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ORIGINAL PAPER

A unified network performance measure with importance identification and the ranking of network components

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Abstract In this paper, we propose the first network performance measure that can be used to assess the efficiency of a network in the case of either fixed or elastic demands. Such a measure is needed for many different applications since only when the performance of a network can be quantifiably measured can the network be appropriately managed. Moreover, as we demonstrate, the proposed performance measure, which captures flow information and behavior, allows one to determine the criticality of various nodes (as well as links) through the identification of their importance and ranking. We present specific networks for which the performance/efficiency is computed along with the importance rankings of the nodes and links. The new measure can be applied to transportation networks, supply chains, financial networks, electric power generation and distribution networks as well as to the Internet and can be used to assess the vulnerability of a network to disruptions.

Keywords Network performance · Network efficiency measure · Network vulnerability · Network component importance ranking · Network equilibrium problems

1 Introduction

Recently, the study of networks, and especially complex networks, has drawn a great deal of interest among researchers from different disciplines; see, for example, Barabási and Albert [1], Newman [23], Boginski et al. [3], and O'Kelly et al. [24].

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Three types of networks, in particular, have received intense attention, especially in regards to the study of reliability and we note the Erdös-Rényi [10] random network model, the Watts-Strogatz [28] small-world model, and the Barabási-Albert [1] scale-free networks.

The importance of studying and identifying the vulnerable components of a network has been linked to events such as 9/11 and to Hurricane Katrina, as well as the biggest blackout in North America that occurred on August 14, 2003. In order to hedge against terrorism and natural disasters [25], a majority of the associated complex network (sometimes also referred to as *network science*) literature focuses on the graph characteristics (e.g. connectivity between nodes) of the associated application in order to evaluate the network vulnerability; see, for example, Chassin and Posse [5].

However, in order to be able to evaluate the vulnerability and the reliability of a network, a measure that can quantifiably capture the efficiency/performance of a network must be developed. For example, in a series of papers, Latora and Marchiori [12–14] discussed the network performance issue by measuring the "global efficiency" in a weighted network as compared to that of the simple non-weighted small-world network. In a weighted network, the network is not only characterized by the edges that connect different nodes, but also by the weights associated with different edges in order to capture the relationships between different nodes. The network efficiency $E(G)$ of a network G is defined by Latora and Marchiori [12–14] as $E = E(G) = \frac{1}{n(n-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}}$, where n is the number of nodes in G and d_{ij} is the shortest path length (the geodesic distance) between nodes i and j. For simplicity, in this paper, we refer to the above Latora and Marchiori measure as the L–M measure.

We believe that the flow on a network is an additional important indicator of network performance as well as network vulnerability. Indeed, flows represent the usage of a network and which paths and links have positive flows and the magnitude of these flows are relevant in the case of network disruptions. However, to the best of our knowledge, there are very few papers to-date that consider network flows in assessing network performance. The results in Zhu et al. [30] are notable since they demonstrate empirically through an application to the airline network of China how a measure with flows and costs outperforms existing measures in yielding more realistic results in terms of, for example, which cities are critical and their rankings in the network. Nevertheless, as we demonstrate in Sect. 3, their measure is only applicable to networks with fixed demands. It is well-known that in many network applications, consumers may be sensitive to prices/costs and, therefore, the demand will no longer be fixed, but will, rather, be elastic, that is, price-dependent. Therefore, a *unified network performance measure* that is consistent across fixed demand as well as elastic demand networks is needed. Moreover, in the case of a disaster, users of the network may be sensitive to the increased associated costs of using the network and the demand may, as a consequence, change.

We note that, recently, Jenelius et al. [11] proposed several link importance indicators and applied them to the road transportation network in northern Sweden. Their indicators, however, are distinct, depending upon whether or not the network becomes disconnected or not. Murray–Tuite and Mahmassani [16] also focused on identifying indices for the determination of vulnerable links in transportation networks but our measure is unified and can be applied to assess the importance of either links or nodes or both and is applicable to both fixed demand and to elastic demand network equilibrium problems.

In this paper, we propose a network performance measure that can be used to evaluate the efficiency of different networks in the case of either fixed or elastic demands. The formal network performance/efficiency measure is presented in the context of network equilibrium, which captures prices and costs and the underlying behavior of "users" of the network.

