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# Finite groups with specific $S_*$-embedded subgroups

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**Abstract**
We examine the $p$-nilpotency and supersolvablitiy of a finite group under the assumption that certain subgroups of prime power order are $S_*$- embedded in the group itself. In particular, we extend and generalize the recent results of Li (Commun. Algebra 50(4):1585–1594, https://doi.org/10.1080/00927872.2021.1986056, 2022) and the related results in the literature.

**Keywords** $p$-nilpotent · Supersolvable · $S$-semipermutable subgroup · $S$-permutably embedded subgroup · $S_*$-embedded subgroup

**Mathematics Subject Classification** 20D10 · 20D20

## 1 Introduction

All the groups are considered to be finite and $G$ always stands for a finite group. $G_p$ denotes a Sylow $p$-subgroup of $G$ where $p$ is a prime number. We use conventional notions and notation as in [1]. Recall that a subgroup $A$ of $G$ is said to be $S$-permutable in $G$ if $AG_p = G_pA$ for every Sylow $p$-subgroup $G_p$ of $G$ [8]. Zhang and Wang [18] called a subgroup $A$ of $G$ $S$-semipermutable in $G$ if $A$ permutes with every Sylow $p$-subgroup $G_p$ of $G$ such that $(p, |A|) = 1$. Ballester-Bolinches and Pedraza-Aguilera [2] called a subgroup $A$ of $G$ $S$-permutably embedded in $G$ if all Sylow $p$-subgroups of $A$ are also Sylow $p$-subgroups of some $S$-permutable subgroup $F$ of $G$. Obviously, the class of all $S$-permutably embedded subgroups is wider than the class of all $S$-permutable subgroups. More recently, Li [9] introduced a new embedding property which covers both of $S$-permutably embedded and $S$-semipermutable concepts as follows: A subgroup $A$ of $G$ is said to be $S_*$-embedded in $G$ if $G$ has an $S$-permutable subgroup $F$ such

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that $AF$ is $S$-permutable in $G$ and $A \cap F \leq A_*$, where $A_*$ is a subgroup contained in $A$ which is either $S$-permutably embedded or $S$-semipermutable in $G$.

By using this new concept, Li [9] studied the group structure when certain subgroups are $S_*$-embedded and obtained new results generalised many classical and recent results in the literature. More precisely, he proved:

**Theorem A** ([8, Main Theorem]) *Assume for each non-cyclic Sylow $p$-subgroup $G_p$ of $G$, either of the following two conditions is held:*

1. *All maximal subgroups of $G_p$ not having a supersolvable supplement in $G$ are $S_*$-embedded in $G$.*
2. *All cyclic subgroups of $G_p$ of prime order or of order 4, that are without a supersolvable supplement in $G$, are $S_*$-embedded in $G$.*

*Then, $G$ is supersolvable.*

Our main object in this paper is to go further in studying the influence of $S_*$-embedded subgroups on the group structure. In fact, we prove:

**Theorem B** *Suppose $A \trianglelefteq G$ such that $G/A$ is $p$-nilpotent for some prime divisor $p$ of $|G|$. If $A$ has a Sylow $p$-subgroup $A_p$ such that $N_G(A_p)$ is $p$-nilpotent and all maximal subgroups of $A_p$, that are without a $p$-nilpotent supplement in $G$, are $S_*$-embedded in $G$, then $G$ is $p$-nilpotent.*

**Theorem C** *Suppose $A \trianglelefteq G$ such that $G/A$ is supersolvable. If all maximal subgroups of each non-cyclic Sylow subgroup of $A$, that are without a supersolvable supplement in $G$, are $S_*$-embedded in $G$, then $G$ is supersolvable.*

**Theorem D** *Suppose $A \trianglelefteq G$ such that $G/A$ is supersolvable. If all cyclic subgroups of each non-cyclic Sylow subgroup of $A$ of prime order or of order 4, that are without a supersolvable supplement in $G$, are $S_*$-embedded in $G$, then $G$ is supersolvable.*

## 2 Preliminaries

**Lemma 1** ([9, Theorem 3.2]) *Assume $p$ is some prime divisor of $|G|$ satisfying $(|G|, p-1) = 1$, and $G_p$ is a Sylow $p$-subgroup of $G$. If all maximal subgroups of $G_p$, that are without a $p$-nilpotent supplement in $G$, are $S_*$-embedded in $G$, then $G$ is $p$-nilpotent.*

