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On weakly-Krull domains of integer-valued polynomials

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Abstract
Given an integral domain D with quotient field K, we consider the ring $\mathrm{Int}(D) := \{f \in K[X];\ f(D) \subseteq D\}$ of integer-valued polynomials over D. This paper deals with the question of when $\mathrm{Int}(D)$ is a weakly-Krull domain.

Keywords Integer-valued polynomials · Weakly-Krull domain

Mathematics Subject Classification 13A15 · 13F05 · 13F20

Introduction

Let D be an integral domain with quotient field K.

We recall that a Krull domain is an integral domain D such that $D = \bigcap_{\mathfrak{p}\in X^1(D)} D_{\mathfrak{p}}$, where $X^1(D)$ denotes the set of all height-one prime ideals of D, $D_{\mathfrak{p}}$ is a DVR for each $\mathfrak{p} \in X^1(D)$ and the intersection $\bigcap_{\mathfrak{p}\in X^1(D)} D_{\mathfrak{p}}$ is locally finite (i.e., each nonzero element of D belongs to only finitely many ideals $\mathfrak{p} \in X^1(D)$).

The notion of weakly-Krull domain was first introduced by Anderson *et al.* in [1, 5] as follows: an integral domain D is *weakly-Krull* if $D = \bigcap_{\mathfrak{p}\in X^1(D)} D_{\mathfrak{p}}$ and this intersection is locally finite. Obviously Krull domains are weakly-Krull.

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For the convenience of the reader we begin by recalling some definitions and notation necessary for the comprehension of the discussion about weakly-Krull property. Let $\mathcal{F}(D)$ (resp., $\mathcal{F}_{\text{f.g.}}(D)$) be the set of all nonzero fractional ideals (resp., nonzero finitely generated fractional ideals) of D. For an ideal $I \in \mathcal{F}(D)$, we set $I^{-1} := \{x \in K;\ xI \subseteq D\}$. The *v-operation* is defined on $\mathcal{F}(D)$ by $I_v := (I^{-1})^{-1}$, the *t-operation* is defined by $I_t := \cup\{J_v;\ J \in \mathcal{F}_{\text{f.g.}}(D)$ and $J \subseteq I\}$, and the *w-operation* is defined by $I_w := \{x \in K;\ xJ \subseteq I$ for some $J \in \mathcal{F}_{\text{f.g.}}(D)$ with $J^{-1} = D\}$. It is straightforward that $I \subseteq I_w \subseteq I_t \subseteq I_v$. An ideal $I \in \mathcal{F}(D)$ is a *v-ideal* (or divisorial) (resp., *t-ideal*, *w-ideal*) if $I_v = I$ (resp., $I_t = I$, $I_w = I$). A prime ideal that is also a t-ideal is called a *t-prime* ideal, and an ideal maximal among integral t-ideals is called a *t-maximal* ideal (and it is a prime ideal). We let $t\text{-Max}(D)$ denote the set of all t-maximal ideals of D. It is well-known that for any integral domain D, $D = \cap_{\mathfrak{p}\in t\text{-Max}(D)} D_{\mathfrak{p}}$. We say that an integral domain D has *t-dimension one* if each t-prime ideal of D is of height-one (we then write $t\text{-dim}(D) = 1$). Notice that if $t\text{-dim}(D) = 1$ then $t\text{-Max}(D) = X^1(D)$. Lastly, an integral domain D is said to be of *t-finite character* (resp., *finite character*) if every nonzero element of D belongs to only finitely many t-maximal (resp., maximal) ideals of D.

Representations of domains D as locally finite intersections of a family $\{D_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}$ ($\mathcal{P} \subseteq \text{Spec}(D)$) of its localization overrings may correspond to some factorization properties for ideals of D. For instance, Krull domains are exactly the domains for which each principal ideal is a t-product of prime ideals ([5, Theorem 3.2]) and weakly-Krull domains are exactly the domains in which each principal ideal is a t-product of primary ideals ([5, Theorem 3.1]).

