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Finite groups with two kernels of nonlinear irreducible characters

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Abstract
In this paper, we consider finite groups with few kernels of nonlinear irreducible characters and classify finite solvable groups with two kernels of nonlinear irreducible characters. Nonsolvable groups with this property are also investigated.

Keywords Finite group · Nonlinear character · Two character kernels

Mathematics Subject Classification 20C15

1 Introduction

Throughout the paper, G is always a finite group. Let $\mathrm{Irr}(G)$ be the set of all complex irreducible characters of G. If $\chi \in \mathrm{Irr}(G)$ and $\chi(1) > 1$, then χ is called a nonlinear irreducible character of G. The set of nonlinear irreducible characters of G is denoted by $\mathrm{NL}(G)$ in this note. For a character χ of G, $\ker(\chi) = \{g \mid \chi(g) = \chi(1), g \in G\}$ is defined as the kernel of χ. Clearly, χ is a nonlinear irreducible character of G iff $G' \nsubseteq \ker(\chi)$. Let $\mathrm{Kern}(G) = \{\ker(\chi) \mid \chi \in \mathrm{NL}(G)\}$ denote the set of kernels of the nonlinear irreducible characters of G. Since every normal subgroup is the intersection of some irreducible character kernels, $\mathrm{Kern}(G)$ heavily influences the normal structure of G. For example, Qian and Wang in [6] classified the finite p-groups G for which $\mathrm{Kern}(G)$ is a chain with respect to inclusion.

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For $\mathrm{Kern}(G)$, a useful fact is that the intersection of the kernels of all nonlinear irreducible characters of G is trivial, i.e., $\cap_{K \in \mathrm{Kern}(G)} K = \{1\}$ (see Theorem 35 of [1, page 94]). With this fact, one can imagine that if a nonabelian group G has few kernels of nonlinear irreducible characters, then G has few normal subgroups not containing G' and the structure of G should be very limited. Specifically, if $\mid \mathrm{Kern}(G) \mid = 1$, then $\mathrm{Kern}(G) = \{1\}$. This is the situation that all of the nonlinear irreducible characters of G are faithful. Berkovich and Zhmud' in [2] said that G is a J_0-group if $\mid \mathrm{Kern}(G) \mid = 1$. And they gave the structures of J_0-groups as follows.

Lemma 1 *[2, Chapter 29, Lemma 2] Let G be a finite group. Then G is a J_0-group iff G' is the unique minimal normal subgroup of G, and one and only one of the following assertions holds:*

(1) *G is a p-group for some prime p and $\mathrm{Z}(G)$ is cyclic.*
(2) *G is a Frobenius group with Frobenius kernel G'.*
(3) *G is nonsolvable.*

For convenience, we say that G is a J_0-p-group if G is a J_0-group and also a p-group in this note. Similarly, a Frobenius J_0-group means that it is a J_0-group and also a Frobenius group.

Note that if a nonabelian group G has a unique nonlinear nonfaithful irreducible character χ, which implies that $\ker(\chi)$ is the unique nonidentity element of $\mathrm{Kern}(G)$, then by the same reason that $\cap_{K \in \mathrm{Kern}(G)} K = \{1\}$, we know $\mathrm{Kern}(G) = \{1, \ker(\chi)\}$ and so $\mid \mathrm{Kern}(G) \mid = 2$. Iranmanesh and Saeidi in [4] studied this kind of groups and after that A.Saeidi in [7] gave a classification of this kind of solvable groups. Furthermore, Berkovich and Zhmud' in [2, page 252] posted a question about classifying finite groups G such that $\mid \mathrm{Kern}(G) \mid \leq 3$. For this question, H.Doostie and A.Saeidi in [3] determined finite p-groups G with $\mid \mathrm{Kern}(G) \mid \leq 3$. In this paper, we aim at advancing on the solution of the question, and provide a classification of the groups G satisfying $\mid \mathrm{Kern}(G) \mid = 2$. More precisely, our main results are the following.