**Lemma 2** ([9, Lemma 2.5]) *Let $A$ be an $S_*$-embedded $p$-subgroup of $G$.*

1. *If $A \leq B \leq G$, then $A$ is $S_*$-embedded in $B$.*
2. *If $L \trianglelefteq G$ and $L \leq A$, then $A/L$ is $S_*$-embedded in $G/L$.*
3. *If $L \trianglelefteq G$ and $(|L|, |A|) = 1$, then $AL/L$ is $S_*$-embedded in $G/L$.*
4. *If $L \trianglelefteq G$ and $A \leq L$, then $G$ has an $S$-permutable subgroup $F$ contained in $L$ such that $AF$ is $S$-permutable in $G$ and $A \cap F \leq A_*$.*

**Lemma 3** ([12, Theorem A]) *Suppose that $A$ is an $S$-permutable $p$-subgroup of $G$. Then $O^p(G)$ is contained in $N_G(A)$.*

**Lemma 4** ([17, Lemma 2.1(d)]) *Suppose that $A$ is a $p$-subgroup of $G$ and $B$ is a normal $p$-subgroup of $G$. If $A$ is either $S$-permutably embedded or $S$-semipermutable in $G$, then $A \cap B$ is also $S$-permutable in $G$.*

**Lemma 5** *Let $A$ be a subgroup of $G$.*

1. [3] *If $A$ is $S$-permutable in $G$, then $A/A_G$ is nilpotent.*
2. [8] *If $A$ is $S$-permutable in $G$, then $A$ is subnormal in $G$.*
3. [16] *If $A$ is a subnormal $p$-subgroup of $G$, then $A$ is contained in $O_p(G)$.*
4. [13] *$A_{SG}$ is an $S$-permutable subgroup of $G$, where $A_{SG}$ is a subgroup of $A$ generated by all $S$-permutable subgroups of $G$ that contained in $A$.*
5. [12] *If $A$ and $B$ are $S$-permutable subgroups of $G$, then $A \cap B$ is also $S$-permutable in $G$.*

**Lemma 6** ([10, 18]) *Suppose that $A$ is either $S$-permutably embedded or $S$-semipermutable in $G$. If $A$ is a $p$-subgroup contained in $O_p(G)$, then $A$ is $S$-permutable in $G$.*

**Lemma 7** ([9, Lemma 2.1(3)]) *Suppose that $p$ is some prime divisor of $|G|$ satisfying $(|G|, p-1) = 1$. If $G$ is $p$-supersolvable, then $G$ is $p$-nilpotent.*

**Lemma 8** ([15, Lemma 2.4]) *If $A$ is a maximal subgroup of $G$ and $B$ is a normal $p$-subgroup of $G$ such that $G = AB$, then $A \cap B \trianglelefteq G$.*

**Lemma 9** ([11, Theorem 3.5]) *If $A \trianglelefteq G$ such that $G/A$ is supersolvable and all maximal subgroups of any Sylow subgroup of $A$ are normal in $G$, then $G$ is supersolvable.*

**Lemma 10** ([8]) *Let $A$ be an $S$-permutable subgroup of $G$.*

1. *If $A \leq B \leq G$, then $A$ is $S$-permutable in $B$.*
2. *If $L \trianglelefteq G$, then $AL/L$ is $S$-permutable in $G/L$.*

**Lemma 11** ([5]) *Suppose that $A$ and $B$ are normal supersolvable subgroups of $G$ with $G = AB$. If the indices $|G : A|$ and $|G : B|$ are relatively prime, then $G$ is supresolvable.*

**Lemma 12** ([16]) *If $A$ is a subnormal subgroup of $G$ such that the number $|G : A|$ is not divisible by $p$, then every Sylow $p$-subgroup $G_p$ of $G$ is contained in $A$.*

## 3 Proofs

**Theorem 1** *Suppose $G_p$ is a Sylow $p$-subgroup of $G$ for some prime divisor $p$ of $|G|$. If $N_G(G_p)$ is $p$-nilpotent and all maximal subgroups of $G_p$, that are without a $p$-nilpotent supplement in $G$, are $S_*$-embedded in $G$, then $G$ is $p$-nilpotent.*