In [5, Lemma 2.1(1)] it is showed that weakly-Krull domains are exactly the *one t-dimensional domains with t-finite character*.

The polynomials with coefficients in K that take values from D into D itself form a commutative D-algebra denoted by $\text{Int}(D)$. More precisely $\text{Int}(D) := \{f \in K[X];\ f(D) \subseteq D\}$ is *the ring of integer-valued polynomials over D*.

Some Krull-like properties for $\text{Int}(D)$ have been investigated in past literature. For instance, if D is a Krull domain, then $\text{Int}(D)$ is a Krull domain if and only if $\text{Int}(D)$ is Mori if and only if $\text{Int}(D) = D[X]$ ([9, Corollary 2.7]). This result has been somehow generalized in [16] by showing that for a domain D of w-dimension one, $\text{Int}(D)$ is Strong Mori if and only if D is Strong Mori and $\text{Int}(D) = D[X]$.

It is well-known that $\mathbb{Z}$ is a (weakly-)Krull domain and $\text{Int}(\mathbb{Z})$ is a two-dimensional Prüfer domain (see, for instance, [7]). But $\text{Int}(\mathbb{Z})$ is not weakly-Krull since it is of t-dimension two. Thus the weakly-Krull property is not preserved, in general, upon passage from D to $\text{Int}(D)$. This led us to ask when $\text{Int}(D)$ is a weakly-Krull domain.

1 Results and applications

By definition, weakly-Krull property is based on a locally finite intersection representation of D. Hence, we start analyzing more generally when $\text{Int}(D)$ has a representation as locally finite intersection of a family of its localizations $\{\text{Int}(D)_{\mathfrak{P}};\ \mathfrak{P} \in \mathcal{P}\}$, for $\mathcal{P} \subseteq \text{Spec}(\text{Int}(D))$.

Proposition 1.1 *Let D be an integral domain with quotient field K. If* $\mathrm{Int}(D)$ *is a locally finite intersection of a family of its localizations then D so is.*

Proof Assume that $\mathrm{Int}(D) = \cap_{\mathfrak{P}\in\mathcal{P}}\mathrm{Int}(D)_{\mathfrak{P}}$, where $\mathcal{P} \subseteq \mathrm{Spec}(\mathrm{Int}(D))$ is the representation of $\mathrm{Int}(D)$ as given in the statement and set $\mathcal{P}' := \{\mathfrak{P} \cap D; \mathfrak{P} \in \mathcal{P}\} \subseteq \mathrm{Spec}(D)$. Then $D = \mathrm{Int}(D) \cap K = \cap_{\mathfrak{P}\in\mathcal{P}}(\mathrm{Int}(D)_{\mathfrak{P}} \cap K) \supseteq \cap_{\mathfrak{p}\in\mathcal{P}'} D_{\mathfrak{p}} \supseteq D$ (in fact, $\mathrm{Int}(D)_{\mathfrak{P}} \cap K \supseteq D_{\mathfrak{P}\cap D}$). Thus $D = \cap_{\mathfrak{p}\in\mathcal{P}'} D_{\mathfrak{p}}$. Obviously this intersection is locally finite because the intersection $\cap_{\mathfrak{P}\in\mathcal{P}}\mathrm{Int}(D)_{\mathfrak{P}}$ is locally finite. □

As mentioned in the introduction, weakly-Krull domains are exactly the one t-dimensional domains with t-finite character. By [3, Corollary 3.4] an integral domain D has t-finite character if and only if $D[X]$ has the t-finite character. About the (t-) finite character for $\mathrm{Int}(D)$ we have the following corollary.

