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Bifurcation analysis in a diffusive predator–prey system with nonlinear growth rate and protection zone

Wenbin Yang[1]

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Abstract
This paper is concerned with the steady-state bifurcations arising from a reaction–diffusion predator–prey system with nonlinear growth rate and a protection zone. Some sufficient conditions for the existence of positive steady-state solutions are given. Our proof is based on the local and global bifurcation theory and some a priori estimates.

Keywords Reaction–diffusion system · Predator–prey interaction · Protection zone · Steady-state bifurcation

Mathematics Subject Classification 35K57 · 35B32 · 92D25

1 Introduction

In this paper, we will investigate the following predator-prey model with a protection zone:

$$\begin{cases} u_t = \Delta u + u(a-u) - \dfrac{b(x)uv}{1+mu}, \quad x \in \Omega, t > 0, \\ v_t = \Delta v + v\left(\dfrac{\alpha}{1+\beta v} - d\right) + \dfrac{c(x)uv}{1+mu}, \quad x \in \Omega \backslash \overline{\Omega}_0, t > 0, \\ \partial_\nu u = 0, \quad x \in \partial\Omega, t > 0, \quad \partial_\nu v = 0, \quad x \in \partial(\Omega \backslash \overline{\Omega}_0), t > 0, \\ u(x,0) = u_0(x) \geq 0, \quad x \in \Omega, \quad v(x,0) = v_0(x) \geq 0, \quad x \in \Omega \backslash \overline{\Omega}_0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in the Euclidean space $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$ and Ω_0 is a subdomain of Ω with smooth boundary $\partial\Omega_0$. $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator in $\mathbb{R}^N$, $\partial_\nu = \partial/\partial\nu$ and ν is the unit outer normal vector

✉ Wenbin Yang
yangwenbin-007@163.com

1 School of Science, Xi'an University of Posts and Telecommunications, Xi'an 710121, Shaanxi, China

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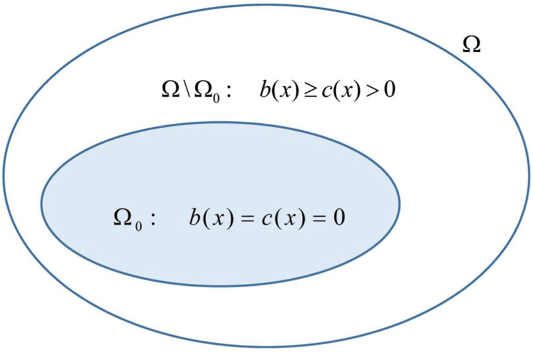

Fig. 1 The diagrammatic sketches of $b(x)$ and $c(x)$ in (1.1)

on $\partial\Omega$ or $\partial(\Omega\backslash\overline{\Omega}_0)$. The homogeneous Neumann boundary condition is assumed so that no individual crosses the habitat boundary. In addition, a, m, d, α, β are positive constants; $b(x) \in L^\infty(\Omega)$, $b(x) \geq 0$ in Ω, $b(x) \equiv 0$ on $\overline{\Omega}_0$ and for any compact subset A of $\Omega\backslash\overline{\Omega}_0$, there exists $\delta_A > 0$ such that

$$\delta_A \leq b(x), \quad \forall x \in A;$$

$c(x) \in L^\infty(\Omega\backslash\overline{\Omega}_0)$ and $0 < c(x) \leq b(x)$ in $\Omega\backslash\overline{\Omega}_0$.

In (1.1), the predator species v cannot enter the subregion Ω_0 of the habitat Ω, whereas the prey species u can enter and leave Ω_0 freely. Namely, Ω_0 is a predation-free zone for the prey species and such a subregion Ω_0 is called a protection zone. One can think that there is a barrier along $\partial\Omega_0$ that blocks the predator but not the prey (see [1, 2] for further details). The diagrammatic sketches of $b(x)$ and $c(x)$ in (1.1) could be shown by Fig. 1.

To present the main results, we collect some basic notations and well-known results. Let $q(x) \in C(\overline{\Omega})$, and denote $\lambda_1^D(q, O)$ and $\lambda_1^N(q, O)$ to be the first eigenvalues of $-\Delta + q(x)$ in O subject to the homogeneous Dirichlet boundary condition and Neumann boundary condition, respectively [3–5]. It is well known that $\lambda_1^N(q, O)$ is increasing in q, that is, $\lambda_1^N(q_1, O) < \lambda_1^N(q_2, O)$ if $q_1, q_2 \in C(\overline{O}), q_1 \leq q_2$ and $q_1 \not\equiv q_2$ on $\overline{O}$. A similar assertion is also true for $\lambda_1^D(q, O)$. Moreover,

$$\lambda_1^N(q, O) < \lambda_1^D(q, O) \text{ for } q \in C(\overline{O}),$$

and

$$\lambda_1^N(q, \Omega) < \lambda_1^N(q, \Omega_0), \lambda_1^D(q, \Omega) < \lambda_1^D(q, \Omega_0) \text{ with } q \in C(\overline{\Omega}) \text{ and } \Omega_0 \subset\subset \Omega.$$

For convenience, we denote $\lambda_1^D(O) = \lambda_1^D(0, O)$ and $\lambda_1^N(O) = \lambda_1^N(0, O)$.

It is well known that the effects of the protection zone on the dynamical behavior are significantly different from non-protection zone case [6–8]. Many theoretical results show that there exists a critical patch size Ω_0 for the protection zone. If Ω_0 is below this size, each model behaves similarly to the non-protection zone case, but every model undergoes profound changes in dynamical behavior once Ω_0 is above the critical patch size, and in such a case, the endangered species is always saved from extinction.

In this paper, we study the existence of positive stationary solutions of (1.1) with a protection zone, and mainly investigate the effect of nonlinear growth rate $\left(\dfrac{\alpha}{1+\beta v}\right)$

on the existence of bifurcation of stationary positive solutions. The stationary problem associated with (1.1) is given by

$$\begin{cases} \Delta u + u(a-u) - \dfrac{b(x)uv}{1+mu} = 0, \quad x \in \Omega, \\ \Delta v + v\left(\dfrac{\alpha}{1+\beta v} - d\right) + \dfrac{c(x)uv}{1+mu} = 0, \quad x \in \Omega\backslash\overline{\Omega}_0, \\ \partial_\nu u = 0, \quad x \in \partial\Omega, \quad \partial_\nu v = 0, \quad x \in \partial(\Omega\backslash\overline{\Omega}_0). \end{cases} \tag{1.2}$$

For convenience, $\Omega\backslash\overline{\Omega}_0$ will be remembered as Ω_1 below.

The paper is organized as follows. In Sect. 2, we will derive some a priori estimates of positive solutions to the semilinear system (1.2). In Sect. 3, we will obtain positive solutions of the semilinear system by using the local and global bifurcation theory. Finally, the bifurcation stability and global bifurcation are included in Sect. 4.

2 A priori estimates and asymptotic stability of semi-trivial solution

The following lemmas can be helpful to obtain the bounds of positive solutions to (1.2).