The paper is organized as follows. In Sect. 2, an elastic demand network model and a fixed demand network model, which is a special case of the former, are recalled. In Sect. 3, the network performance measure is introduced, and its relationships to several existing measures identified. In Sect. 4, the new measure is applied to three network examples. The paper concludes with a summary and future research directions in Sect. 5.

2 The network equilibrium models with elastic and fixed demands

In this Section, we recall the network equilibrium model with elastic demands with given inverse demand or disutility functions (see [7]). We then provide a special case in which the demands are assumed fixed and known. These models were originally proposed in the context of transportation but, given their wide applicability, the presentation below is for any network equilibrium problem. Indeed, Nagurney [19], Liu and Nagurney [15], and Wu et al. [29] have shown, respectively, that supply chain networks, financial networks, and electric power generation and distribution networks can be reformulated and solved as transportation network problems over appropriately constructed abstract networks or supernetworks [20]. Moreover, it has been realized (cf. [21] and the references therein) that the Internet also exhibits behavior similar to that of transportation network equilibrium problems, including the occurrence of the Braess [4] paradox.

2.1 Network equilibrium model with elastic demands

We consider a network $\mathcal{G}$ with the set of directed links L with K elements, the set of origin/destination (O/D) pairs W with n_W elements, and the set of acyclic paths joining the O/D pairs by P with n_P elements.

We denote the set of paths joining O/D pair w by P_w. Links are denoted by a, b, etc; paths by p, q, etc., and O/D pairs by w_1, w_2, etc.

We denote the nonnegative flow on path p by x_p and the flow on link a by f_a and we group the path flows into the vector $x \in R_+^{n_P}$ and the link flows into the vector $f \in R_+^K$. The link flows are related to the path flows through the following conservation of flow equations:

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in L, \tag{1}$$

where $\delta_{ap} = 1$ if link a is contained in path p, and $\delta_{ap} = 0$, otherwise. Hence, the flow on a link is equal to the sum of the flows on paths that contain that link.

The user cost on a path p is denoted by C_p and the user cost on a link a by c_a. We denote the demand associated with using O/D pair w by d_w and the disutility by λ_w.

The user costs on paths are related to user costs on links through the following equations:

$$C_p = \sum_{a \in L} c_a \delta_{ap}, \quad \forall p \in P, \tag{2}$$

that is, the user cost on a path is equal to the sum of user costs on links that make up the path.

For the sake of generality, we allow the user link cost function on each link to depend upon the entire vector of link flows, so that

$$c_a = c_a(f), \quad \forall a \in L. \tag{3}$$

We also assume that the link cost functions are continuous.

The following conservation of flow equations must also hold:

$$\sum_{p \in P_w} x_p = d_w, \quad \forall w \in W, \tag{4}$$

which means that the sum of path flows on paths connecting each O/D pair must be equal to the demand for that O/D pair.

Also, we assume, as given, the disutility (that is, the inverse demand) functions for the O/D pairs, which are assumed to be continuous, such that

$$\lambda_w = \lambda_w(d), \quad \forall w \in W, \tag{5}$$

where d is the vector of demands.

Definition 1 Network equilibrium–elastic demands

A path flow and demand pattern $(x^*, d^*) \in \mathcal{K}^1$, where $\mathcal{K}^1 \equiv \{(x, d) | (x, d) \in R_+^{n_P + n_W}$ and (4) holds$\}$, is said to be a network equilibrium, in the case of elastic demands, if, once established, no user has any incentive to alter his "travel" decisions. The state is expressed by the following condition which must hold for each O/D pair $w \in W$ and every path $p \in P_w$:

$$C_p(x^*) \begin{cases} = \lambda_w(d^*), & \text{if} \quad x_p^* > 0, \\ \geq \lambda_w(d^*), & \text{if} \quad x_p^* = 0. \end{cases} \tag{6}$$

Condition (6) states that all utilized paths connecting an O/D pair have equal and minimal user costs and these costs are equal to the disutility associated with using that O/D pair. As established in Dafermos [7], the network equilibrium condition (6) is equivalent to the following variational inequality problem.