Corollary 1.2 *Let D be an integral domain. If* $\mathrm{Int}(D)$ *has the t-finite (resp., finite) character then D has it too.*

Proof We can apply Proposition 1.1 where $\mathcal{P}$ is the set of t-maximal ideals of $\mathrm{Int}(D)$. By [20, Corollary 2.2] we have that the set $\mathcal{P}'$ of Proposition 1.1 is exactly the set of t-maximal ideals of D and so D has the t-finite character. If $\mathcal{P} = \mathrm{Max}(\mathrm{Int}(D))$ (i.e., $\mathrm{Int}(D)$ has the finite character), then $\mathcal{P}' = \mathrm{Max}(D)$, hence D has the finite character too. □

Remarks 1.3

(a) An alternative (direct) proof of Corollary 1.2 can be given, by contraposition, as follows:

Suppose that D has not the t-finite character. Then there is a nonzero element of D which is contained in infinitely many t-maximal ideals of D. By [20, Corollary 2.2], each t-maximal ideal of D is the contraction of a t-maximal ideal of $\mathrm{Int}(D)$. Then $\mathrm{Int}(D)$ has not the t-finite character. Now, if D has not the finite character, then there is a nonzero element $x \in D$ that is contained in infinitely many maximal ideas of D. Then, the same x is contained in infinitely many maximal ideals of $\mathrm{Int}(D)$: for any $\mathfrak{m} \in \mathrm{Max}(D)$, choose the ideal $\mathfrak{M}_0 := \{f \in \mathrm{Int}(D); f(0) \in \mathfrak{m}\}$. Thus $\mathrm{Int}(D)$ has not the finite character.

(b) t- FINITE CHARACTER

It is well-known that $D[X]$ has the t-finite character if and only if D has it ([3, Corollary 3.4]). This is not true, in general, for $\mathrm{Int}(D)$. In fact, if D has the t-finite character $\mathrm{Int}(D)$ may not have it. For instance $\mathrm{Int}(\mathbb{Z})$ does not have the t-finite character (and $\mathbb{Z}$ has it). Indeed $\mathrm{Int}(\mathbb{Z})$ is Prüfer, so each ideal is a t-ideal and the t-finite character is equivalent to the finite character on maximal ideals. A Prüfer domain with the finite character is Krull-type and, from [18, Theorem 2.30], we would have that $\mathrm{Int}(\mathbb{Z}) = \mathbb{Z}[X]$ which is not true. This example shows that t-dimension one and t-finite character, properties that combined together characterize weakly-Krull domains, maybe both verified in D but not in $\mathrm{Int}(D)$.

Conversely, there are also examples of nontrivial integer-valued polynomial rings $\mathrm{Int}(D)(\neq D[X])$ that have the t-finite character. We recall that an integral domain is *Mori* if it satisfies the ascending chain condition on integral divisorial ideals. In [9] the authors give an example of a one-dimensional Mori domain D such that $\mathrm{Int}(D)$ is Mori and with $\mathrm{Int}(D) \neq D[X]$. It is known that Mori domains have the t-finite character ([5, Lemma 2.1(1)]), hence $\mathrm{Int}(D)$ has it.

(c) FINITE CHARACTER

It is known ([2, Proposition 18]) that the polynomial ring $D[X]$ never has the finite character on maximal ideals, unless D is a field. We can see, however, that $\mathrm{Int}(D)$ may have the finite character on maximal ideals. Indeed, consider a one-dimensional, local, non unibranched Noetherian domain D as given in [7, § 5, page 110]. If $\mathfrak{m}$ is the maximal ideal of D, then the prime spectrum of $\mathrm{Int}(D)$ is made of the primes above $\mathfrak{m}$ and the primes above (0). In this case the prime ideals of $\mathrm{Int}(D)$ above $\mathfrak{m}$ are finitely many. The set of nonzero primes above (0) has the finite character (since they correspond to the nonzero primes of $K[X]$ which is Dedekind). So, we have that $\mathrm{Int}(D)$ has the finite character on maximal ideals.