Theorem 1 *Let G be a finite solvable group. Then G has two kernels of nonlinear irreducible characters if and only if one of the following cases occurs:*

(1) *G is of order p^4 and nilpotency class 3, where p is a prime.*
(2) *G is a 2-group, $\mid G' \mid = 2$ and $\mathrm{Z}(G) \cong C_2 \times C_{2^r}$ $(r \geq 1)$, and $r > 1$ implies that $G' \subseteq \Phi(\mathrm{Z}(G))$.*
(3) *$G = H \times K$, H is a J_0-p-group and K is a group of order q, where p, q are two different primes.*
(4) *$\mathrm{Z}(G)$ and G' are all the minimal normal subgroups of G and $G/\mathrm{Z}(G)$ is a Frobenius J_0-group.*
(5) *The Fitting subgroup $\mathrm{F}(G) \subsetneq G'$ is the unique minimal normal subgroup of G and the complements of $\mathrm{F}(G)$ in G are solvable J_0-groups.*
(6) *The Frattini subgroup $\Phi(G) \subsetneq G'$ is the unique minimal normal subgroup of G and $G/\Phi(G)$ is a Frobenius J_0-group.*

Theorem 2 *Let G be a nonsolvable group with two kernels of nonlinear irreducible characters, then*

(1) *G has faithful nonlinear irreducible characters;*
(2) *G/K is a J_0-group, where K is the same kernel of the nonlinear nonfaithful irreducible characters of G; and either K is the unique minimal normal subgroup of G or else $K = \mathrm{Z}(G)$ is of prime order and $G/\mathrm{Z}(G)$ is a nonsolvable J_0-group.*

2 Preliminaries

In this section, we give some results needed for the proofs of our main results. The notations and terminologies are standard and can be found in [1, 5]. In addition, for a group G, we write $\mathrm{Soc}(G)$ and $\exp(G)$ in the following to denote the socle of G and the exponent of G, respectively. C_n means a cyclic group of order n and p is always a prime number.

We begin with a result on $\mathrm{Kern}(G)$, which has been mentioned in the introduction. Since the result will be used repeatedly, we restate it here.

Lemma 2 *([1, Chapter 4, Theorem 35]) Let G be a nonabelian group. Then $\bigcap_{K\in\mathrm{Kern}(G)} K = \{1\}$.*

The next result due to H. Doostie and A.Saeidi is a classification of finite p-groups G with $|\mathrm{Kern}(G)| \leq 3$.

Lemma 3 *([3, Theroem 1.1]) Let G be a nonabelian p-group and $t = |\mathrm{Kern}(G)|$ be the number of nonlinear irreducible character kernels of G. Then,*

(1) *$t = 1$ if and only if $|G'| = p$ and $\mathrm{Z}(G)$ is cyclic.*
(2) *$t = 2$ if and only if one of the following cases occurs:*
   (a) *G is of order p^4 and nilpotency class 3.*
   (b) *$|G'| = 2$ and $\mathrm{Z}(G) \cong C_2 \times C_{2^r} (r \geq 1)$, and $r > 1$ implies that $G' \subseteq \Phi(\mathrm{Z}(G))$.*
(3) *$t = 3$ if and only if one of the following cases occurs:*
   (a) *G is of order p^5 and nilpotency class 4.*
   (b) *G is of order 32 and nilpotency class 3 and $\mathrm{Z}(G)$ is cyclic.*
   (c) *$|G'| = 3$, $\mathrm{Z}(G) \cong C_3 \times C_{3^r} (r \geq 1)$ and $r > 1$ implies that $G' \subseteq \Phi(\mathrm{Z}(G))$.*
   (d) *$G' = \mathrm{Z}(G) \cong C_2 \times C_2$ and $N \leq \mathrm{Z}(G)$, for each $N \lhd G$ not containing G'.*
   (e) *$|G'| = 2$, $\mathrm{Z}(G) \cong C_2 \times C_4$ and $G' \neq \Phi(\mathrm{Z}(G))$.*
   (f) *G is a 2-group of nilpotency class 2, $|G'| = 4$, $\mathrm{Z}(G)$ is cyclic and $|N\mathrm{Z}(G) : \mathrm{Z}(G)| \leq 2$, for each $N \lhd G$ not containing G'.*

The following lemma gives a criterion for identifying whether G has faithful irreducible characters by using the structure of $\mathrm{Soc}(G)$.