Theorem 1 *A path flow and demand pattern* $(x^*, d^*) \in \mathcal{K}^1$ *is an equilibrium according to Definition 1 if and only if it satisfies the variational inequality: determine* $(x^*, d^*) \in \mathcal{K}^1$ *such that*

$$\sum_{w \in W} \sum_{p \in P_w} C_p(x^*) \times \left[x_p - x_p^*\right] - \sum_{w \in W} \lambda_w(d^*) \times \left[d_w - d_w^*\right] \geq 0, \quad \forall (x, d) \in \mathcal{K}^1. \tag{7}$$

2.2 Network equilibrium model with fixed demands

Assume now that the demands are fixed and known. We then have that Definition 1 simplifies to:

Definition 2 Network equilibrium–fixed demands

A path flow pattern $x^* \in \mathcal{K}^2$, where $\mathcal{K}^2 \equiv \{x | x \in R_+^{n_P}$ and (4) holds with d_w known and fixed for each $w \in W\}$, is said to be a network equilibrium, in the case of fixed demands, if the following condition holds for each O/D pair $w \in W$ and each path $p \in P_w$:

$$C_p(x^*) \begin{cases} = \lambda_w, & \text{if} \quad x_p^* > 0, \\ \geq \lambda_w, & \text{if} \quad x_p^* = 0. \end{cases} \tag{8}$$

The interpretation of condition (8) is that all used paths connecting an O/D pair have equal and minimal costs (see also [2,27]). As proved in [6,26], the fixed demand network equilibrium condition (8) is equivalent to the following variational inequality problem.

Theorem 2 *A path flow pattern* $x^* \in \mathcal{K}^2$ *is a network equilibrium according to Definition 2 if and only if it satisfies the variational inequality problem: determine* $x^* \in \mathcal{K}^2$ *such that*

$$\sum_{w \in W} \sum_{p \in P_w} C_p(x^*) \times \left[x_p - x_p^*\right] \geq 0, \quad \forall x \in \mathcal{K}^2. \tag{9}$$

Clearly, (9) can be obtained directly from (7) by noting that $d_w^* = d_w$, with the d_w's being fixed and known a priori for all $w \in W$.

Existence of a solution to variational inequality (9) is guaranteed from the standard theory of variational inequalities (see e.g. [18]) under the assumption that the link cost functions and, hence, the path cost functions are continuous since the feasible set $\mathcal{K}^2$ is compact. Uniqueness of an equilibrium link flow pattern, in turn, is then guaranteed under the assumption that the user link cost functions are strictly monotone. In the case of variational inequality (7) stronger conditions need to be imposed to obtain existence of a solution. We note that, in particular, strong monotonicity of the link cost functions and minus the disutility functions will guarantee uniqueness of the corresponding equilibrium link flow and demand pattern (see also [18]). Algorithms for the solution of variational inequalities (7) and (9) can be found in [18,22], and the references therein.

Thus, an appropriate and unified network performance/efficiency measure should be as appropriate for the case of elastic demands as it is for fixed demands.

3 A unified network performance measure

Before we introduce a unified network performance measure we first state an important property that such a measure should have.

Network performance property
The performance/efficiency measure for a given network should be nonincreasing with respect to the equilibrium disutility for each O/D pair, holding the equilibrium disutilities for the other O/D pairs constant.

Given this desirable property of a network performance measure, we propose a new, unified network performance measure as follows:

Definition 3 A unified network performance measure
The network performance/efficiency measure, $\mathcal{E}(G, d)$, for a given network topology G and the equilibrium (or fixed) demand vector d, is defined as follows:

$$\mathcal{E} = \mathcal{E}(G, d) = \frac{\sum_{w \in W} \frac{d_w}{\lambda_w}}{n_W}, \tag{10}$$

where recall that n_W is the number of O/D pairs in the network, and d_w and λ_w denote, for simplicity, the equilibrium (or fixed) demand and the equilibrium disutility for O/D pair w, respectively.

Interestingly, we demonstrate in the following theorem that, under certain assumptions, our measure collapses to the L–M measure, which, however, considers neither explicit demands nor flows!

Theorem 3 *If positive demands exist for all pairs of nodes in the network G, and each of these demands is equal to 1 and if d_{ij} is set equal to λ_w, where $w = (i, j)$, for all $w \in W$ then the proposed network efficiency measure (10) and the L–M measure are one and the same.*

Proof Let n be the number of nodes in G. Hence, the total number of O/D pairs, n_W, is equal to $n(n-1)$ given the assumption that there exist positive demands for all pairs of nodes in G. Furthermore, by assumption, we have that $d_w = 1, \ \forall w \in W$, $w = (i, j)$, and $d_{ij} = \lambda_w$, where $i \neq j, \ \forall i, j \in G$. Then the L–M measure becomes as follows:

$$E = E(G) = \frac{1}{n(n-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}} = \frac{\sum_{i \neq j \in G} \frac{1}{d_{ij}}}{n_W} = \frac{\sum_{w \in W} \frac{d_w}{\lambda_w}}{n_W} = \mathcal{E}(G, d) = \mathcal{E}. \tag{11}$$

