, $J_1(\hat{s}) > 0$. But for $\bar{s} = (0, -1)^{\mathrm{T}}$, we have $J_6(\bar{s}) = \cdots = J_2(\bar{s}) = 0$, $J_1(\bar{s}) < 0$. By Theorem 3.3, we know that there is no global minimizer for $p(x)$.

Clearly, if there is a global minimizer which stays in $\|x\| \le a$ for some $a > 0$, then (1.1) is equivalent to following constrained optimization

$$\begin{aligned} &\min p(x) \\ &s.t.\ \ \|x\| \le a. \end{aligned} \tag{3.17}$$

Then we have following results.

Theorem 3.5 *If there is a positive integer $j : 1 \le j \le m$ such that $I_i = 0$ when $j+1 \le i \le m$ and $I_j > 0$, then the global optimization* (1.1) *is equivalent to following constrained optimization*

$$\begin{aligned} &\min p(x) \\ &s.t.\ \ \|x\| \le \sigma_j \end{aligned} \tag{3.18}$$

where $\sigma_j := \max\left\{1, \frac{j!(n+\frac{n^2}{2!}+\cdots+\frac{n^{j-1}}{(j-1)!})M_{j-1}}{I_j}\right\}.$

Theorem 3.6 *Let $p(x) : R^n \longrightarrow R$ be a m-degree polynomial as in* (2.2)*. If there is a positive integer $j : 1 \le j \le m$ such that $I_i = 0$ when $j + 1 \le i \le m$ and $I_j > 0$, then as $N \longrightarrow \infty$, the sequence $\{\inf Q_{\sigma_j}^N\}$ is monotonously increasing and*

$$\inf Q_{\sigma_j}^N \to p^* := \min p(x), \quad x \in R^n. \tag{3.19}$$

where $Q_{\sigma_j}^N$ is defined as in (2.4) *when $a = \sigma_j$*

Proof By Theorem 3.5, we see that the global minimizers of $p(x)$ stay in $\|x\| \le \sigma_j$. On the other hand, the condition "there is a positive integer $j : 1 \le j \le m$ such that $I_i = 0$ when $j + 1 \le i \le m$ and $I_j > 0$" implies that m is even. Then (3.19) follows from the conclusion of Theorem A (in Sect. 2).

In next section, we will use Theorems 3.5 and 3.6 to deal with Example 1, 2 in [1] (see pp. 803–804, [1]).

Now we present the following algorithm as a conclusion.

Algorithm 3.7. Given a polynomial $p(x)$.

Step 1: Input all coefficients of $p(x)$.

Step 2: Solving (3.4) for I_j, $1 \le j \le m$. If $I_j = 0$, $1 \le j \le m$, then $p^* = p(0)$. If there exists a $j : 1 \le j \le m$ such that $I_i = 0$ when $j+1 \le i \le m$ and $I_j > 0$, go to step 3. If $I_m < 0$, go to step 4. If, for a $(s_1^*, s_2^*, \ldots, s_n^*)^T$ with $(s_1^*)^2 + (s_2^*)^2 + \cdots + (s_n^*)^2 = 1$, there is a $j : 1 \le j \le m$ such that $J_{j+1}(s^*) = \cdots = J_m(s^*) = 0$ and $J_j(s^*) < 0$, go to step 4.
Step 3: Compute M_{j-1} as in (3.6) and σ_j as in Theorem 3.1. Solving (3.18).
Step 4: $\inf p(x) = -\infty$.

Although not all cases are covered in Algorithm 3.7, these cases in Algorithm 3.7 should cover many problems of interest. For example, we can use Theorem 3.1 and Corollary 3.2 to deal with the polynomial having the form:

$$p(x) = x_1^{2m} + \cdots + x_n^{2m} + g(x_1, x_2, \ldots, x_n), \tag{3.20}$$

where $g(x_1, x_2, \ldots, x_n)$ is a real-valued polynomial of degree less than or equal to $2m - 1$. In next section, we demonstrate this issue by some examples.