***Proof*** If $p = 2$, then, by Lemma 1, $G$ is $p$-nilpotent. So, it can be assumed that $p > 2$. Suppose the result is not true providing $G$ as a counterexample of minimal order. Following the method of the first part of the proof of [6, Theorem 2.3], we conclude that all maximal subgroups of $G_p$ are $S_*$-embedded in $G$. Now, we build up the proof by the following steps:

(1) $O_{p'}(G) = 1$.
Assume that $O_{p'}(G) \neq 1$. Let $X/O_{p'}(G)$ be a maximal subgroup of the Sylow $p$-subgroup $G_pO_{p'}(G)/O_{p'}(G)$ of $G/O_{p'}(G)$. Then $G_p$ has a maximal subgroup $X_1$ such that $X = X_1O_{p'}(G)$. Since $X_1$ is $S_*$-embedded in $G$, it follows that $X_1O_{p'}(G)/O_{p'}(G)$ is $S_*$-embedded in $G/O_{p'}(G)$ by Lemma 2(3). Also, we have $N_{G/O_{p'}(G)}(G_pO_{p'}(G)/O_{p'}(G)) = N_G(G_p)O_{p'}(G)/O_{p'}(G)$ is $p$-nilpotent. Our choice of $G$ implies that $G/O_{p'}(G)$ is $p$-nilpotent. Consequently, $G$ is $p$-nilpotent which is a contradiction.

(2) If $G_p \leq A < G$, then $A$ is $p$-nilpotent.
Clearly, $N_A(G_p) \leq N_G(G_p)$ is $p$-nilpotent, and by Lemma 2(1) all maximal subgroups of $G_p$ are $S_*$-embedded in $A$. The minimal choice of $G$ implies that $A$ is $p$-nilpotent.

(3) $G$ is $p$-solvable.
By Thompson's result [14, Corollary], $G_p$ has a non-trivial characteristic subgroup $R$ such that $N_G(R)$ is not $p$-nilpotent. Let $L$ be any characteristic subgroup of $G_p$ such that $O_p(G) < L \leq G_p$. Since $L$ char $G_p \trianglelefteq N_G(G_p)$, we have $L \trianglelefteq N_G(G_p)$. Hence $G_p \leq N_G(G_p) \leq N_G(L) < G$ which implies that $N_G(L)$ is $p$-nilpotent by step (2). If $O_p(G) = 1$, then $1 < R \leq G_p$ and hence $N_G(R)$ is $p$-nilpotent; a contradiction. Thus, we may assume that $O_p(G) \neq 1$. Clearly, $N_{G/O_p(G)}(G_p/O_p(G)) = N_G(G_p)/O_p(G)$ is $p$-nilpotent. By using Lemma 2(2), it is easy to see that $G/O_p(G)$ satisfies the hypothesis of the theorem. Our choice of $G$ implies that $G/O_p(G)$ is $p$-nilpotent, and thereby $G$ is $p$-solvable.

(4) There exists a unique minimal normal subgroup $L$ of $G$ such that $G/L$ is $p$-nilpotent and $G = L \rtimes X$, where $X$ is a maximal subgroup of $G$. Moreover, $L = C_G(L) = F(G) = O_p(G)$.
Let $L$ be a minimal normal subgroup of $G$. Then $L$ is an elementary abelian $p$-group by steps (1) and (3). This implies $L \leq O_p(G) \leq G_p$. If $L = G_p$, then clearly, $G/L$ is $p$-nilpotent. Thus, we may assume that $L < G_p$. In view of Lemma 2(2), we can see that the hypothesis still holds for $G/L$. Our choice of $G$ implies that $G/L$ is also $p$-nilpotent. Since the class of all $p$-nilpotent groups is a saturated formation, it follows that $L$ is the unique minimal normal subgroup of $G$ and $L \nleq \Phi(G)$. Hence, there exists a maximal subgroup $X$ of $G$ such that $G = L \rtimes X$, $L = C_G(L) = F(G) = O_p(G)$.