The conclusion, thus, follows. □

Note that, from the definition, λ_w is the equilibrium disutility or "shortest path" for O/D pair w and d_{ij} is the shortest path length (the geodesic distance) between nodes i and j. Therefore, the assumption of d_{ij} being equal to λ_w is reasonable. Our measure, however, is a more general measure since it also captures the flows on the network through the disutilities, costs, and the demands.

Furthermore, we note that in the L–M measure, there is no information regarding the demand for each O/D pair. Therefore, $n(n-1)$ can be interpreted as the total possible number of O/D pairs regardless of whether there exists a demand for a pair of nodes or not. However, because our measure is an average network efficiency measure, it does not make sense to count a pair of nodes which has no associated demand in the computation of the network efficiency. Therefore, the number of O/D pairs, n_W, is more appropriate as a divisor in our measure than $n(n-1)$. Of course, if there is a positive associated demand between all pairs of nodes in the network then $n_W = n(n-1)$.

Zhu et al. [30] introduced another measure, which we denote by $\hat{E}(G)$, which they then applied to gauge the efficiency of the Chinese airline transportation network with fixed demands. Their network performance measure is characterized by the average social travel cost, which is represented below (with their notation adapted to ours, for clarity):

$$\hat{E}(G) = \frac{\sum_{w \in W} \lambda_w d_w}{\sum_{w \in W} d_w}. \tag{12}$$

The Zhu et al. [30] measure is an average disutility weighted by the demands. It can be used for networks with fixed demands, provided that the network does not get disconnected, in which case the measure becomes undefined. Indeed, a very important feature of our measure is that there is no assumption made that the network needs to be connected. In contrast, the Zhu et al. [30] measure requires such an assumption because, otherwise, their network performance measure will become infinity. In our measure, the elimination of a link is treated by removing that link from the network while the removal of a node is managed by removing the links entering or exiting that node. In the case that the removal results in no path connecting an O/D pair, we simply assign the demand for that O/D pair (either fixed or elastic) to an abstract path with a cost of infinity.

For a network with fixed demands, it is easy to verify that the above approach makes our measure well-defined. Now, let's check if our measure works for a network with elastic demands. In a network with elastic demands, when there is a disconnected O/D pair w, we have, from the above discussion, that the associated "path cost" of the abstract path, say, r, $C_r(x^*)$, is equal to infinity. If the disutility functions are known as discussed in Sect. 2.1, according to equilibrium condition (6), we then have that $C_r(x^*) > \lambda_w(d^*)$, and, hence, $x_r^* = 0$, so that $d_w^* = 0$, which leads to the conclusion of $d_w^*/\lambda_w = 0$. Therefore, the disconnected O/D pair w makes zero "contribution" to the efficiency measure and our measure is well-defined in both the fixed and elastic demand cases. The above procedure(s) to handle disconnected O/D pairs, will be illustrated in the examples in Sect. 4, when we compute the importance of the network components and their rankings.

We believe that this feature of the unified performance measure is important. In reality, it is relevant to investigate the efficiency of a large-scale network even in the case of disconnected O/D pairs. A measure with such adaptability and flexibility can enable the study of the performance of a wider range of networks, especially when evaluating networks under disruptions. Moreover, it also allows us to investigate the criticality of various network components without worrying about the connectivity assumption. Notably, Latora and Marchiori [12] also mentioned this important characteristic which gives their measure an attractive property over the measure used for the small-world model.

Furthermore, as will be shown in the analysis in Sect. 3.1, the Zhu et al. [30] measure cannot capture network performance/efficiency in the case of elastic demands.

3.1 A property of an earlier network performance measure

A network, be it a transportation network, or a supply chain network, or an economic/financial network, is characterized by its topology, its demand, and associated costs. In order to evaluate the importance of nodes and links of a network, the examination of only the topology of the network is insufficient. We also need to evaluate the flows and the induced costs in the network.

A reasonable measure should capture the efficiency deterioration with the increase of path costs in a network. Let's first examine if the measure in (12) has such a feature, even in the simplest separable case in which λ_w is a function only of d_w for all $w \in W$.