We now recall that a prime ideal $\mathfrak{p}$ of an integral domain D is called *int prime* if $\mathrm{Int}(D) \nsubseteq D_{\mathfrak{p}}[X]$ and it is called *polynomial prime* if $\mathrm{Int}(D) \subseteq D_{\mathfrak{p}}[X]$. If $\mathfrak{p}$ is a polynomial prime we also have that $\mathrm{Int}(D)_{\mathfrak{p}} = D_{\mathfrak{p}}[X]$ (where $\mathrm{Int}(D)_{\mathfrak{p}} := \mathrm{Int}(D)_{D\setminus\mathfrak{p}}$), by [7]. We remark that if $\mathfrak{p}$ has infinite residue field (for instance, if it is not maximal), $\mathfrak{p}$ is a polynomial prime ([7, Proposition I.3.4]).

The following two lemmas can be found in [18] but, for the sake of completeness, we include their proofs.

Lemma 1.4 *Let D be a weakly-Krull domain. Then each int prime ideal of D is of height-one.*

Proof Let $\mathfrak{m}$ be an int prime ideal of D. By way of contradiction, assume that $\mathfrak{m}$ is of height at least two. Thus $\mathfrak{m} \notin X^1(D)$. By the local finiteness of the intersection $\bigcap_{\mathfrak{p}\in X^1(D)} D_{\mathfrak{p}}$, it follows from [19, Lemma 1.5] that $D_{\mathfrak{m}} = \bigcap_{\mathfrak{p}\in X^1(D)}(D_{\mathfrak{p}})_{\mathfrak{m}}$. For each $\mathfrak{p} \in X^1(D)$, we have $\mathfrak{m} \neq \mathfrak{p}$ and $\mathrm{Int}((D_{\mathfrak{p}})_{\mathfrak{m}}) = (D_{\mathfrak{p}})_{\mathfrak{m}}[X]$. Moreover, $(D_{\mathfrak{p}})_{\mathfrak{m}} = \bigcap_{\mathfrak{q}\subseteq\mathfrak{p}\cap\mathfrak{m}} D_{\mathfrak{q}}$ where the primes $\mathfrak{q}$ are polynomial primes ($\mathfrak{q} \subsetneq \mathfrak{m}$ and it is not maximal). Thus, by [8, Corollaires (3), page 303], we have:

$$\begin{aligned}\mathrm{Int}(D_{\mathfrak{m}}) &= \bigcap_{\mathfrak{p}\in X^1(D)} \mathrm{Int}((D_{\mathfrak{p}})_{\mathfrak{m}}) = \bigcap_{\mathfrak{p}\in X^1(D)} \left(\bigcap_{\mathfrak{q}\subseteq\mathfrak{p}\cap\mathfrak{m}} \mathrm{Int}(D)_{\mathfrak{q}}\right) = \bigcap_{\mathfrak{p}\in X^1(D)} \left(\bigcap_{\mathfrak{q}\subseteq\mathfrak{p}\cap\mathfrak{m}} D_{\mathfrak{q}}[X]\right)\\ &= \bigcap_{\mathfrak{p}\in X^1(D)} (D_{\mathfrak{p}})_{\mathfrak{m}}[X] = D_{\mathfrak{m}}[X].\end{aligned}$$

Therefore $\mathrm{Int}(D)_{\mathfrak{m}} = \mathrm{Int}(D_{\mathfrak{m}}) = D_{\mathfrak{m}}[X]$, which contradicts the hypothesis that $\mathfrak{m}$ is an int prime. □

Lemma 1.5 *Let D be an integral domain. If t-$\dim(\mathrm{Int}(D)) = 1$ then D is either a field or of t-dimension* 1.

Proof Suppose that D is not a field and, by way of contradiction, that t-$\dim(D) > 1$. Then, there exist at least two nonzero t-primes of D, $\mathfrak{q}$ and $\mathfrak{p}$, such that $(0) \subsetneq \mathfrak{q} \subsetneq$

$\mathfrak{p}$. Obviously, the ideal $\mathfrak{q}$ is not maximal and so it is a polynomial prime. Hence $\mathfrak{q}D_{\mathfrak{q}}[X] \cap \mathrm{Int}(D)$ is a t-prime of $\mathrm{Int}(D)$, because it is the contraction of the t-prime $\mathfrak{q}D_{\mathfrak{q}}[X]$ (where $\mathrm{Int}(D)_{\mathfrak{q}} = D_{\mathfrak{q}}[X]$).