Lemma 4 ([1, Chapter 9, Corollary 6]) A group G possesses a faithful irreducible character if and only if $\mathrm{Soc}(G)$ is generated by a G-class. In particular, a nonabelian group G possesses a faithful nonlinear irreducible character if and only if $\mathrm{Soc}(G)$ is generated by a G-class.

Two subgroups F, H of a group G are said to be nonincident if $F \neq F \cap H \neq H$. In the chapter 29 of [2], a nonabelian group G is said to be a J-group if any two different elements of $\mathrm{Kern}(G)$ are nonincident. The following result is a classification of the nilpotent J-groups.

Lemma 5 *([2, Chapter 29, Lemma 5]) If G is a nilpotent J-group, then it is a p-group. A p-group is a J-group if and only if $G' \leq \mathrm{Z}(G)$ and $\exp(G') = p$.*

Assume that G is a finite group with $\mathrm{Kern}(G) = \{K_1, K_2\}$. Clearly, there are two cases for $\mathrm{Kern}(G)$: $1 \notin \mathrm{Kern}(G)$ and $1 \in \mathrm{Kern}(G)$. For the first case, using Lemma 4 and Lemma 5 we can determine the structure of G as the following Theorem 3 shows. We note that our Theorem 3 generalizes the conclusion of Lemma 3.1 in [3]. And for the second case, we also have a result (see Theorem 4) which will be used repeatedly in the proofs of our main Theorems 5 and 6.

Theorem 3 *Let G be a finite group. Then $\mathrm{Kern}(G) = \{K_1, K_2\}$ and $K_i \neq 1$ for $i = 1, 2$ if and only if G is a 2-group, $| G' |= 2$ and $\mathrm{Z}(G) \cong C_2 \times C_{2^r} (r \geq 1)$, and $r > 1$ implies that $G' \subseteq \Phi(\mathrm{Z}(G))$.*

Proof If $\mathrm{Kern}(G) = \{K_1, K_2\}$ and $1 \neq K_i$, for $i = 1, 2$, then by Lemma 2 we know $K_1 \cap K_2 = 1$. Particularly, K_1, K_2 are nonincident and so G is a J-group. Next, we claim that G is nilpotent.

Let $N \leq K_1$ be a minimal normal subgroup of G. Clearly, $K_2 \cap N = 1$ since $K_1 \cap K_2 = 1$. Observe that $\mathrm{NL}(G/N) = \{\chi \mid \chi \in \mathrm{NL}(G), \ker(\chi) = K_1\}$. Using Lemma 2 for G/N, we can get that $K_1 = N$, which is minimal normal in G. Similarly, K_2 is minimal normal too.

Suppose G' is not minimal normal in G, take $N \trianglelefteq G$ be a minimal normal subgroup of G, then $G' \nleq N$ and G/N is nonabelian. Consider $\mathrm{NL}(G/N)$ as above, we can get either $N = K_1$ or $N = K_2$. Thus, K_1, K_2 are all the minimal normal subgroups of G and so $K_1 \times K_2$ is the socle of G. Take $1 \neq x \in K_1$, $1 \neq y \in K_2$, then $xy \in K_1K_2$. Let $\langle xy\rangle^G = \langle (xy)^g \mid g \in G\rangle$ be the subgroup generated by the G-class of xy. Clearly, $\langle xy\rangle^G \trianglelefteq G$ and $K_i \neq \langle xy\rangle^G \leq K_1K_2$ for $i = 1, 2$. Recall that K_1, K_2 are all the minimal normal subgroups, then there exists K_i such that $K_i \lneqq \langle xy\rangle^G$. Without loss of generality, let $K_1 \lneqq \langle xy\rangle^G$, then

$$\langle xy\rangle^G = \langle xy\rangle^G \cap K_1K_2 = K_1(\langle xy\rangle^G \cap K_2) = K_1K_2 = \mathrm{Soc}(G).$$

Using Lemma 4, we know that G has a faithful nonlinear irreducible character. This contradicts $K_i \neq 1$ for $i = 1, 2$. Hence, G' is minimal normal in G too.