Assume that the disutility functions are known as described in Sect. 2.1. Let's take the partial derivative of $\hat{E}(G)$ in (12) with respect to λ_w for a network with elastic demands with the equilibrium disutilities for all the other O/D pairs being held constant, which yields the following:

$$\frac{\partial \hat{E}(G)}{\partial \lambda_w} = \frac{d_w \cdot (\sum_{w\in W} d_w) - (\sum_{w\in W} \lambda_w(d_w)d_w) \cdot (\lambda_w^{'}(d_w))^{-1}}{(\sum_{w\in W} d_w)^2} + \frac{\lambda_w(d_w) \cdot (\lambda_w^{'}(d_w))^{-1}}{\sum_{w\in W} d_w}. \quad (13)$$

It is reasonable to assume that $\lambda_w(d_w) \geq 0$, $d_w \geq 0$, and $\lambda_w^{'}(d_w) < 0$, $\forall w \in W$. Obviously, the first term in (13) is nonnegative and the second term is nonpositive. Therefore, the sign of $\frac{\partial \hat{E}(G)}{\partial \lambda_w}$ depends on the equilibrium demand and the disutility function for each w, which leads to the conclusion that the measure presented in (12) is not appropriate for elastic demand networks.

Now let's check if the new measure given by (10) has the desired network performance property specified earlier. Let's assume that the disutility functions are known as introduced in Sect. 2.1. The disutility function for each $w \in W$ is assumed to depend, for the sake of generality, on the entire demand vector. With the assumption of the equilibrium disutilities for all the other O/D pairs being held constant, the partial derivative of $\mathcal{E}(G, d)$ in (10) with regard to λ_w for the network with elastic demands is then given as follows:

$$\frac{\partial \mathcal{E}(G, d)}{\partial \lambda_w} = \frac{\frac{-d_w}{(\lambda_w(d))^2} + \sum_{v \in W} \frac{(\frac{\partial \lambda_w(d)}{\partial d_v})^{-1}}{\lambda_v(d)}}{n_W}. \tag{14}$$

Given the assumption that $d_w \geq 0$, $\lambda_w \geq 0$, and $\frac{\partial \lambda_w}{\partial d_v} < 0$, $\forall v \in W$, it is obvious that $\mathcal{E}(G, d)$ in (14) is a nonincreasing function of λ_w, $\forall w \in W$.

Let's now interpret the new proposed measure given by (10) in terms of transportation networks. The equilibrium O/D pair disutility, λ_w, is proportional to the (travel) time between each O/D pair w. d_w is the equilibrium demand (in terms of total vehicles) between each O/D pair w. Therefore, d_w/λ_w is the (vehicle) throughput between O/D pair w. $\mathcal{E}(G, d)$ is the average (vehicle) throughput on the network G with demand vector d. Here, instead of using the average social travel disutility/cost as in (12) to quantify the performance of the network, an average throughput measure is proposed. The higher the throughput that a network has, the better its performance and the more efficient it is. For general networks, the performance/efficiency measure $\mathcal{E}$ defined in (10) is actually the average demand to price ratio. When G and d are fixed, a network is more efficient if it can satisfy a higher demand at a lower price!

3.2 The importance of network components

With our network performance/efficiency measure, we are ready to investigate the importance of network components by studying their impact on the network efficiency through their removal. The network efficiency can be expected to deteriorate when a critical network component is eliminated from the network. Such a component can include a link or a node or a subset of nodes and links depending on the network problem under study. Furthermore, the removal of a critical network component will cause more severe damage than that of a trivial one. Hence, similar to the definition of importance of network components in the paper of Latora and Marchiori [14], we define the importance of a network component as follows:

Definition 4 Importance of a network component

The importance of a network component $g \in G$, $I(g)$, is measured by the relative network efficiency drop after g is removed from the network:

$$I(g) = \frac{\triangle \mathcal{E}}{\mathcal{E}} = \frac{\mathcal{E}(G, d) - \mathcal{E}(G - g, d)}{\mathcal{E}(G, d)} \tag{15}$$

where $G - g$ is the resulting network after component g is removed from network G.

The upper bound of the importance of a network component is 1. The higher the value, the more important a network component is.