Lemma 2.1 (Maximum principle, [9, Proposition 2.2]) Assume that $g \in C(\overline{O} \times \mathbb{R})$.
(i) If $w \in C^2(O) \cap C^1(\overline{O})$ satisfies

$$\Delta w(x) + g(x, w(x)) \geq 0 \text{ in } O,\ \partial_\nu w \leq 0 \text{ on } \partial O,$$

and $w(x_0) = \max_{\overline{O}} w$, then $g(x_0, w(x_0)) \geq 0$.
(ii) If $w \in C^2(O) \cap C^1(\overline{O})$ satisfies

$$\Delta w(x) + g(x, w(x)) \leq 0 \text{ in } O,\ \partial_\nu w \geq 0 \text{ on } \partial O,$$

and $w(x_0) = \min_{\overline{O}} w$, then $g(x_0, w(x_0)) \leq 0$.

Theorem 2.2 *Suppose that (u, v) is a positive solution of (1.2). Then:*

(1) *If $\alpha < d$, then*

$$\max\{a - \|b(x)\|_\infty \tfrac{a(a+d-\alpha)}{d-\alpha}, 0\} \leq u \leq a \text{ in } \overline{\Omega},\ 0 < v \leq \tfrac{a(a+d-\alpha)}{d-\alpha} \text{ in } \overline{\Omega_1}.$$

(2) *If $\alpha \geq d$, then*

$$\max\{a - \|b(x)\|_\infty \tfrac{\alpha-d}{d\beta}, 0\} \leq u \leq a \text{ in } \overline{\Omega},\ \tfrac{\alpha-d}{d\beta} \leq v \text{ in } \overline{\Omega_1}.$$

Furthermore, if $d > \frac{a\|c(x)\|_\infty}{1+am}$, then $v \leq \frac{(\alpha-d)(1+am)+a\|c(x)\|_\infty}{\beta[d(1+am)-a\|c(x)\|_\infty]}$ in $\overline{\Omega_1}$.

Proof (1) Case $\alpha < d$. By adding two equations in (1.2), we have

$$\begin{aligned} -\Delta(u+v) &= u(a-u) + v\left(\frac{\alpha}{1+\beta v} - d\right) + [c(x) - b(x)]\frac{uv}{1+mu} \\ &\leq au + (\alpha - d)v = au - (d-\alpha)v \\ &= (a+d-\alpha)u - (d-\alpha)(u+v) \\ &\leq a(a+d-\alpha) - (d-\alpha)(u+v) \end{aligned}$$

By Lemma 2.1, $u+v \le \frac{a(a+d-\alpha)}{d-\alpha}$. Consequently, $v \le \frac{a(a+d-\alpha)}{d-\alpha}$. From the equation for u in (1.2), we get

$$-\Delta u = u(a-u) - \frac{b(x)uv}{1+mu} \ge u\left[a - u - \|b(x)\|_\infty \frac{a(a+d-\alpha)}{d-\alpha}\right],$$

and then the maximum principle implies that $u \ge a - \|b(x)\|_\infty \frac{a(a+d-\alpha)}{d-\alpha}$.

(2) Case $\alpha \ge d$. From the equation for v in (1.2), we obtain

$$-\Delta v = v\left(\frac{\alpha}{1+\beta v} - d\right) + \frac{c(x)uv}{1+mu} \ge v\left(\frac{\alpha}{1+\beta v} - d\right),$$

which implies that $v \ge \frac{\alpha-d}{d\beta}$. Similarly,

$$\begin{aligned} -\Delta u &= u(a-u) - \frac{b(x)uv}{1+mu} \ge u(a - u - \|b(x)\|_\infty \tfrac{\alpha-d}{d\beta}), \\ -\Delta v &= v\left(\frac{\alpha}{1+\beta v} - d\right) + \frac{c(x)uv}{1+mu} \le v\left(\frac{\alpha}{1+\beta v} - d + \frac{a\|c(x)\|_\infty}{1+am}\right). \end{aligned}$$

Then $u \ge a - \|b(x)\|_\infty \frac{\alpha-d}{d\beta}$, and $v \le \frac{(\alpha-d)(1+am)+a\|c(x)\|_\infty}{\beta[d(1+am)-a\|c(x)\|_\infty]}$ if $d > \frac{a\|c(x)\|_\infty}{1+am}$. □

Theorem 2.3 *Suppose that* (u, v) *is a positive solution of* (1.2). *Then:*

(1) *If* $\alpha < d$, *then* $\dfrac{\alpha(d-\alpha)}{(d-\alpha)(1+a\beta)+a^2\beta} \le d \le \alpha - \lambda_1^N\left(-\dfrac{ac(x)}{1+am}, \Omega_1\right)$.

(2) *If* $\alpha \ge d$, *then* $a \ge \lambda_1^N\left(\dfrac{(\alpha-d)b(x)}{d\beta(1+am)}, \Omega\right)$.

Proof Suppose that (u, v) is a positive solution of (1.2).

(1) Case $\alpha < d$. From the equation for v in (1.2), we obtain

$$-d = \lambda_1^N\left(-\frac{\alpha}{1+\beta v} - \frac{c(x)u}{1+mu}, \Omega_1\right).$$

By the monotonicity of eigenvalues,

$$\begin{aligned} -d &\le \lambda_1^N\left(-\frac{\alpha}{1+\beta\frac{a(a+d-\alpha)}{d-\alpha}}, \Omega_1\right) \\ &\le \lambda_1^N\left(-\frac{\alpha(d-\alpha)}{(d-\alpha)(1+a\beta)+a^2\beta}, \Omega_1\right) = -\frac{\alpha(d-\alpha)}{(d-\alpha)(1+a\beta)+a^2\beta}, \end{aligned}$$

and

$$\begin{aligned} -d &\ge \lambda_1^N\left(-\alpha - \frac{c(x)u}{1+mu}, \Omega_1\right) \\ &\ge \lambda_1^N\left(-\alpha - \frac{ac(x)}{1+am}, \Omega_1\right) = -\alpha + \lambda_1^N\left(-\frac{ac(x)}{1+am}, \Omega_1\right). \end{aligned}$$

Thus, we have

$$\frac{\alpha(d-\alpha)}{(d-\alpha)(1+a\beta)+a^2\beta} \le d \le \alpha - \lambda_1^N\left(-\frac{ac(x)}{1+am}, \Omega_1\right).$$

(2) Case $\alpha > d$. From the equation for v in (1.2), similar to the above, we obtain that

$$\begin{aligned} a &= \lambda_1^N\left(u + \frac{b(x)v}{1+mu}, \Omega\right) \\ &\ge \lambda_1^N\left(\frac{b(x)\frac{\alpha-d}{d\beta}}{1+am}, \Omega\right) = \lambda_1^N\left(\frac{(\alpha-d)b(x)}{d\beta(1+am)}, \Omega\right). \end{aligned}$$

□

Remark 2.4 The results given in Theorem 2.3 lead us directly to the conclusions that:

(1) Assume that $\alpha < d$. If $d < \frac{\alpha(d-\alpha)}{(d-\alpha)(1+a\beta)+a^2\beta}$ or $d > \alpha - \lambda_1^N\left(-\frac{ac(x)}{1+am}, \Omega_1\right)$, then (1.2) has no positive solution.