4 Some examples

Let $p(x) : R^n \to R$ be a $2m$-degree polynomial as (2.2) with global minimum $p^* = \min p(x)$, $x \in R^n$. In [1], after introducing the following convex LMI optimization problem (or positive semi-definite program) Q:

$$\begin{cases} \inf_y \ \sum_\alpha p_\alpha y_\alpha \\ M_m(y) \ge 0 \end{cases}$$

Lasserre showed that (i): if $p(x) - p^*$ is a sum of squares of polynomials then $p^* := \min p(x) = \min Q$; (ii): if the dual problem Q^* of Q has a feasible solution and $\min P = \min Q$, then $p(x) - p^*$ is a sum of squares of polynomials (see also Nesterov [5]). In other words, to check if $p(x) - p^*$ is a sum of squares of polynomials, one has to obtain p^* first. But it is usually hard to obtain p^*. Let's take a look at two examples given in [1] (see pp. 803–804, [1]). We present these two polynomials $p_1(x), p_2(x) : R^2 \to R$ as follows.

$$\begin{aligned} p_1(x) &= (x_1^2+1)^2 + (x_2^2+1)^2 + (x_1+x_2+1)^2 \\ p_2(x) &= (x_1^2+1)^2 + (x_2^2+1)^2 - 2(x_1+x_2+1)^2 \end{aligned} \tag{4.1}$$

Under previously knowing $p_1^* = \min p_1(x)$ and $p_2^* = \min p_2(x)$ respectively , it is proved in [1] that $p_1(x) - p_1^*$ can not be written as a sum of squares of polynomials and $p_2(x) - p_2^*$ can be written as a sum of squares of polynomials. Now we will show that one can use Theorems 3.5 and 3.6 in last section to obtain p_1^* and p_2^*.

Example 4.1 To obtain p_1^*. Since $p_1(x) = x_1^4 + x_2^4 + 3x_1^2 + 3x_2^2 + 2x_1x_2 + 2x_1 + 2x_2 + 3$, we see that for $p_1(x)$

$$I_4 = 4! \min \left\{ s_1^4 + s_2^4 : \ \ s_1^2 + s_2^2 = 1 \right\} = 12, \tag{4.2}$$

$$M_3 = \max \{6, 2, 2, 3\} = 6, \tag{4.3}$$

and

$$\sigma_4 = \frac{4!(2 + \frac{2^2}{2} + \frac{2^3}{6})6}{12} = 64. \tag{4.4}$$

By Corollary 3.2, Theorem 3.5, the global optimization (1.1) for $p_1(x)$ is equivalent to

$$\begin{aligned} &\min \ (x_1^2 + 1)^2 + (x_2^2 + 1)^2 + (x_1 + x_2 + 1)^2 \\ &s.t. \ x_1^2 + x_2^2 \le (64)^2. \end{aligned} \tag{4.5}$$

By Theorem 3.6, with $a = 64$ and $K_a = \{\|x\| \le 64\}$, solving Q_a^4, we obtain $p_1^* = 2.355$.

Example 4.2 To obtain p_2^*. Since $p_2(x) = x_1^4 + x_2^4 - 4x_1x_2 - 4x_1 - 4x_2$, we see that for $p_2(x)$

$$I_4 = 4! \min \left\{ s_1^4 + s_2^4 : \ \ s_1^2 + s_2^2 = 1 \right\} = 12, \tag{4.6}$$

$$M_3 = \max\{4, 0\} = 4, \tag{4.7}$$

and

$$\sigma_4 = \frac{4! \left(2 + \frac{2^2}{2} + \frac{2^3}{6}\right) 4}{12} = \frac{128}{3}. \tag{4.8}$$

By Corollary 3.2, Theorem 3.5, the global optimization (1.1) for $p_1(x)$ is equivalent to

$$\begin{aligned} &\min \ (x_1^2 + 1)^2 + (x_2^2 + 1)^2 - 2(x_1 + x_2 + 1)^2 \\ &s.t. \ x_1^2 + x_2^2 \le \left(\frac{128}{3}\right)^2. \end{aligned} \tag{4.9}$$

By Theorem 3.6, with $a = \frac{128}{3}$ and $K_a = \left\{\|x\| \le \frac{128}{3}\right\}$, solving Q_a^5, we obtain $p_2^* = -11.4581$.