(5) Finishing the proof.
Let $X_p$ be a Sylow $p$-subgroup of $X$ such that $G_p = LX_p$, and $P_1$ be a maximal subgroup of $G_p$ containing $X_p$. Then $G_p = LX_p = LP_1$. We may assume that $L \neq G_p$ (Otherwise, $G = N_G(L) = N_G(G_p)$ is $p$-nilpotent; a contradiction). Since $P_1$ is $S_*$-embedded in $G$, then there exists an $S$-permutable subgroup $F$ of $G$ such that $P_1F$ is $S$-permutable in $G$ and $(P_1 \cap F) \leq (P_1)_*$. If $F = 1$, then $P_1$ is $S$-permutable in $G$ which implies that $P_1 \trianglelefteq G_pO^p(G) = G$ by using Lemma 3 and since $P_1 \neq 1$, we have $L \leq P_1$ by step (4) which means $P_1 = G_p$; a contradiction. Thus, we may assume that $F \neq 1$. Assume that $F_G \neq 1$. Then $L \leq F_G \leq F$ which implies that $P_1 \cap L \leq P_1 \cap F \leq (P_1)_* \cap L$ and so $P_1 \cap L = (P_1)_* \cap L$. Lemma 4 yields $P_1 \cap L$ is $S$-permutable in $G$. If $P_1 \cap L \neq 1$, then $L \leq (P_1 \cap L)^{O^p(G)G_p} \leq (P_1)^{G_p} = P_1$ by Lemma 3; a contradiction. So, assume $P_1 \cap L = 1$. This implies that $|L| = p$. Since $X \cong G/L = G/C_G(L) \lesssim Aut(L)$,

we have $|X| \mid (p-1)$. It follows that $L$ is a Sylow $p$-subgroup of $G$. Hence $L = G_p$; a contradiction. Thus, $F_G = 1$. In view of Lemma 5(1), we have $F$ is a nilpotent group. Then from step (1), $F$ is a $p$-subgroup. By Lemma 5(2) and Lemma 5(3), we have $(P_1)_* \leq P_1 \leq P_1 F \leq O_p(G)$ which implies that $(P_1)_* \leq (P_1)_{sG}$ by using Lemma 6 and Lemma 5(4). Now, we have $P_1 \cap F \leq (P_1)_* \leq (P_1)_{sG}$ which implies $P_1 \cap F \leq (P_1)_{sG} \cap F$. Hence, $P_1 \cap F = (P_1)_{sG} \cap F$. In view of Lemma 5(5), we have $P_1 \cap F$ is $S$-permutable in $G$. If $P_1 \cap F = 1$, then $|F| = p$. It follows that $P_1 F$ is a Sylow $p$-subgroup of $G$. Hence, $P_1 F = G_p = O_p(G) = L$; a contradiction. Thus, it can be assumed that $P_1 \cap F \neq 1$. Then, by using Lemma 3 and step (4), we have $L \leq (P_1 \cap F)^G = (P_1 \cap F)^{O^p(G)G_p} \leq (P_1)^{G_p} = P_1$. Hence $G_p = LP_1 = P_1$; a final contradiction. □

***Proof of Theorem B.*** Assume the result is not true providing $G$ as a counterexample of minimal order. Lemma 2(1) with Theorem 1, imply $A$ is $p$-nilpotent. Let $A_{p'}$ be the normal $p$-complement of $A$. Since $A_{p'}$ char $A \trianglelefteq G$, then $A_{p'} \trianglelefteq G$. Assume that $|A_{p'}| > 1$. Clearly, $A/A_{p'} \trianglelefteq G/A_{p'}$ and $(G/A_{p'})/(A/A_{p'}) \cong G/A$ is $p$-nilpotent. Let $X/A_{p'}$ be a maximal subgroup of the Sylow $p$-subgroup $A_p A_{p'}/A_{p'}$ of $A/A_{p'}$. Then $A_p$ has a maximal subgroup $X_1$ such that $X = X_1 A_{p'}$. If $X_1$ possesses a $p$-nilpotent supplement $D$ in $G$, then $DA_{p'}/A_{p'}$ is a $p$-nilpotent supplement of $X_1 A_{p'}/A_{p'}$ in $G/A_{p'}$. If $X_1$ is $S_*$-embedded in $G$, it follows that $X_1 A_{p'}/A_{p'}$ is $S_*$-embedded in $G/A_{p'}$ by using Lemma 2(3). Our choice of $G$ implies that $G/A_{p'}$ is $p$-nilpotent. Consequently, $G$ is $p$-nilpotent which is a contradiction. Thus, it can be assumed $A_{p'} = 1$. Then $A = A_p$, which implies $G = N_G(A) = N_G(A_p)$ is $p$-nilpotent; a final contradiction. □

***Remark 1*** The condition $N_G(G_p)$ is $p$-nilpotent in Theorem 1 and Theorem B is necessary. For example, consider $G = A_5$ and $p = 3$. Then all maximal subgroups of any Sylow 3-subgroup $G_3$ of $G$ are $S_*$-embedded in $G$, but $G$ is not 3-nilpotent.