(2) Assume that $\alpha \ge d$. If $a < \lambda_1^N\left(\frac{(\alpha-d)b(x)}{d\beta(1+am)}, \Omega\right)$, then (1.2) has no positive solution.

Now we start our analysis with a standard linearization argument. For any $\alpha > 0$, (1.2) has two semi-trivial solutions: $(a, 0)$ and $(0, \frac{\alpha-d}{d\beta})$ if $\alpha > d$. From the strong maximum principle, any non-negative solution (u, v) of (1.2) is either $(0, 0)$, or semi-trivial, or positive.

The linearized system of (1.2) about the equilibrium point (u, v) can be characterized by the Jacobian matrix

$$J(u,v) = \begin{pmatrix} \Delta + a - 2u - \frac{b(x)v}{(1+mu)^2} & -\frac{b(x)u}{1+mu} \\ \frac{c(x)v}{(1+mu)^2} & \Delta + \frac{\alpha}{(1+\beta v)^2} - d + \frac{c(x)u}{1+mu} \end{pmatrix}.$$

At the equilibrium point $(a, 0)$ and $(0, \frac{\alpha-d}{d\beta})$ if $\alpha > d$, the corresponding Jacobian matrix $J(u, v)$ are

$$J(a,0) = \begin{pmatrix} \Delta - a & -\frac{ab(x)}{1+am} \\ 0 & \Delta + \alpha - d + \frac{ac(x)}{1+am} \end{pmatrix},$$

and

$$J(0, \frac{\alpha-d}{d\beta}) = \begin{pmatrix} \Delta + a - \frac{\alpha-d}{d\beta}b(x) & 0 \\ \frac{\alpha-d}{d\beta}c(x) & \Delta + \frac{d(d-\alpha)}{\alpha} \end{pmatrix}.$$

Theorem 2.5 *We have the asymptotic stability results of* (1.2)*:*

(1) *The semi-trivial solution* $(a, 0)$ *is locally stable when* $\alpha < d+\lambda_1^N\left(-\dfrac{ac(x)}{1+am}, \Omega_1\right)$ *and unstable when* $\alpha > d + \lambda_1^N\left(-\dfrac{ac(x)}{1+am}, \Omega_1\right)$.

(2) *Assume that* $\alpha > d$. *The semi-trivial solution* $(0, \frac{\alpha-d}{d\beta})$ *is locally stable when* $a < \lambda_1^N(\frac{\alpha-d}{d\beta}b(x), \Omega)$ *and unstable when* $a > \lambda_1^N\left(\frac{\alpha-d}{d\beta}b(x), \Omega\right)$.

Proof (1) The linearized eigenvalue problem of (1.2) at $(a, 0)$ is

$$\begin{cases} \Delta h - ah - \dfrac{ab(x)}{1+am}k + \mu h = 0, \quad x \in \Omega, \qquad h = 0, \quad x \in \partial\Omega, \\ \Delta k + [\alpha - d + \dfrac{ac(x)}{1+am}]k + \mu k = 0, \quad x \in \Omega_1, \; k = 0, \quad x \in \partial\Omega_1, \end{cases} \tag{2.1}$$

where μ is eigenvalue and (h, k) is the corresponding eigenfunction.

If $k = 0$, then $h \neq 0$, and $\mu \geq a > 0$. If $k \neq 0$, then

$$\mu \geq \lambda_1^N(d - \alpha - \frac{ac(x)}{1+am}, \Omega_1) = d - \alpha + \lambda_1^N\left(-\frac{ac(x)}{1+am}, \Omega_1\right) > 0$$

when $\alpha < d + \lambda_1^N\left(-\dfrac{ac(x)}{1+am}, \Omega_1\right)$. This shows that $(a, 0)$ is locally stable when $\alpha < d + \lambda_1^N\left(-\dfrac{ac(x)}{1+am}, \Omega_1\right)$.

When $\alpha > d + \lambda_1^N\left(-\dfrac{ac(x)}{1+am}, \Omega_1\right)$, we assume that μ_0 and k_0 are the principal eigenvalue and the corresponding positive eigenfunction of

$$\Delta k + \left[\alpha - d + \frac{ac(x)}{1+am}\right]k + \mu k = 0, \quad x \in \Omega_1, \qquad k = 0, x \in \partial\Omega_1.$$

Then $\mu_0 = d + \lambda_1^N\left(-\dfrac{ac(x)}{1+am}\right) - \alpha < 0$, and the following problem

$$\Delta h - ah - \frac{ab(x)}{1+am}k_0 + \mu_0 h = 0, \quad x \in \Omega, \qquad h = 0, x \in \partial\Omega$$

has a unique solution h_0 because the operator $-\Delta + a - \mu_0$ is invertible. This shows that (μ_0, h_0, k_0) satisfies (2.1), i.e. the eigenvalue problem (2.1) has a negative eigenvalue μ_0 and so $(a, 0)$ is unstable.

(2) The linearized eigenvalue problem of (1.2) at $(0, \frac{\alpha-d}{d\beta})$ is

$$\begin{cases} \Delta h + (a - \frac{\alpha-d}{d\beta}b(x))h + \mu h = 0, \quad x \in \Omega, \qquad h = 0, \quad x \in \partial\Omega, \\ \Delta k + \dfrac{d(d-\alpha)}{\alpha}k + \frac{\alpha-d}{d\beta}c(x)h + \mu k = 0, \quad x \in \Omega_1, \; k = 0, \quad x \in \partial\Omega_1, \end{cases} \tag{2.2}$$

where μ is eigenvalue and (h, k) is the corresponding eigenfunction.

If $h = 0$, then $k \neq 0$, and $\mu \geq \dfrac{d(\alpha - d)}{\alpha} > 0$. If $h \neq 0$, then

$$\mu \geq \lambda_1^N\left(-a + \frac{\alpha - d}{d\beta}b(x), \Omega\right) = -a + \lambda_1^N\left(\frac{\alpha - d}{d\beta}b(x), \Omega\right) > 0$$

when $a < \lambda_1^N(\frac{\alpha-d}{d\beta}b(x), \Omega)$. This shows that $(0, \frac{\alpha-d}{d\beta})$ is locally stable when $a < \lambda_1^N(\frac{\alpha-d}{d\beta}b(x), \Omega)$.