If $\mathfrak{p}$ is a polynomial prime then we argue similarly as done for $\mathfrak{q}$ and we have the chain of t-primes in $\mathrm{Int}(D)$:

$$(0) \subsetneq \mathfrak{q}D_{\mathfrak{q}}[X] \cap \mathrm{Int}(D) \subsetneq \mathfrak{p}D_{\mathfrak{p}}[X] \cap \mathrm{Int}(D).$$

If $\mathfrak{p}$ is an int prime, as $\mathfrak{P}_0 := \{f \in \mathrm{Int}(D);\ f(0) \in \mathfrak{p}\}$ contains $\mathrm{Int}(D, \mathfrak{p}) := \{f \in \mathrm{Int}(D);\ f(D) \subseteq \mathfrak{p}\}$, it follows from [10, Propositions 1.2 and 1.4] that $\mathfrak{P}_0$ is a t-prime ideal of $\mathrm{Int}(D)$. Thus, from the inclusions $\mathfrak{q}D_{\mathfrak{q}}[X] \cap \mathrm{Int}(D) \subseteq \mathrm{Int}(D, \mathfrak{q}) \subsetneq \mathrm{Int}(D, \mathfrak{p}) \subsetneq \mathfrak{P}_0$, we have the chain of t-primes in $\mathrm{Int}(D)$:

$$(0) \subsetneq \mathfrak{q}D_{\mathfrak{q}}[X] \cap \mathrm{Int}(D) \subsetneq \mathfrak{P}_0.$$

Therefore, in each case, we have obtained chain of t-primes in $\mathrm{Int}(D)$ of length 2 and this contradicts the fact that t-dim$(\mathrm{Int}(D)) = 1$. □

We recall that D is a *UMT-domain* if every nonzero prime ideal of $D[X]$ which contracts to zero in D is a t-maximal ideal. UMT domains were introduced and studied by E.G. Houston and M. Zafrullah in [14]. Later, they have been used to characterize when the polynomial rings $D[X]$ are weakly-Krull.

Lemma 1.6 ([4, Proposition 4.11]) *Let D be an integral domain. Then $D[X]$ is weakly-Krull if and only if D is a weakly-Krull UMT-domain.*

In the following Theorem we show that $\mathrm{Int}(D)$ may be weakly-Krull only in the trivial case.

Theorem 1.7 *Let D be an integral domain. The ring $\mathrm{Int}(D)$ is weakly-Krull if and only if $\mathrm{Int}(D) = D[X]$ and D is a weakly-Krull UMT-domain.*

Proof Assume that $\mathrm{Int}(D)$ is weakly-Krull. From Lemma 1.6, it is sufficient to show that $\mathrm{Int}(D) = D[X]$. If $\mathrm{Int}(D) \neq D[X]$, from [7, Lemma I.3.6] there exists a maximal ideal $\mathfrak{m}$ of D such that $\mathrm{Int}(D)_{\mathfrak{m}} \neq D_{\mathfrak{m}}[X]$, thus $\mathfrak{m}$ is an int prime ideal of D, whence it is a t-ideal ([10, Propositon 1.2]). It is well-known that weakly Krull domains have t-dimension one, then t-dim$(\mathrm{Int}(D)) = 1$, t-dim$(D) = 1$ (Lemma 1.5) and $\mathfrak{m}$ is height-one. Set $\mathfrak{M}_0 := \{f \in \mathrm{Int}(D);\ f(0) \in \mathfrak{m}\}$. Since $\mathfrak{M}_0$ contains $\mathrm{Int}(D, \mathfrak{m})$, it follows from [10, Proposition 1.4] that $\mathfrak{M}_0$ is an int prime of $\mathrm{Int}(D)$ and it is of height-one (Lemma 1.4).