We work toward the proof of Theorem C:

**Theorem 2** *Suppose that $p$ is some prime divisor of $|G|$ satisfies $(|G|, p-1) = 1$, and $A \trianglelefteq G$ such that $G/A$ is $p$-nilpotent. If $A$ has a Sylow $p$-subgroup $A_p$ such that all maximal subgroups of $A_p$, that are without a $p$-nilpotent supplement in $G$, are $S_*$-embedded in $G$, then $G$ is $p$-nilpotent.*

***Proof*** Assume the result is not true providing $G$ as a counterexample of minimal order. By Lemma 2(1) and Lemma 1, we have that $A$ is $p$-nilpotent. Let $A_{p'}$ be the normal $p$-complement of $A$. Since $A_{p'}$ char $A \trianglelefteq G$, then $A_{p'} \trianglelefteq G$. Assume that $|A_{p'}| > 1$. Clearly, $A/A_{p'} \trianglelefteq G/A_{p'}$ and $(G/A_{p'})/(A/A_{p'}) \cong G/A$ is $p$-nilpotent. Let $X/A_{p'}$ be a maximal subgroup of the Sylow $p$-subgroup $A_p A_{p'}/A_{p'}$ of $A/A_{p'}$. Then $A_p$ has a maximal subgroup $X_1$ such that $X = X_1 A_{p'}$. If $X_1$ possesses a $p$-nilpotent supplement $D$ in $G$, then $DA_{p'}/A_{p'}$ is a $p$-nilpotent supplement of $X_1 A_{p'}/A_{p'}$ in $G/A_{p'}$. If $X_1$ is $S_*$-embedded in $G$, then $X_1 A_{p'}/A_{p'}$ is $S_*$-embedded in $G/A_{p'}$ by Lemma 2(3). So $G/A_{p'}$ is $p$-nilpotent due to the minimal choice of $G$. It follows, $G$ is $p$-nilpotent; a contradiction. Thus, it can be assumed $A_{p'} = 1$ which yields $A = A_p$ is a $p$-group. Let $L/A_p$ be the normal $p$-complement of $G/A_p$. Schur-Zassenhaus Theorem implies that $L$ has a Hall $p'$-subgroup $L_{p'}$ such that $L = A_p \rtimes L_{p'}$. Since $L$ is $p$-nilpotent by Lemma 2(1) and Lemma 1, it follows that $L = A_p \times L_{p'}$. Hence, we have $L_{p'}$ is the normal $p$-complement of $G$, and thereby $G$ is $p$-nilpotent; a contradiction. □

Lemma 7 and Theorem 2 lead to the following corollary:

**Corollary 1** *Suppose $A \trianglelefteq G$ provided $G/A$ is $p$-nilpotent, where $p$ is the smallest prime divisor of $|G|$. If $A$ has a Sylow $p$-subgroup $A_p$ such that all maximal subgroups of $A_p$ that are without a $p$-supersolvable supplement in $G$ are $S_*$-embedded in $G$, then $G$ is $p$-nilpotent.*

***Proof*** Clearly, $(|G|, p-1) = 1$ as $p$ is the smallest prime divisor of $|G|$. Lemma 7 suggests all maximal subgroups of $A_p$ that are without a $p$-nilpotent supplement in $G$ are $S_*$-embedded in $G$. Using Theorem 2, gives $G$ is $p$-nilpotent. □

Now we can prove Theorem C:

***Proof of Theorem C.*** Assume the result is not true providing $G$ as a counterexample of minimal order. In view of Theorem A(1), we have $A$ is supersolvable. Let $G_p$ be a Sylow $p$-subgroup of $G$, where $p$ is the largest prime divisor of $|G|$. We distinguish two cases.