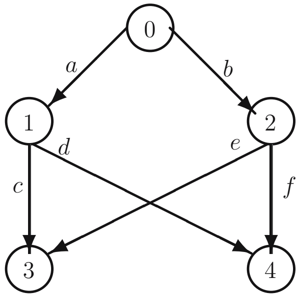

Fig. 1 Network for Examples 1 and 3

4 Numerical examples

In this section, three examples of networks are presented for which the unified network performance/efficiency measure is computed. The first two examples, reported in Sect. 4.1, are fixed demand examples, whereas the third example, given in Sect. 4.2, is an elastic demand example. Moreover, the importance of individual nodes and links are determined, ranked, and compared by using our measure, the L–M measure, and the Zhu et al. [30] measure for Example 1. In addition, for completeness, the importance of individual nodes and links are determined and their rankings are reported by using our measure and the L–M measure for Example 3. In the following examples, we assume that d_{ij} in the L–M measure is equal to λ_w where $w = (i, j)$ for $w \in W$. (Note that if a pair of nodes i, j becomes disconnected, then according to the L–M measure, $d_{ij} = \infty$ and, hence, $\frac{1}{d_{ij}} = 0$, in this case.) Example 2 is a larger example for which we compute the importance values and the importance rankings of the links.

4.1 Fixed demand examples

We now present two fixed demand network examples.

Example 1 A network with fixed demands

Example 1 is a fixed demand network problem as described in Sect. 2.2 and with the topology given in Fig. 1.

There are two O/D pairs in the above network given by $w_1 = (0, 3)$ and $w_2 = (0, 4)$. There are two paths connecting each O/D pair:

for O/D pair w_1:

$$p_1 = (a, c), \quad p_2 = (b, e),$$

for O/D pair w_2:

$$p_3 = (a, d), \quad p_4 = (b, f).$$

The link cost functions are as follows:

$$c_a(f_a) = f_a, \ c_b(f_b) = f_b, \ c_c(f_c) = f_c, \ c_d(f_d) = f_d, \ c_e(f_e) = f_e, \ c_f(f_f) = f_f.$$

The demands for the O/D pairs w_1 and w_2 are: $d_{w_1} = 100$ and $d_{w_2} = 20$. The equilibrium solution (cf. (8)) for this network is:

$$x^*_{p_1} = 50, \quad x^*_{p_2} = 50, \quad x^*_{p_3} = 10, \quad x^*_{p_4} = 10,$$
$$\lambda_{w_1} = 110, \quad \lambda_{w_2} = 70.$$

Our network performance/efficiency measure for Example 1 is then given by:

$$\mathcal{E}(G,d) = \frac{1}{n_W}\left[\frac{d_{w_1}}{\lambda_{w_1}} + \frac{d_{w_2}}{\lambda_{w_2}}\right] = \frac{\frac{100}{110} + \frac{20}{70}}{2} = 0.5974.$$

The L–M measure for Example 1 is:

$$E(G) = \frac{1}{n(n-1)}\left[\frac{1}{d_{01}} + \frac{1}{d_{02}} + \frac{1}{d_{03}} + \frac{1}{d_{04}} + \frac{1}{d_{13}} + \frac{1}{d_{14}} + \frac{1}{d_{23}} + \frac{1}{d_{24}}\right]$$
$$= \frac{1}{20}\left[\frac{1}{60} + \frac{1}{60} + \frac{1}{110} + \frac{1}{70} + \frac{1}{50} + \frac{1}{10} + \frac{1}{50} + \frac{1}{10}\right] = 0.0148.$$

The Zhu et al. (2006) measure, in turn, is:

$$\hat{E}(G) = \frac{(d_{w_1}\lambda_{w_1} + d_{w_2}\lambda_{w_2})}{(d_{w_1} + d_{w_2})} = \frac{(100 \times 110 + 20 \times 70)}{(20 + 100)} = 103.33.$$

The importance of links and nodes and their ranking are reported, respectively, in Tables 1 and 2; see also Latora and Marchiori [14] and Zhu et al. [30]. Note that the importance of network components according to Zhu et al. [30] is similar to that in (15) but with $\hat{E}(G)$ substituted for $\mathcal{E}(G,d)$, $\hat{E}(G-g)$ for $\mathcal{E}(G-g,d)$ and the deduction order being changed, whereas Latora and Marchiori [14] define the importance of a network component as: $I(g) = E(G) - E(G-g) = \Delta E$ but they use $I(g) = \frac{\Delta E}{E}$ in their calculations and we do, as well, below, when we compare our measure to the L–M measure.