When $a > \lambda_1^N(\frac{\alpha-d}{d\beta}b(x), \Omega)$, we assume that μ_0 and h_0 are the principal eigenvalue and the corresponding positive eigenfunction of

$$\Delta h + \left(a - \frac{\alpha - d}{d\beta}b(x)\right)h + \mu h = 0, \quad x \in \Omega, \qquad h = 0, \quad x \in \partial\Omega.$$

Then $\mu_0 = -a + \lambda_1^N(\frac{\alpha-d}{d\beta}b(x), \Omega) < 0$, and the following problem

$$\Delta k + \frac{d(d - \alpha)}{\alpha}k + \frac{\alpha - d}{d\beta}c(x)h_0 + \mu_0 k = 0, \quad x \in \Omega_1, \qquad k = 0, \quad x \in \partial\Omega_1$$

has a unique solution h_0 because of the operator $-\Delta + \frac{\alpha-d}{d\beta} - \mu_0$ is reversible. This shows that (μ_0, h_0, k_0) satisfies (2.2), i.e. the eigenvalue problem (2.2) has a negative eigenvalue μ_0 and so $(0, \frac{\alpha-d}{d\beta})$ is unstable. □

Remark 2.6 For the two curves of solutions in the space of

$$\Gamma_u = \{(\alpha; a, 0) : \alpha > 0\} \quad and \quad \Gamma_v = \{(\alpha; 0, \frac{\alpha - d}{d\beta}) : \alpha > d\}, \tag{2.3}$$

Theorem 2.5 implies that bifurcation could occur along the semi-trivial branches (2.3), if (1): $\alpha > d + \lambda_1^N\left(-\dfrac{ac(x)}{1 + am}, \Omega_1\right)$, or (2): $\alpha > d$ and $a > \lambda_1^N(\frac{\alpha-d}{d\beta}b(x), \Omega)$.

3 Bifurcation from semi-trivial solution

In this section, we will investigate the bifurcation solutions of (1.2) by the bifurcation theory. We fix d and take α as the main bifurcation parameter. In order to main bifurcation parameter of (1.2) which bifurcate from semi-trivial solution $(a, 0)$ and $(0, \frac{\alpha-d}{d\beta})$ with $\alpha \geq d$. First, we set up the abstract framework for our bifurcation analysis. For $p > 1$, we define

$$X = W^{2,p}(\Omega) \times W^{2,p}(\Omega_1) \doteq X_1 \times X_2, \quad Y = L^p(\Omega) \times L^p(\Omega_1) \doteq Y_1 \times Y_2,$$

where $W^{2,p}(O) = \{w \in W^{2,p}(O) : \partial_\nu w = 0 \text{ on } \partial O\}$.

For a given operator L, we denote the kernel and range of L with $\mathcal{N}(L)$ and $\mathcal{R}(L)$, respectively.

Theorem 3.1 *We have the results:*

(1) *If* $d > -\lambda_1^N(-\frac{ac(x)}{1+am}, \Omega_1)$, *then*

- (a) α *is a bifurcation point where a continuum* Γ_1 *of positive solutions to* (1.2) *bifurcates from* Γ_u *at* $(\overline{\alpha}; a, 0)$ *if and only if* $\alpha = d + \lambda_1^N(-\frac{ac(x)}{1+am}, \Omega_1) \doteq \overline{\alpha}$.
- (b) *all positive solutions of* (1.2) *near* $(\overline{\alpha}; a, 0) \in \mathbb{R} \times X$ *can be expressed as* $(\overline{\alpha}(s); u(s), v(s))$ *with* $s \in (0, \delta)$, *where* $(\overline{\alpha}(s); u(s), v(s))$ *is a smooth function with respect to s and satisfies* $(\overline{\alpha}(s); u(s), v(s)) = (\overline{\alpha}; a, 0)$ *and the bifurcation is supercritical.*

(2) *If* $\alpha > d$ *and* $\lambda_1^N\left(\dfrac{(\alpha-d)b(x)}{d\beta(1+am)}, \Omega\right) \le a < \lambda_1^D(\Omega_0)$, *then*

- (a) *there exists a unique* $\widehat{\alpha}(a)$ *such that* $a = \lambda_1^N(\frac{\alpha-d}{d\beta}b(x), \Omega)$. *Moreover,* $\widehat{\alpha}(a) \to d$ *as* $a \to 0^+$ *and* $\widehat{\alpha}(a) \to \infty$ *as* $a \to \lambda_1^D(\Omega_0)^-$.
- (b) α *is a bifurcation point where an continuum* Γ_2 *of positive solutions to* (1.2) *bifurcates from* Γ_v *at* $(\widehat{\alpha}; 0, \frac{\widehat{\alpha}-d}{d\beta})$ *if and only if* $\alpha = \widehat{\alpha}$.
- (c) *all positive solutions of* (1.2) *near* $(\widehat{\alpha}; 0, \frac{\widehat{\alpha}-d}{d\beta}) \in \mathbb{R} \times X$ *can be expressed as* $(\widehat{\alpha}(s); u(s), v(s))$ *with* $s \in (0, \delta)$, *where* $(\widehat{\alpha}(s); u(s), v(s))$ *is a smooth function with respect to s and satisfies* $(\overline{\alpha}(s); u(s), v(s)) = (\widehat{\alpha}; 0, \frac{\widehat{\alpha}-d}{d\beta})$ *and the bifurcation is supercritical (subcritical), if* $m_0 > 1$ (< 1), *where*

$$m_0 = \frac{\int_\Omega \frac{m(\widehat{\alpha}-d)}{d\beta} b(x)\varphi_1^3 dx - \int_\Omega b(x)\varphi_1^2\varphi_2 dx}{\int_\Omega \varphi_1^3 dx}.$$

(3) *If* $a \ge \lambda_1^D(\Omega_0)$, *then for any* $\alpha > d$, $a > \lambda_1^N(\frac{\alpha-d}{d\beta}b(x), \Omega)$ *and no bifurcation of positive solutions can occur along* Γ_v.

Proof (1) Take the variables $w = a - u$ and define $F(\alpha; w, v) : \mathbb{R} \times X \to Y$ by

$$F(\alpha; w, v) = \begin{pmatrix} \Delta w + w^2 - aw + \dfrac{b(x)(a-w)v}{1+m(a-w)} \\ \Delta v + v\left(\dfrac{\alpha}{1+\beta v} - d\right) + \dfrac{c(x)(a-w)v}{1+m(a-w)} \end{pmatrix}.$$

By using a simple calculation, we obtain

$$F_{(w,v)}(\alpha; 0, 0)[h, k] = \begin{pmatrix} \Delta h - ah + \dfrac{ab(x)}{1+am}k \\ \Delta k + [\alpha - d + \dfrac{ac(x)}{1+am}]k \end{pmatrix}$$

$$F_{\alpha(w,v)}(\alpha; 0, 0)[h, k] = \begin{pmatrix} 0 \\ k \end{pmatrix},$$

and

$$F_{(w,v)(w,v)}(\alpha;0,0)[h,k]^2 = \begin{pmatrix} 2h^2 - \frac{2b(x)}{(1+am)^2}hk \\ -\frac{2c(x)}{(1+am)^2}hk - 2\alpha\beta k^2 \end{pmatrix},$$

Let $\alpha = d + \lambda_1^N(-\dfrac{ac(x)}{1+am}, \Omega_1) \doteq \overline{\alpha}$. Then $F_{(w,v)}(\alpha;0,0)[h,k] = 0$ has a solution with $h > 0$. Thus $\overline{\alpha}$ is the only bifurcation point along Γ_u where positive solutions of (1.2) bifurcates.