Recall that G', K_1, K_2 are the minimal normal subgroups of G. So $G' \cap K_i = 1$ and $K_i \leq \mathrm{Z}(G)$ ($i = 1, 2$). Consider the factor group $G/(K_1K_2)$. Suppose $G/(K_1K_2)$ has a nonlinear irreducible character χ, then $\ker(\chi) \in \mathrm{Kern}(G)$ and $K_1K_2 \leq \ker(\chi)$, a contradiction. Thus, $G/(K_1K_2)$ is abelian and so $G' \leq K_1 \times K_2 \leq \mathrm{Z}(G)$, which implies that G is nilpotent.

Now we know that G is a nilpotent J-group. Then using Lemma 5 for G we can get that G is a p-group. Furthermore, since G has no faithful irreducible characters, by item (b) of Theorem 2.32 in [5], we also know that $\mathrm{Z}(G)$ is not cyclic. Finally, by the necessity of item (2) of Lemma 3, we know (b) happens since the center of every p-group of maximal nilpotency class is cyclic. That is: G is a 2-group, $|G'| = 2$ and $Z(G) \cong C_2 \times C_{2^r}\,(r \geq 1)$, and $r > 1$ implies that $G' \subseteq \Phi(Z(G))$.

For the converse, if G is a 2-group, $|G'| = 2$ and $Z(G) \cong C_2 \times C_{2^r}\,(r \geq 1)$, and $r > 1$ implies that $G' \subseteq \Phi(Z(G))$, then using the sufficiency of item (2) of Lemma 3, we know $|\operatorname{Kern}(G)| = 2$. And since $Z(G)$ is not cyclic, by item (a) of Theorem 2.32 in [5] we also know $1 \notin \operatorname{Kern}(G)$. The proof is complete. □

By the above theorem, the following corollary is clear.

Corollary 1 *Let G be a finite group with $|\operatorname{Kern}(G)| = 2$. Suppose G is not a 2-group, then $1 \in \operatorname{Kern}(G)$.*

Theorem 4 *Let G be a finite group. Then $\operatorname{Kern}(G) = \{1, K\}$ if and only if G/K is a J_0-group and K is the unique nontrivial normal subgroup of G which does not contain G'. In particular, K is minimal normal in G and G has no other minimal normal subgroups except possibly G'.*

Proof Suppose $\operatorname{Kern}(G) = \{1, K\}$, then for every nonlinear nonfaithful irreducible character χ of G, $\ker(\chi) = K$. In particular, $\operatorname{NL}(G/K) = \{\chi \mid \chi \in \operatorname{NL}(G), \ker(\chi) = K\}$. Thus, every nonlinear irreducible character of G/K is faithful and so G/K is a J_0-group. Let N be any nontrivial normal subgroup of G which does not contain G', then G/N is nonabelian and so $N \leq K$. Using Lemma 2 for G/N, we can deduce that $N = K$. Hence, K is the unique nontrivial normal group of G which does not contain G'. In particular, if $M \neq G'$ is a minimal normal subgroup of G, then $G' \nleq M$ and so $M = K$. Therefore, K is minimal normal in G and G has no other minimal subgroups except possibly G'.