Case 1. $G_p \leq A$.
Then $G_p \trianglelefteq A$ as $A$ is supersolvable. Since $G_p$ char $A \trianglelefteq G$, it follows that $G_p \trianglelefteq G$. Now, we show that $G/G_p$ is supersolvable. Clearly, $(A/G_p) \trianglelefteq (G/G_p)$ and $(G/G_p)/(A/G_p) \cong (G/A)$ is supersolvable. Let $X/G_p$ be a maximal subgroup of the Sylow $q$-subgroup $A_q G_p/G_p$ of $A/G_p$. Then $A_q$ has a maximal subgroup $X_1$ such that $X = X_1 G_p$. If $X_1$ has a supersolvable supplement $B$ in $G$, then $BG_p/G_p$ is a supersolvable supplement of $X_1 G_p/G_p$ in $G/G_p$. If $X_1$ is $S_*$-embedded in $G$, then $X_1 G_p/G_p$ is $S_*$-embedded in $G/G_p$ by using Lemma 2(3). The minimal choice of $G$ yields $G/G_p$ is supersolvable and $G_p$ is not cyclic. Let $N$ be a minimal normal subgroup of $G$ contained in $G_p$. It is also easy to see that $G/N$ is supersolvable. Further, since the class of all supersolvable groups is a saturated formation, it follows that $N$ is the unique minimal normal subgroup of $G$ contained in $G_p$ and $N \not\leq \Phi(G)$. Hence $G$ possesses a maximal subgroup $X$ such that $G = NX$ and $N \cap X = 1$. Since $G_p \cap X$ is normalized by $G$ by using Lemma 8, it follows that $N = G_p$ which yields $G_p$ is an elementray abelian $p$-group. Now, Let $N_1$ be a maximal subgroup of $N$. If $N_1$ has a supersolvable supplement $B$ in $G$, then $G = N_1 B = NB$ and $N = N \cap N_1 B = N_1(N \cap B)$, which implies that $N \cap B \neq 1$. Since $N \cap B \trianglelefteq G$ and $N$ is a minimal normal subgroup of $G$, we have $N \cap B = N$. Consequently, $N \leq B$ which implies that $G = B$ is supersolvable which is a contradiction. Thus, we can assume that $N_1$ is $S_*$-embedded in $G$. In view of Lemma 2(4), $G$ possesses an $S$-permutable subgroup $F$ contained in $G_p$ such that $N_1 F$ is $S$-permutable in $G$ and $N_1 \cap F \leq (N_1)_*$. If $F = 1$, then $N_1$ is $S$-permutable in $G$ and $N_1 \trianglelefteq G_p O^p(G) = G$ by using Lemma 3 and so $|G_p| = p$; a contradiction. Thus, $F \neq 1$. Since $G_p$ is an elemntary abelian $p$-group, then $F \trianglelefteq G_p$. Applying Lemma 3 again, we get $F \trianglelefteq G_p O^p(G) = G$.

This implies $F = G_p = N$. Hence $N_1 \cap F = N_1 = (N_1)_*$, is $S$-permutable in $G$ by Lemma 6; again a contradiction as above.

Case 2. $G_p \nleq A$.
In this case we distinguish the following two subcases.

Subcase (i). $G_pA < G$.
Clearly, $A \trianglelefteq G_pA$ and $G_pA/A \cong G_p/G_p \cap A$ is supersolvable. By Lemma 2(1), all maximal subgroups of any non-cyclic Sylow subgroup of $A$ not having a supersolvable supplement in $G$ are $S_*$-embedded in $G_pA$. Therefore, $G_pA$ is supersolvable due to the minimal choice of $G$. Since $G_pA/A$ is a Sylow $p$-subgroup of $G/A$, where $p$ is the largest prime divisor of $|G|$ and $G/A$ is supersolvable, it follows that $G_pA/A \trianglelefteq G/A$ and so $G_pA \trianglelefteq G$. Since $G_p$ char $G_pA \trianglelefteq G$, then $G_p \trianglelefteq G$. Hence $G_p \cap A \trianglelefteq G$, where $G_p \cap A$ is a Sylow $p$-subgroup of $A$. By using the same arguments as in Case 1, we have $G/G_p \cap A$ is supersolvable and $G_p \cap A$ is a minimal normal subgroup of $G$. Set $N = G_p \cap A$. By our choice of $G$ together with Lemma 9, $N$ has a maximal subgroup $N_1$ such that $N_1 \ntrianglelefteq G$. If $N_1$ possesses a supersolvable supplement $B$ in $G$, we have $G$ is supersolvable as in Case 1; a contradiction. Thus, we can assume that $N_1$ is $S_*$-embedded in $G$. In view of Lemma 2(4), there exists an $S$-permutable subgroup $F$ of $G$ contained in $N$ such that $N_1F$ is $S$-permutable in $G$ and $N_1 \cap F \leq (N_1)_*$. By the maximality of $N_1$ in $N$, we have either $N_1F = N_1$ or $N_1F = N$. If the former holds, then $N_1$ is $S$-permutable in $G$. So, $N_1 \trianglelefteq NO^p(A) = A$ and hence $A \leq N_G(N_1)$ by using Lemma 10(1) and Lemma 3. Applying Lemma 3 again, we get $O^p(G) \leq N_G(N_1) < G$. Hence $G$ possesses a maximal subgroup $X$ such that $O^p(G) \leq N_G(N_1) \leq X < G$ with $|G : X| = p$. Since $X/O^p(G) \trianglelefteq G/O^p(G)$, then $X \trianglelefteq G$. Due to the minimal choice of $G$, we have $X$ is supersolvable. Therefore, $G = G_pX$ and consequently $G$ is supersolvable by using Lemma 11; a contradiction. Thus, we can assume that $N_1F = N$. If $F \trianglelefteq G$, then $F = N$ which implies $N_1 \cap F = N_1 = (N_1)_*$ is $S$-permutable in $G$ by Lemma 6; a contradiction as above. Thus, $F \ntrianglelefteq G$. Since $N$ is an elementary abelian $p$-group, then $F \trianglelefteq N$. Hence $F \trianglelefteq NO^p(A) = A$ by Lemma 10(1) and Lemma 3 and so $A \leq N_G(F)$. Applying Lemma 3 again, there exists a maximal normal subgroup $X$ of $G$ such that $O^p(G) \leq N_G(F) \leq X < G$ with $|G : X| = p$. It follows that $X$ is supersolvable by our choice of $G$. So, $G = G_pX$ and consequently $G$ is supersolvable again by Lemma 11; a contradiction.