It is easy to verify that the kernel $\mathcal{N}(F_{(w,v)}(\overline{\alpha};0,0)) = span\{(\varphi_1,\varphi_2)\}$, where $(\varphi_1,\varphi_2) \neq (0,0)$ satisfies

$$\begin{cases} \Delta\varphi_1 - a\varphi_1 + \dfrac{ab(x)}{1+am}\varphi_2 = 0, \quad x \in \Omega, \\ \Delta\varphi_2 + \left[\overline{\alpha} - d + \dfrac{ac(x)}{1+am}\right]\varphi_2 = 0, \quad x \in \Omega_1, \\ \partial_\nu\varphi_1 = 0, \quad x \in \partial\Omega, \quad \partial_\nu\varphi_2 = 0, \quad x \in \partial\Omega_1. \end{cases} \tag{3.1}$$

We can choose $\varphi_2 > 0$ as the corresponding positive eigenfunction of $\lambda_1^N(d - \dfrac{ac(x)}{1+am}, \Omega_1)$ with $\int_{\Omega_1}\varphi_2^2 dx = 1$, and then $\varphi_1 = (-\Delta + a)^{-1}\left(\dfrac{ab(x)}{1+am}\varphi_2\right) > 0$.

It is easy to check that the range

$$\mathcal{R}(F_{(w,v)}(\overline{\alpha};0,0)) = \left\{(f,g)^T \in Y : \int_{\Omega_1} g(x)\varphi_2 dx = 0\right\},$$

and

$$F_{\alpha(w,v)}(\overline{\alpha};0,0)[\varphi_1,\varphi_2] = (0,\varphi_2)^T \notin \mathcal{R}(F_{(w,v)}(\overline{\alpha};0,0))$$

since $\int_{\Omega_1}\varphi_2^2 dx = 1 > 0$. By applying the results of [10] or [11], the set of solutions to (1.2) near $(\overline{\alpha}; a, 0)$ is a smooth curve

$$\Gamma_1 = \{(\overline{\alpha}(s); a - w(s), v(s)) : s \in [0,\delta)\},$$

with $\delta > 0$ small, $\overline{\alpha}(0) = d + \lambda_1^N(-\frac{ac(x)}{1+am}, \Omega_1)$, $w(s) = s\varphi_1 + o(|s|)$, $v(s) = s\varphi_2 + o(|s|)$. By [12, Corollary 2.3],

$$\overline{\alpha}'(0) = -\frac{\langle l, F_{(w,v)(w,v)}(\overline{\alpha};0,0)[\varphi_1,\varphi_2]^2\rangle}{2\langle l, F_{\alpha(w,v)}(\overline{\alpha};0,0)[\varphi_1,\varphi_2]\rangle} = \int_{\Omega_1}\left(\frac{c(x)\varphi_1\varphi_2^2}{(1+am)^2} + \alpha\beta\varphi_2^3\right)dx > 0,$$

where l is a linear functional on Y^2 defined as $\langle l, [f,g]\rangle = \int_{\Omega_1} g(x)\varphi_2 dx$. This yields that the bifurcation Γ_u at $(\widetilde{\alpha};0,0)$ is supercritical.

(2) We take the variables $v = \frac{\alpha-d}{d\beta} + w$ and define $G(\alpha; u, w) : \mathbb{R} \times X \to Y$ by

$$G(\alpha; u, w) = \begin{pmatrix} \Delta u + u(a-u) - \dfrac{b(x)u(\frac{\alpha-d}{d\beta}+w)}{1+mu} \\ \Delta w + (\frac{\alpha-d}{d\beta}+w)\left(\dfrac{\alpha}{1+\beta(\frac{\alpha-d}{d\beta}+w)} - d\right) + \dfrac{c(x)u(\frac{\alpha-d}{d\beta}+w)}{1+mu} \end{pmatrix}.$$

By using a simple calculation, we obtain

$$G_{(u,v)}(\alpha; 0, 0)[h, k] = \begin{pmatrix} \Delta h + ah - \frac{\alpha-d}{d\beta}b(x)h \\ \Delta k + \dfrac{d(d-\alpha)}{\alpha}k + \frac{\alpha-d}{d\beta}c(x)h \end{pmatrix}$$

$$G_{\alpha(u,v)}(\alpha; 0, 0)[h, k] = \begin{pmatrix} -\frac{b(x)}{d\beta}h \\ \frac{c(x)}{d\beta}h + \overline{C}k \end{pmatrix},$$

where $\overline{C} = \frac{d^2(\beta-1)(\beta^2-2\beta+2)(\widehat{\alpha}-d)-d\alpha^2\beta^3}{\widehat{\alpha}^3\beta^3}$, and

$$G_{(u,v)(u,v)}(\alpha; 0, 0)[h, k]^2 = \begin{pmatrix} (-2 + \frac{2m(\alpha-d)}{d\beta}b(x))h^2 - 2b(x)hk \\ -\frac{2m(\alpha-d)}{d\beta}c(x)h^2 + 2c(x)hk - \frac{2\beta d^3}{\alpha^3}k^2 \end{pmatrix}.$$

Let $a = \lambda_1^N(\frac{\alpha-d}{d\beta}b(x), \Omega)$. Then $G_{(u,w)}(\alpha; 0, 0)[h, k] = 0$ has a solution with $h > 0$.

Let $\lambda_1^N(\frac{\alpha-d}{d\beta}b(x), \Omega)$ be the principal eigenvalue of

$$-\Delta u + \frac{\alpha-d}{d\beta}b(x)u = \lambda u, \quad x \in \Omega, \qquad \partial_\nu u = 0, \quad x \in \partial\Omega. \tag{3.2}$$

By the proof of Theorem 2.1 in [1], we obtain that for any $\alpha > d$, $\lambda_1^N(\frac{\alpha-d}{d\beta}b(x), \Omega)$ is strictly increasing respect to α, $\lambda_1^N(\frac{\alpha-d}{d\beta}b(x), \Omega) < \lambda_1^D(\Omega_0)$, and

$$\lim_{\alpha\to\infty} \lambda_1^N(\frac{\alpha-d}{d\beta}b(x), \Omega) = \lambda_1^D(\Omega_0).$$

Now if $a \geq \lambda_1^D(\Omega_0)$, then for any $\alpha \geq d$, $a > \lambda_1^N(\frac{\alpha-d}{d\beta}b(x), \Omega)$. Hence, by the analyses above, no bifurcation of positive solutions can occur along Γ_v.

If $a < \lambda_1^D(\Omega_0)$, then there exits a unique $\widehat{\alpha}(a)$ such that $a = \lambda_1^N(\frac{\alpha-d}{d\beta}b(x), \Omega)$ due to the continuity and monotonicity of $\lambda_1^N(\frac{\alpha-d}{d\beta}b(x), \Omega)$. We easily see that $\widehat{\alpha}(a) \to d$ as a decreases to 0, and $\widehat{\alpha}(a) \to \infty$ as a increases to $\lambda_1^D(\Omega_0)$.