Example 2 A larger fixed demand network

The second example consisted of 20 nodes, 28 links, and 8 O/D pairs, and is depicted in Fig. 2.

A similar transportation network had been used previously in [17] where it is referred to as Network 20; see also Dhanda et al. [9]. For simplicity, and easy reproducibility, we considered separable user link cost functions, which were adapted from Network 20 in [17] with the cross-terms removed.

The O/D pairs were: $w_1 = (1, 20)$ and $w_2 = (1, 19)$ and the travel demands: $d_{w_1} = 100$, and $d_{w_2} = 100$. The link cost functions are given in Table 3.

We utilized the projection method [6, 18] with the embedded Dafermos and Sparrow [8] equilibration algorithm (see also, e.g., [17]) to compute the equilibrium solutions and to determine the network efficiency according to (10) and well as the importance values and the importance rankings of the links according to (15).

Table 1 Importance and ranking of links in example 1

Link	Importance value from our measure	Importance ranking from our measure	Importance value from the L–M measure	Importance ranking from L–M measure	Importance value from the Zhu et al. measure	Importance ranking from the Zhu et al. measure
a	0.5000	1	N/A	N/A	1.0000	1
b	0.5000	1	N/A	N/A	1.0000	1
c	0.1630	2	N/A	N/A	0.6774	2
d	0.0422	3	0.5119	1	0.0242	3
e	0.1630	2	N/A	N/A	0.6774	2
f	0.0422	3	0.5119	1	0.0242	3

Table 2 Importance and ranking of nodes in Example 1

Node	Importance value from our measure	Importance ranking from our measure	Importance value from the L-M measure	Importance ranking from L–M measure	Importance value from the Zhu et al. measure	Importance ranking from the Zhu et al. measure
0	1.0000	1	N/A	N/A	N/A	N/A
1	0.5000	2	0.7303	1	1.0000	1
2	0.5000	2	0.7303	1	1.0000	1
3	0.1630	3	-0.5166	3	N/A	N/A
4	0.1630	3	0.6967	2	N/A	N/A

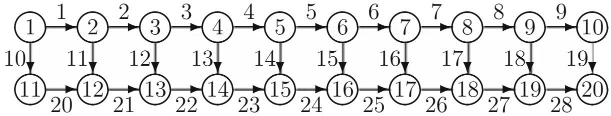

Fig. 2 Network for Example 2

The computed efficiency measure for this network is: $\mathcal{E} = 0.002518$. The computed importance values of the links and their rankings for this transportation network are reported in Table 3.

From the results in Table 3, it is clear that transportation planners and network security officials should pay most attention to links: 1, 2, and 26, 27, since these are the top four links in terms of importance rankings. On the other hand, the elimination of links: 11, 13, 14, 15, and 17 should have no impact on the network performance/efficiency.

4.2 An elastic demand network example

Example 3 A network with elastic demands

Table 3 Example 2 - Links, link cost functions, importance values, and importance rankings

Link a	Link cost function $c_a(f_a)$	Importance value	Importance ranking
1	$0.00005f_1^4 + 5f_1 + 500$	0.9086	3
2	$0.00003f_2^4 + 4f_2 + 200$	0.8984	4
3	$0.00005f_3^4 + 3f_3 + 350$	0.8791	6
4	$0.00003f_4^4 + 6f_4 + 400$	0.8672	7
5	$0.00006f_5^4 + 6f_5 + 600$	0.8430	9
6	$7f_6 + 500$	0.8226	11
7	$0.00008f_7^4 + 8f_7 + 400$	0.7750	12
8	$0.00004f_8^4 + 5f_8 + 650$	0.5483	15
9	$0.00001f_9^4 + 6f_9 + 700$	0.0362	17
10	$4f_{10} + 800$	0.6641	14
11	$0.00007f_{11}^4 + 7f_{11} + 650$	0.0000	22
12	$8f_{12} + 700$	0.0006	20
13	$0.00001f_{13}^4 + 7f_{13} + 600$	0.0000	22
14	$8f_{14} + 500$	0.0000	22
15	$0.00003f_{15}^4 + 9f_{15} + 200$	0.0000	22
16	$8f_{16} + 300$	0.0001	21
17	$0.00003f_{17}^4 + 7f_{17} + 450$	0.0000	22
18	$5f_{18} + 300$	0.0175	18
19	$8f_{19} + 600$	0.0362	17
20	$0.00003f_{20}^4 + 6f_{20} + 300$	0.6641	14
21	$0.00004f_{21}^4 + 4f_{21} + 400$	0.7537	13
22	$0.00002f_{22}^4 + 6f_{22} + 500$	0.8333	10
23	$0.00003f_{23}^4 + 9f_{23} + 350$	0.8598	8
24	$0.00002f_{24}^4 + 8f_{24} + 400$	0.8939	5
25	$0.00003f_{25}^4 + 9f_{25} + 450$	0.4162	16
26	$0.00006f_{26}^4 + 7f_{26} + 300$	0.9203	2
27	$0.00003f_{27}^4 + 8f_{27} + 500$	0.9213	1
28	$0.00003f_{28}^4 + 7f_{28} + 650$	0.0155	19