Conversely, Assume that G/K is a J_0-group and K is the unique nontrivial normal group of G which does not contain G'. Take any $\chi \in \operatorname{NL}(G)$, then $\ker(\chi) \trianglelefteq G$ and $G' \nsubseteq \ker(\chi)$. So, if $\ker(\chi) \neq 1$, then $\ker(\chi) = K$. On the other hand, by Lemma 2 again, we know that G must have faithful nonlinear irreducible characters and it follows that $\operatorname{Kern}(G) = \{1, K\}$. We are done. □

3 Proofs of main results

Let G be a solvable group with $|\operatorname{Kern}(G)| = 2$, we discuss the structure of G in two situations: the cases when G is nilpotent and not nilpotent. First, we handle the nilpotent case.

Theorem 5 *Let G be a finite nilpotent group. Then $|\operatorname{Kern}(G)| = 2$ if and only if one of the following cases occurs:*

(1) *G is of order p^4 and nilpotency class 3.*
(2) *G is a 2-group, $|G'| = 2$ and $Z(G) \cong C_2 \times C_{2^r}\,(r \geq 1)$, and $r > 1$ implies that $G' \subseteq \Phi(Z(G))$.*
(3) *$G = H \times K$, H is a J_0-p-group and K is a group of order q, where p, q are two different primes.*

Proof Let G be a nilpotent group with $|\operatorname{Kern}(G)| = 2$. If G is a p-group, then by the necessity of Lemma 3(2), (1) or (2) follows. Otherwise, G is a nilpotent group whose order has at least two different prime divisors. By Corollary 1, it follows that G has faithful nonlinear irreducible characters and then we can assume that $\operatorname{Kern}(G) = \{1, K\}$. By Theorem 4, we see that G/K is a nilpotent J_0-group. Furthermore, using Lemma 1, we know that G/K is a J_0-p-group for some prime number p. Note that K is minimal normal in G by Theorem 4 and G is nilpotent, $K \leq Z(G)$ holds. So $K \cong C_q$ for some prime number q. Recall that $|G|$ has at least two different prime divisors. So $q \neq p$. Now let H be a Sylow p-subgroup of G, then $H \cong G/K$ is a J_0-p-group and $G = H \times K$, (3) follows.

Conversely, if (1) or (2) happens, then by the sufficiency of Lemma 3(2), we are done. If (3) happens, then for any $\chi \in \mathrm{NL(G)}$, there exist $\varphi \in \mathrm{NL(H)}$ and $\lambda \in \mathrm{Irr(K)}$ such that $\chi = \varphi \times \lambda$. If λ is not the principal character 1_K of K, then by Problem 4.3 of [5] we know $\ker(\chi) = 1$; otherwise, if $\lambda = 1_K$, then $\ker(\chi) = \ker(\varphi) \times \ker(1_K) = K$. Thus, if (3) happens, then $\operatorname{Kern}(G) = \{1, K\}$. We complete the proof. □

We proceed now toward the nonnilpotent case.

Theorem 6 *G is a solvable nonnilpotent group with $|\operatorname{Kern}(G)| = 2$ if and only if one of the following cases occurs:*

(1) *$Z(G)$ and G' are all the minimal normal subgroups of G and $G/Z(G)$ is a Frobenius J_0-group.*
(2) *The Fitting subgroup $F(G) \subsetneq G'$ is the unique minimal normal subgroup of G and the complements of $F(G)$ in G are solvable J_0-groups.*
(3) *The Frattini subgroup $\Phi(G) \subsetneq G'$ is the unique minimal normal subgroup of G and $G/\Phi(G)$ is a Frobenius J_0-group.*

Proof If (1) happens, then G is solvable but not nilpotent since $G/Z(G)$ is a Frobenius J_0-group. And for any $\chi \in \mathrm{NL(G)}$, if $\ker(\chi) \neq 1$, then $Z(G) \leq \ker(\chi)$ and $\chi \in \mathrm{NL}(G/Z(G))$. Note that $G/Z(G)$ is a Frobenius J_0-group, so χ as the nonlinear irreducible character of $G/Z(G)$ is faithful. Thus, for any $\chi \in \mathrm{NL}(G)$, either $\ker(\chi) = 1$ or $\ker(\chi) = Z(G)$. So $\operatorname{Kern}(G) = \{1, Z(G)\}$.