Subcase (ii). $G = G_pA$.
Lemma 12 and Corollary 1 yield $A$ contains all Sylow $q$-subgroups of $G$ with $q \neq p$ and $G$ is a $q_r$-nilpotent, where $q_r$ is the smallest prime divisor of $|G|$. This implies that $G$ has a Sylow tower group of supersolvable type. Therefore, $G_p \trianglelefteq G$ as $p$ is the largest prime divisor of $|G|$. Applying Lemma 11, $G$ is supersolvable; a final contradiction.□

In order to show Theorem D, we need the following useful theorem:

**Theorem 3** *Suppose $A$ is a normal $p$-subgroup of $G$ such that $G/A$ is supersolvable. If all cyclic subgroups of $A$ of order $p$ or of order 4 not having a supersolvable supplement in $G$ are $S_*$-embedded in $G$, then $G$ is supersolvable.*

***Proof*** Assume the result is not true providing $G$ as a counterexample of minimal order. It is easy to see that all proper subgroups of $G$ are supersolvable by using Lemma 2(1). Hence $G$ is a minimal non-supersolvable group. By Doerk's result [4] $G$ has a Sylow $p$-subgroup $G_p$ such that $G_p \trianglelefteq G$ for a prime divisor $p$ of $|G|$, $G_p/\Phi(G_p)$ is a minimal normal subgroup of $G/\Phi(G_p)$, and the exponent of $G_p$ is either $p$ or 4. Clearly, $A \leq G_p$ (Otherwise, $G \cong G/G_p \cap A \lesssim G/G_p \times G/A$ is supersolvable; a contradiction). We build up the proof by the following two steps.

(1) $A = G_p$.
Since $A\Phi(G_p)/\Phi(G_p) \trianglelefteq G/\Phi(G_p)$, we have either $A\Phi(G_p) = G_p$ or $A \leq \Phi(G_p)$. If the latter holds, then $G/\Phi(G_p)$ is supersolvable. It follows that, from $\Phi(G_p) \leq \Phi(G)$, $G/\Phi(G)$ is supersolvable and so $G$ is also supersolvable by a well-known result of Huppert [7, p.713]; a contradiction. Thus, $A\Phi(G_p) = G_p$ and thereby $A = G_p$ as required.