At $(\alpha; u, w) = (\widehat{\alpha}; 0, 0)$, $\mathcal{N}(G_{(w,v)}(\widehat{\alpha}; 0, 0)) = span\{(\phi_1, \phi_2)\}$. We can choose $\phi_1 > 0$ with $\int_\Omega \phi_1^2 dx = 1$ and $\phi_2 = (-\Delta + \frac{d(\alpha-d)}{\alpha})^{-1}(\frac{\alpha-d}{d\beta}c(x)\phi_1) > 0$. Then

$$\mathcal{R}(G_{(u,w)}(\widehat{\alpha}; 0, 0)) = \{(f, g)^T \in Y : \int_\Omega f(x)\phi_1 dx = 0\},$$

and

$$G_{\alpha(u,w)}(\widehat{\alpha}; 0, 0)[\phi_1, \phi_2] = \left(-\frac{b(x)}{d\beta}\phi_1, \frac{c(x)}{d\beta}\phi_1 + \overline{C}\phi_2\right)^T \notin \mathcal{R}(F_{(w,v)}(\widehat{\alpha}; 0, 0))$$

since $-\frac{1}{d\beta}\int_\Omega b(x)\phi_1^2 dx \neq 0$.

By applying the results of [10] or [11], the set of solutions to (1.2) near $(\widehat{\alpha}; 0, \frac{\widehat{\alpha}-d}{d\beta})$ is a smooth curve

$$\Gamma_2 = \{(\widehat{\alpha}(s); u(s), \frac{\widehat{\alpha}-d}{d\beta} + w(s)) : s \in [0, \delta)\},$$

with $\delta > 0$ small, $\widehat{\alpha}(0) = \widehat{\alpha}$, $u(s) = s\phi_1 + o(|s|)$, $w(s) = s\phi_2 + o(|s|)$. By [12, Corollary 2.3],

$$\begin{aligned}\widehat{\alpha}'(0) &= -\frac{\langle l, G_{(u,w)(u,w)}(\overline{\alpha}; 0, 0)[\phi_1, \phi_2]^2\rangle}{2\langle l, G_{\alpha(u,w)}(\overline{\alpha}; 0, 0)[\phi_1, \phi_2]\rangle} \\ &= \frac{\int_\Omega(\frac{m(\widehat{\alpha}-d)}{d\beta}b(x) - 1)\phi_1^3 dx - \int_\Omega b(x)\phi_1^2\phi_2 dx}{\frac{1}{d\beta}\int_\Omega b(x)\phi_1^2 dx},\end{aligned}$$

where l is a linear functional on Y^2 defined as $\langle l, [f, g]\rangle = \int_\Omega f(x)\phi_1 dx$. □

4 Bifurcation stability and global bifurcation

Theorem 4.1 *Recall* $\overline{\alpha}, \widehat{\alpha}, (\varphi_1, \varphi_2)$ *and* (ϕ_1, ϕ_2) *in Theorem* 3.1.

(1) *Suppose that* $d > -\lambda_1^N(-\frac{ac(x)}{1+am}, \Omega_1)$. *If*

$$\frac{1}{(1+am)^2}\int_{\Omega_1} c(x)\varphi_2^3 dx < \overline{\alpha}\beta\int_{\Omega_1}\varphi_1\varphi_2^2 dx,$$

then the local bifurcation coexistence state $(u(s), v(s))$ *bifurcating from* $(\overline{\alpha}; a, 0)$ *is linearly stable.*

(2) *If* $\alpha > d$, *then the local bifurcation coexistence state* $(u(s), v(s))$ *bifurcating from* $(\widehat{\alpha}; 0, \frac{\widehat{\alpha}-d}{d\beta})$ *is nondegenerate and linearly stable.*

Proof For convenience, we use the notation $\overline{\alpha}(s) = \alpha$, $(u(s), v(s)) = (u, v)$ in Theorem 3.1. The linearized problem of (1.2) at (u, v) can be written as

$$\mathcal{L}(s)(h, k) = \gamma(s)(h, k) \tag{4.1}$$

where

$$\mathcal{L}(s) = \begin{pmatrix} \Delta + a - 2u - \dfrac{b(x)v}{(1+mu)^2} & -\dfrac{b(x)u}{1+mu} \\ \dfrac{c(x)v}{(1+mu)^2} & \Delta + \dfrac{\alpha}{(1+\beta v)^2} - d + \dfrac{c(x)u}{1+mu} \end{pmatrix}.$$

It easy to see that, as $s \to 0$,

$$\mathcal{L}(s) \to \mathcal{L}_0 \doteq \begin{pmatrix} \Delta - a & -\dfrac{ab(x)}{1+am} \\ 0 & \Delta + \overline{\alpha} - d + \dfrac{ac(x)}{1+am} \end{pmatrix}.$$

By the proof in Theorem 3.1, we know that 0 is the principal eigenvalue of $\mathcal{L}_0$ with the corresponding eigenfunction (φ_1, φ_2), where φ_1 and φ_2 are defined in Theorem 3.1.

By the perturbation theory of linear operators [13], we know that, when s is sufficiently small, $\mathcal{L}(s)$ has a unique eigenvalue $\gamma(s)$ satisfying $\lim_{s\to 0} \gamma(s) = 0$ and all the other eigenvalues of $\mathcal{L}(s)$ have negative real parts and are apart from 0. Now we determine the sign of $Re(\gamma(s))$ as $s > 0$ is sufficiently small. Let (h, k) be the corresponding eigenfunction to $\gamma(s)$ such that $(h, k) \to (\varphi_1, \varphi_2)$.

Multiplying the second equation of (4.1) by v and integrating over Ω_1, we get

$$\int_{\Omega_1} v\Delta k + \frac{\alpha v}{(1+\beta v)^2}k - dvk + \frac{c(x)uv}{1+mu}k + \frac{c(x)v^2}{(1+mu)^2}hdx = \int_{\Omega_1} \gamma(s)vkdx. \tag{4.2}$$

Multiplying the second equation of (1.2) by k integrating over Ω_1, we have

$$\int_{\Omega_1} k\Delta v + v\left(\frac{\alpha}{1+\beta v} - d\right)k + \frac{c(x)uv}{1+mu}kdx = 0. \tag{4.3}$$

The fact combined with (4.2) and (4.3) to yields

$$\gamma(s)\int_{\Omega_1} vkdx = \int_{\Omega_1} \frac{c(x)v^2}{(1+mu)^2}h - \frac{\alpha\beta v^2}{(1+\beta v)^2}kdx. \tag{4.4}$$

Note that $\overline{\alpha}(0) = d + \lambda_1^N(-\frac{ac(x)}{1+am}, \Omega_1) \doteq \overline{\alpha}$, $w(s) = s\varphi_1 + o(|s|)$, $v(s) = s\varphi_2 + o(|s|)$. Dividing by s^2 and letting $s \to 0^+$ in (4.4), it is deduced that

$$\lim_{s\to 0^+} \frac{\gamma(s)}{s} = \frac{1}{(1+am)^2}\int_{\Omega_1} c(x)\varphi_2^3 dx - \overline{\alpha}\beta \int_{\Omega_1} \varphi_1\varphi_2^2 dx,$$

which implies that the bifurcation coexistence state $(u(s), v(s))$ $(\overline{\alpha}; a, 0)$ is linearly stable, if

$$\frac{1}{(1+am)^2}\int_{\Omega_1} c(x)\varphi_2^3 dx < \overline{\alpha}\beta \int_{\Omega_1} \varphi_1\varphi_2^2 dx.$$