Using similar argument as above, we can deduce that G is solvable nonnilpotent whenever (2) or (3) happens. Moreover, if (2) or (3) happens, then we have $\operatorname{Kern}(G) = \{1, F(G)\}$ and $\operatorname{Kern}(G) = \{1, \Phi(G)\}$, respectively.

Conversely, suppose that G is a solvable nonnilpotent group with $|\operatorname{Kern}(G)| = 2$, then by Corollary 1 we know $1 \in \operatorname{Kern}(G)$ and so we can assume that $\operatorname{Kern}(G) = \{1, K\}$. By Theorem 4 we know that K is a minimal normal subgroup, so either $K \cap G' = 1$ or $K < G'$ happens. In the following we consider these two cases.

Case 1. $K \cap G' = 1$

If $K \cap G' = 1$, then by Theorem 4, K and G' are all the minimal normal subgroups of G. In particular, since $K \cap G' = 1$, K as a subgroup of $Z(G)$ is simple. On the other hand, since G is not nilpotent, we have $G' \nleq Z(G)$. We also from Theorem 4 know that K is the unique nontrivial normal subgroup of G which does not contain G'. Hence $K = Z(G)$ and $G/Z(G)$ is a solvable J_0-group. Note that $G/Z(G)$ can not be a p-group since G is not nilpotent. So from Lemma 1 we know that $G/Z(G)$ is a Frobenius J_0-group and (1) follows.

Case 2. $K < G'$

If $K < G'$, then K is the unique minimal normal subgroup of G by Theorem 4. In the following, we consider two subcases according to whether or not G' is nilpotent.

If G' is not nilpotent, then the Fitting subgroup $\mathrm{F}(G)$ of G doesn't contain G' and so $\mathrm{F}(G) = K$ is the unique minimal normal subgroup of G. In particular, $\Phi(G) = 1$ and so that $\mathrm{F}(G)$ has complements in G. Furthermore, $G/\mathrm{F}(G)$ is a solvable J_0-group and (2) follows.

Finally, if G' is nilpotent, then $G' \leq \mathrm{F}(G)$. Suppose $\Phi(G) = 1$, then $\mathrm{F}(G) = K$ since K is the unique minimal normal subgroup of G. And so that $G' \leq K$ happens, contradicting with $K < G'$. Thus, $\Phi(G) \neq 1$. Observe that $G' \nleq \Phi(G)$ since G is not nilpotent. So by the fact that K is the unique nontrivial normal subgroup of G which does not contain G', we know $\Phi(G) = K$. And we have that $G/\Phi(G)$ is a solvable J_0-group by Theorem 4. Recall that G is not nilpotent, $G/\Phi(G)$ can not be a p-group. Now using Lemma 1, we know that $G/\Phi(G)$ is a Frobenius J_0-group and (3) follows. We are done. □

Proof of Theorem 1 From Theorems 5, 6 and 1 is immediately available.

Proof of Theorem 2 Let G be a nonsolvable group with $|\ \mathrm{Kern}(G)\ | = 2$. (1) follows immediately from Corollary 1. And so we can assume $\mathrm{Kern}(G) = \{1, K\}$. Clearly, K is the same kernel of the nonlinear nonfaithful irreducible characters of G. Using Theorem 4, we have that G/K is a J_0-group, K is minimal normal in G and G has no other minimal normal subgroups except possibly G'.

Suppose G' is minimal normal in G too, then $K \cap G' = 1$ and so $K \leq \mathrm{Z}(G)$ is of prime order. Note that G is nonsolvable, $G' \nleq \mathrm{Z}(G)$. By Theorem 4 again, it follows that $K = \mathrm{Z}(G)$ and $G/\mathrm{Z}(G)$ is a nonsolvable J_0-group. Now (2) holds and we complete the proof.

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