Remark 4.3 In using Theorem 3.1, if $I_j > 0$, we need to know whether $I_k = 0$, for all $k > j$. But this can be done unless one may compute the exact values of I_k for $k > j$. The approximation by the methodology in [1] will not work for this process. Therefore, the only case for which the methodology developed in section 3 is guaranteed to work, is when $I_m > 0$, in which the approximation procedure in [1] would provide an approximation $\hat{I}_m > 0$, sufficient for computing a valid σ_m. We present this procedure in next section.

5 Solving (1.1) in the case $I_m > 0$

In the case $I_m > 0$, we introduce a way here to deal with the unconstrained optimization (1.1) of a polynomial, by solving (3.4) with Theorem B first and then solving (3.18) with Theorem A.

5.1 Approximating I_m by Theorem B

By means of Theorem B, we are going to solve

$$\min \quad J_m(s) = \sum_{\{1 \le k_1, k_2, \ldots, k_m \le n\}} \frac{\partial^l p(0)}{\partial x_{k_1} \partial x_{k_2} \cdots \partial x_{k_l}} s_{k_1} s_{k_2} \cdots s_{k_m}$$
$$s.t. \ \|s\|^2 = s_1^2 + s_2^2 + \cdots + s_n^2 = 1.$$

Define

$$\begin{aligned} g_1(s) &= s_1^2 + \cdots + s_n^2 - 1, \\ g_2(s) &= 1 - (s_1^2 + \cdots + s_n^2), \end{aligned} \tag{5.1}$$

and

$$K = \{s : g_i(s) \ge 0, i = 1, 2\}. \tag{5.2}$$

Clearly, K is compact. Take $u_0(s) \equiv 0$, $u_1(s) \equiv 1$, $u_2(s) \equiv 2$ and define

$$\begin{aligned} u(s) &:= u_0(s) + g_1(s)u_1(s) + g_2(s)u_2(s) \\ &= 1 - (s_1^2 + \cdots + s_n^2). \end{aligned} \tag{5.3}$$

It's easy to see from (5.3) that $\{s : u(s) \ge 0\}$ is compact. Therefore, Assumption 2.1 holds for the following constrained optimization $J_K^{(m)}$ which is equivalent to (3.4),

$$\min_K J_m(s). \tag{5.4}$$

We replace $p(x)$ in Theorem B by $J_m(s)$. Then by Theorem B , I_m can be approximated by a sequence of convex LMI problems Q_K^N:

$$\begin{cases} \inf_y \ \sum_\alpha J_\alpha^{(m)} y_\alpha \\ M_N(y) \ge 0 \\ M_{N-1}(g_i y) \ge 0, \quad i = 1, 2, \end{cases} \tag{5.5}$$

where $\{J_\alpha^{(m)}\}$ with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\sum_i \alpha_i \leq m$, is got by rewriting

$$J_m(s) = \sum_\alpha J_\alpha^{(m)} s^\alpha, \tag{5.6}$$

with $s^\alpha := s_1^{\alpha_1} \cdots s_n^{\alpha_n}$ and $\sum_i \alpha_i \leq m$.

5.2 Solving (3.18) by Theorem A

Since $I_m > 0$, by Theorem 3.5 we see that solving (1.1) is equivalent to solving (3.18). Then we use Corollary 3.2 to get

$$\sigma_m = \max\left\{1, \frac{m!(n + \frac{n^2}{2!} + \cdots + \frac{n^{m-1}}{(m-1)!})M_{m-1}}{I_m}\right\}.$$

By Theorem 3.6, we can use Theorem A to approximate (3.18) with $a = \sigma_m$. We approximate (3.18) by a sequence of convex LMI problems Q_a^N:

$$\begin{cases} \inf_y \ \sum_\alpha p_\alpha y_\alpha \\ M_N(y) \geq 0 \\ M_{N-1}(qy) \geq 0, \end{cases} \tag{5.7}$$

where $a = \sigma_m$ and $q(x) = a^2 - \|x\|^2$.

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