(2) Analogously, multiplying the first equation of (4.1) by u and the first equation of (1.2) by h, integrating over Ω, we get

$$\int_\Omega u\Delta h + auh - 2u^2h - \frac{b(x)uv}{(1+mu)^2}h - \frac{b(x)u^2}{1+mu}k dx = \int_\Omega \gamma(s)uh dx,$$
$$\int_\Omega h\Delta u + u(a-u)h - \frac{b(x)uv}{1+mu}h dx = 0.$$

Then we have

$$\lim_{s\to 0^+}\frac{\gamma(s)}{s} = -\int_\Omega b(x)\phi_1^2\phi_2 dx - \int_\Omega \phi_1^3 dx < 0.$$

This implies that then the local bifurcation coexistence state $(u(s), v(s))$ bifurcating from $(\widehat{\alpha}; 0, \frac{\widehat{\alpha}-d}{d\beta})$ is nondegenerate and linearly stable. The proof is completed. □

Next, we will investigate the global bifurcation of (1.2). We fix the parameters $a > 0$ and $d > \frac{a\|c(x)\|_\infty}{1+am}$ (See theorem 2.2) and take α as the main bifurcation parameter. By the unilateral global bifurcation theorem developed by LKpez-GKmez, one can see [11] or [14] for the details, we study the global bifurcation at $(\overline{\alpha}; a, 0)$.

Let $P_O = \{w \in W^{2,p}(O) : w > 0, x \in \overline{O}\}$. Then $P^2 = P_\Omega \times P_{\Omega_1}$ is the nature positive cone in X. From the proof of Theorem 3.1, it follows that all the conditions in [11, Theorem 6.4.3] hold. This yields that there exists a component $\mathcal{C}^+ \supset \Gamma_u$ of solution to (1.2) bifurcating at $(\overline{\alpha}; a, 0)$ and $\mathcal{C}^+$ satisfies one of the following alternatives:

(i) $\mathcal{C}^+$ is unbounded in $\mathbb{R} \times X$;

(ii) There exists a real number $\widetilde{\alpha} \neq \overline{\alpha}$, such that $(\widetilde{\alpha}; a, 0) \in \mathcal{C}^+$;

(iii) $\mathcal{C}^+$ contains a point $(\alpha; u, v) \in \Gamma_v$ or $\in \Gamma_0 = \{(\alpha; 0, 0) : \alpha \in R\}$, such that $(\alpha; u, v) \in \mathcal{C}^+$.

By Theorems 2.2 and 2.3, the alternative (i) do not occur. By Theorem 3.1 (1)(a), i.e., α is a bifurcation point where an continuum Γ_1 of positive solutions to (1.2) bifurcates from Γ_u at $(\overline{\alpha}; a, 0)$ if and only if $\alpha = \overline{\alpha}$, we know that the alternative (ii) don't occur. So, the alternative (iii) must occur. Now, we claim that $\mathcal{C}^+$ ends at some point $(\widehat{\alpha}; 0, \frac{\widehat{\alpha}-d}{d\beta})$ on Γ_v for some $\widehat{\alpha} > d$.

In fact, we assume on the contrary that $\mathcal{C}^+$ ends at some point $(\alpha; 0, 0)$. Let $u(s) = s\psi_1(s) + o(s)$ and $v(s) = s\psi_2(s) + o(s)$ for $0 < s \ll 1$, then $\lim_{s\to 0^+} u(s)/s = \psi_1$, $\lim_{s\to 0^+} u(s)/s = \psi_2$, where ψ_1 and ψ_2 are the positive functions in Ω and Ω_1 respectively. By dividing the first equation of (1.2) by s and letting $s \to 0^+$, we obtain that

$$\Delta\psi_1 + a\psi_1 = 0, x \in \Omega, \qquad \partial_\nu\psi_1 = 0, x \in \partial\Omega. \tag{4.5}$$

Thus, we get $a = 0$, which contradicts $a > 0$.

Combined the arguments above with the local bifurcation results (Theorem 3.1), we obtain the following theorem.

Theorem 4.2 *Suppose that* $0 < a < \lambda_1^D(\Omega_0)$ *and* $d > \frac{a\|c(x)\|_\infty}{1+am}$ *be fixed. Then there exists a continuum* $\mathcal{C}^+$ *of the positive solutions connecting* $(\overline{\alpha}; a, 0)$ *to* $(\widehat{\alpha}; 0, \frac{\widehat{\alpha}-d}{d\beta})$ *with* $\widehat{\alpha} > d$ *and satisfying*

$$Proj_\alpha \mathcal{C}^+ = (\overline{\alpha}, \widehat{\alpha}),$$

which implies that (1.2) *possesses at least a positive solution for any* $\alpha \in (\overline{\alpha}, \widehat{\alpha})$.

Remark 4.3 Theorem 3.1) shows that (1) if $d > \frac{a\|c(x)\|_\infty}{1+am}$, then there exist bifurcation of positive solutions along Γ_u; (2) if $a > \lambda_1^D(\Omega_0)$, then there exists no bifurcation of positive solutions along Γ_v.

Remark 4.4 For fixed $a > 0$, the term $a > \lambda_1^D(\Omega_0)$ in Remark 4.3 can be interpreted as the fact that the protection zone Ω_0 is large. In addition, if $d > \frac{a\|c(x)\|_\infty}{1+am}$, then by the same proof process similar to theorem 4.2, there exists a continuum $\mathcal{C}^+$ of the positive solutions emanating from $(\overline{\alpha}; a, 0)$ and satisfying $Proj_\alpha \mathcal{C}^+ = (\overline{\alpha}, +\infty)$.

Remark 4.5 (Numerical example) Letting $\Omega = (0, 5\pi)$, we consider the effect of degenerate on the positive solution to (1.1), i.e., the following two cases (See Figure 2):

(1) Functions $b(x)$ and $c(x)$ are independent of x: $b(x) \equiv 0.05$ and $c(x) \equiv 0.03$;

(2) Functions $b(x)$ and $c(x)$ are dependent of x:

$$b(x) = \begin{cases} 0.05 & \pi \le x \le 4\pi \\ 0 & otherwise \end{cases} \quad \text{and } c(x) = \begin{cases} 0.03 & \pi \le x \le 4\pi \\ 0 & otherwise \end{cases}.$$

5 Discussions

In this paper we propose a reaction–diffusion predator–prey model with a protection zone for the prey and nonlinear growth rate for the predator. It is shown that the protection zone will affect the existence of positive steady-state solutions or steady-state bifurcations form (1.1). By Remark 2.4 and Theorem 3.1, the existence and non-existence results are summarized below:

(A1) Assume that $\alpha < d$. If $d < \dfrac{\alpha(d-\alpha)}{(d-\alpha)(1+a\beta)+a^2\beta}$ or $d > \alpha - \lambda_1^N(-\dfrac{ac(x)}{1+am}, \Omega_1)$, then (1.2) has no positive solution.

(A2) Assume that $\alpha > d$ and $a \ge \min\{\lambda_1^D(\Omega_0), \lambda_1^N(\frac{\alpha-d}{d\beta}b(x), \Omega)\}$. Then there is no bifurcation of positive solutions to (1.2) can occur along Γ_v.