(2) Finishing the proof.
Assume that $|A/\Phi(A)| = p$. Then there exists $x$ in $A$ such that $A/\Phi(A) =< x\Phi(A) >$ which implies that $A$ is cyclic and consequently $G$ is supersolvable which is a contradiction. So, $|A/\Phi(A)| = p^n$, $n > 1$ and $A/\Phi(A) =< x_1\Phi(A), x_2\Phi(A), ..., x_n\Phi(A) >$ as $A/\Phi(A)$ is an elementary abelian $p$-group. Hence, we have $A =< x_1, x_2, ..., x_n >$. Set $A_i =< x_i >$ for all $i = 1, 2, ..., n$. So, we have $|A_i| = p$ or 4. Now, the hypothesis of the theorem assures that $A_i$ either has a supersolvable supplement in $G$ say $B$ or $A_i$ is $S_*$-embedded in $G$. If $A_i$ is not $S_*$-embedded in $G$, then $G = A_iB$ and so $A = A \cap G = A \cap A_iB = A_i(A \cap B)$. Obviously, $(A \cap B)\Phi(A)/\Phi(A) \trianglelefteq G/\Phi(A)$ as $A/\Phi(A)$ is abelian. In view of step (1), $A/\Phi(A)$ is a minimal normal subgroup of $G/\Phi(A)$ which implies that either $(A \cap B)\Phi(A) = A$ or $(A \cap B) \leq \Phi(A)$. If the latter holds, then $A = A_i$ is cyclic and so $G$ is supersolvable; a contradiction. Hence $(A \cap B)\Phi(A) = A$ and so $A \cap B = A$ which implies that $G = B$ is supersolvable; contradicts our choice of $G$. Thus, we can assume that $A_i$ is $S_*$-embedded in $G$. In view of Lemma 2(4), then $G$ possesses an $S$-permutable subgroup $F$ contained in $A$ such that $A_iF$ is also $S$-permutable in $G$ and $A_i \cap F \leq (A_i)_*$. By Lemma 10(2) and the fact that $A/\Phi(A)$ is abelian, it is easy to see that $F\Phi(A)/\Phi(A)$ is $S$-permutable in $G/\Phi(A)$ and $F\Phi(A)/\Phi(A) \trianglelefteq A/\Phi(A)$. Applying Lemma 3, we have $F\Phi(A)/\Phi(A) \trianglelefteq (A/\Phi(A))(O^p(G/\Phi(A)) = G/\Phi(A)$. Again, the minimal normality of $A/\Phi(A)$ in $G/\Phi(A)$ implies that either $F\Phi(A) = A$ or $F\Phi(A) \leq \Phi(A)$. If the latter holds, $A_i\Phi(A)/\Phi(A) = A_iF\Phi(A)/\Phi(A)$ is $S$-permutable in $G/\Phi(A)$ by using Lemma 10(2). If $F\Phi(A) = A$, then we have $F = A$. Therefore, $A_i \cap F = A_i = (A_i)_*$ is $S$-permutable in $G$ by Lemma 6, and so $A_i\Phi(A)/\Phi(A)$ is $S$-permutable in $G/\Phi(A)$ by Lemma 10(2). By [13, Lemma 2.11], there exists a maximal subgroup $X\Phi(A)/\Phi(A)$ of $A/\Phi(A)$ such that $X\Phi(A)/\Phi(A) \trianglelefteq G/\Phi(A)$; a final contradiction. □

***Proof of Theorem D.*** Assume the result is not true providing $G$ as a counterexample of minimal order. If the order of $A$ is of prime power, then $G$ is supersolvable by

Theorem 3; a contradiction. Thus, we can assume that the order of $A$ is divisible by at least two distinct primes. By Lemma 2(1) and Theorem A(2), we have $A$ is supersolvable. Hence $A$ possesses a normal Sylow $p$-subgroup $A_p$, where $p$ is the largest prime divisor of $|A|$. Since $A_p$ char $A \trianglelefteq G$, we have $A_p \trianglelefteq G$. Let $U/A_p$ be a cyclic subgroup of the Sylow $q$-subgroup $A_q A_p/A_p$ of $A/A_p$ such that $|U/A_p| = q$ or 4. Then $A_q$ has a cyclic subgroup $R$ such that $U = RA_p$ and $|R| = q$ or 4. If $R$ has a supersolvable supplement $B$ in $G$, then $BA_p/A_p$ is a supersolvable supplement of $RA_p/A_p$ in $G/A_p$. If $R$ is $S_*$-embedded in $G$, then $RA_p/A_p$ is $S_*$-embedded in $G/A_p$ by using Lemma 2(3). Our choice of $G$ yields $G/A_p$ is supersolvable. Applying Theorem 3, we get $G$ is supersolvable; a contradiction. □

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## Declarations

**Conflicts of interest** On behave of all authors, the corresponding author states that there is no conflict of interest.

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