We return now to Example 1 except that, now, we let the demand for O/D pairs w_1 and w_2 be elastic, so that the problem is as described in Sect. 2.1, where, specifically, we have that:

$$\lambda_{w_1}(d_{w_1}) = 100 - d_{w_1}, \quad \lambda_{w_2}(d_{w_2}) = 40 - d_{w_2}.$$

It is easy to calculate the following equilibrium solution (cf. (6)):

$$x^*_{p_1} = 24, \quad x^*_{p_2} = 24, \quad x^*_{p_3} = 4, \quad x^*_{p_4} = 4,$$

$$d^*_{w_1} = 48, \quad d^*_{w_2} = 8$$

Table 4 Importance and ranking of links in Example 3

Link	Importance value from our measure	Importance ranking from our measure	Importance value from the L–M measure	Importance ranking from the L–M measure
a	0.5327	1	N/A	N/A
b	0.5327	1	N/A	N/A
c	0.1475	2	N/A	N/A
d	0.0533	3	0.4516	1
e	0.1475	2	N/A	N/A
f	0.0533	3	0.4516	1

Table 5 Importance and ranking of nodes in Example 3

Node	Importance value from our measure	Importance ranking from our measure	Importance value from the L–M measure	Importance ranking from the L–M measure
0	1.0000	1	N/A	N/A
1	0.5327	2	0.2775	2
2	0.5327	2	0.2775	2
3	0.1475	3	0.3509	1
4	0.1475	3	0.3509	1

so that:

$$\lambda_{w_1} = 52, \quad \lambda_{w_2} = 32.$$

Our network performance/efficiency measure for Example 3 is:

$$\mathcal{E}(G, d) = \frac{1}{n_W}\left[\frac{d_{w_1}}{\lambda_{w_1}} + \frac{d_{w_2}}{\lambda_{w_2}}\right] = \frac{\frac{48}{52} + \frac{8}{32}}{2} = 0.5865.$$

The L–M measure for Example 3 is:

$$E(G) = \frac{1}{n(n-1)}\left[\frac{1}{d_{01}} + \frac{1}{d_{02}} + \frac{1}{d_{03}} + \frac{1}{d_{04}} + \frac{1}{d_{13}} + \frac{1}{d_{14}} + \frac{1}{d_{23}} + \frac{1}{d_{24}}\right]$$

$$= \frac{1}{20}\left[\frac{1}{28} + \frac{1}{28} + \frac{1}{52} + \frac{1}{32} + \frac{1}{24} + \frac{1}{4} + \frac{1}{24} + \frac{1}{4}\right] = 0.0353.$$

As discussed in Sect. 3.1, the Zhu et al. [30] measure cannot be used to assess networks with elastic demands. Therefore, in Tables 4 and 5, only the importance of links and nodes and their rankings using our measure and the L–M measure are given.

As discussed in Sect. 3, by adding an abstract (and infinite cost) path to a disconnected O/D pair, our measure can be used to study networks with disconnected O/D pairs. This feature enables us to investigate the importance of nodes 3 and 4 in the Examples 1 and 3 while the Zhu et al. [30] measure is then undefined.

5 Conclusions and future research directions

In this paper, we introduced a unified network performance/efficiency measure, which can be applied to evaluate the network efficiency of different types of networks whether the demands on the network are fixed or elastic. The measure assesses the network efficiency by incorporating flows, and costs, along with behavior, all important factors when dealing with network vulnerability and reliability. Future research will utilize the above measure to identify the important/vulnerable components of large-scale networks in a variety of distinct network settings and applications.

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