(A3) If $d > -\lambda_1^N(-\frac{ac(x)}{1+am}, \Omega_1)$, then there is a continuum Γ_1 of positive solutions to (1.2) bifurcates from Γ_u at $(\overline{\alpha}; a, 0)$ where $\alpha = d + \lambda_1^N(-\frac{ac(x)}{1+am}, \Omega_1) \doteq \overline{\alpha}$.

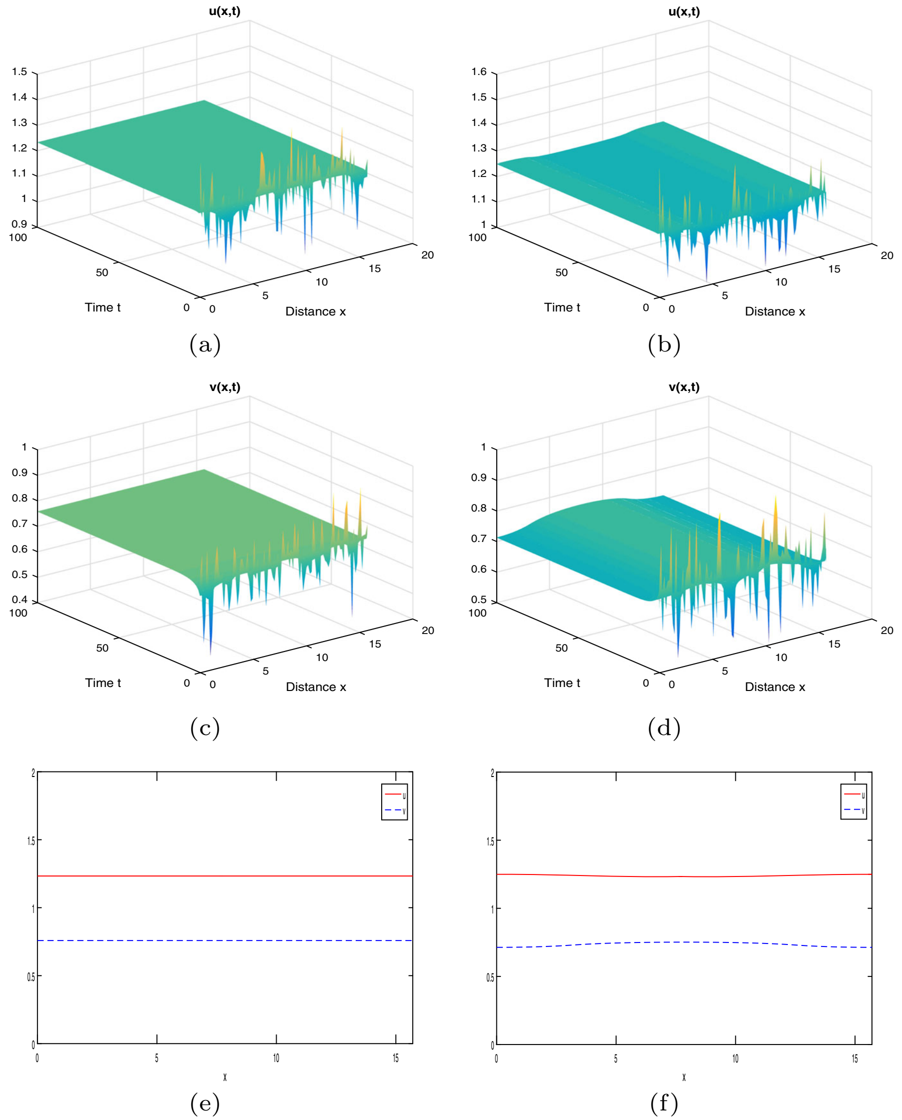

Fig. 2 Numerical simulation of the spatio-temporal positive solutions to (1.1) with $a = 1.25$, $m = 1$, $\alpha = 0.85$, $\beta = 1$ and $d = 0.5$, where the first and second column represent $u(x, t)$ and $v(x, t)$, respectively. **a**–**c**–**e** case 1, the unique positive spatially homogeneous equilibrium (1.2330, 0.7583) is locally asymptotically stable; **b**–**d**–**f** case 2, there exists a spatially heterogeneous positive steady state solution

(A4) If $\alpha > d$ and $\lambda_1^N(\dfrac{(\alpha - d)b(x)}{d\beta(1 + am)}, \Omega) \le a < \lambda_1^D(\Omega_0)$, then there a continuum Γ_2 of positive solutions to (1.2) bifurcates from Γ_v at $(\widehat{\alpha}; 0, \frac{\widehat{\alpha}-d}{d\beta})$, where $\alpha = \widehat{\alpha}$ and which is a unique $\widehat{\alpha}(a)$ such that $a = \lambda_1^N(\frac{\alpha-d}{d\beta}b(x), \Omega)$. Moreover, $\widehat{\alpha}(a) \to d$ as $a \to 0^+$ and $\widehat{\alpha}(a) \to \infty$ as $a \to \lambda_1^D(\Omega_0)^-$.

Note that $b(x) \in L^\infty(\Omega)$, $b(x) \geq 0$ in Ω, $b(x) \equiv 0$ on $\overline{\Omega}_0$ and for any compact subset A of $\Omega\backslash\overline{\Omega}_0$, there exists $\delta_A > 0$ such that $\delta_A \leq b(x), \forall x \in A$; $c(x) \in L^\infty(\Omega\backslash\overline{\Omega}_0)$ and $0 < c(x) \leq b(x)$ in $\Omega\backslash\overline{\Omega}_0$. Now, let $b(x) = c(x) = 1$ in $\Omega\backslash\overline{\Omega}_0$ in order to better analyze the affect of protection zone on the dynamics of (1.1).

Since $-\lambda_1^N(-\dfrac{ac(x)}{1+am}, \Omega_1)$ is increasing in Ω_0, $\lambda_1^D(\Omega_0)$ and $\lambda_1^N(\frac{\alpha-d}{d\beta}b(x), \Omega)$ are decreasing in Ω_0, (A1) and (A2) imply that the smaller the size of protection zone Ω_0, two populations u and v are more likely to coexist. This is also in line with the original intention of constructing ecological nature reserve in reality.

Let $\overline{\alpha} = d + \lambda_1^N(-\frac{ac(x)}{1+am}, \Omega_1)$ in (A3). Define the unique $\widehat{\alpha}(a)$ in (A4) such that $a = \lambda_1^N(\frac{\alpha-d}{d\beta}b(x), \Omega)$ if $\alpha > d$ and $\lambda_1^N(\dfrac{(\alpha-d)b(x)}{d\beta(1+am)}, \Omega) \leq a < \lambda_1^D(\Omega_0)$. (A3) and (A4) show that there is a circular domain Ω_0, at which there is a continuum Γ_1 or Γ_2 of positive solutions to (1.2) bifurcates from Γ_u or Γ_v.

Hence a recommendation for the people setting up the protection zone is to have a circular region with as large as possible area as the protect [15].

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Code availability All codes generated or used during the study are available from the corresponding author by request (W. Yang).

Declarations

Conflict of interest The author declares that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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