Now, consider the prime ideal $\mathfrak{Q} := \{f \in \mathrm{Int}(D);\ f(0) = 0\}$. It is easily seen that $(0) \subsetneq \mathfrak{Q} \subsetneq \mathfrak{M}_0$. Hence, $\mathfrak{M}_0$ has height at least two, which is a contradiction. Thus D has not int prime ideals, so $\mathrm{Int}(D) = D[X]$.

The converse follows from Lemma 1.6. □

We investigate when $\mathrm{Int}(D) = D[X]$ in the case D is weakly-Krull.

Proposition 1.8 *Let D be a weakly-Krull domain. Then $\mathrm{Int}(D) = D[X]$ if and only if $\mathrm{Int}(D_{\mathfrak{p}}) = D_{\mathfrak{p}}[X]$, for each $\mathfrak{p} \in X^1(D)$.*

Proof We observe that since the intersection $\cap_{\mathfrak{p}\in X^1(D)} D_{\mathfrak{p}} = D$ is locally finite, $\mathrm{Int}(D_{\mathfrak{p}}) = \mathrm{Int}(D)_{\mathfrak{p}}$ for each $\mathfrak{p} \in X^1(D)$ (cf. [20, Proposition 2.3]). Then if $\mathrm{Int}(D) = D[X]$, obviously $\mathrm{Int}(D)_{\mathfrak{p}} = D_{\mathfrak{p}}[X]$ and the thesis follows from the equality $\mathrm{Int}(D) = \cap_{\mathfrak{p}\in X^1(D)} \mathrm{Int}(D_{\mathfrak{p}})$.

On the contrary, if $\mathrm{Int}(D_{\mathfrak{p}}) = D_{\mathfrak{p}}[X]$ for each $\mathfrak{p} \in X^1(D)$, then $\mathrm{Int}(D) = \cap_{\mathfrak{p}\in X^1(D)} \mathrm{Int}(D_{\mathfrak{p}}) = \cap_{\mathfrak{p}\in X^1(D)} D_{\mathfrak{p}}[X] = D[X]$. □

Thus to have $\mathrm{Int}(D) = D[X]$ for a weakly-Krull domain D, it is sufficient to know when a one-dimensional local domain D is such that $\mathrm{Int}(D) = D[X]$. This happens, for instance, when the residue field of D over the maximal ideal is infinite (cf. [7, Proposition I.3.4]) or when D is a valuation domain with non-principal maximal ideal (cf. [7, Proposition I.3.16]).

It is well-known that a Noetherian domain D is a UMT-domain if and only if it has t-dimension one (cf. [14, Theorem 3.7]). Thus, from Theorem 1.7 and [4, Corollary 4.12] we get a similar result to [9, Corollary 2.7], which states that $\mathrm{Int}(D)$ is a Krull domain if and only if D is Krull and $\mathrm{Int}(D) = D[X]$.

Corollary 1.9 *Let D be a Noetherian domain. Then* $\mathrm{Int}(D)$ *is weakly-Krull if and only if D is weakly-Krull and* $\mathrm{Int}(D) = D[X]$.

Corollary 1.10 *Let D be a Noetherian domain. Then* $\mathrm{Int}(D)$ *is weakly-Krull if and only if D is weakly-Krull and the residue fields of D over the height-one primes are infinite.*

Proof By [7, Corollary I.3.15], if D is a one-dimensional Noetherian domain, then $\mathrm{Int}(D) = D[X]$ if and only if its residue field is infinite. The thesis follows from Proposition 1.8 and Corollary 1.9. □

Krull domains are exactly the domains which are locally finite intersection of localizations that are DVR (or, equivalently PID). Like weakly-Krull domains, a natural generalization of Krull domains are the *infra-Krull* domains which are locally finite intersections of one-dimensional Noetherian domains that are localizations. Infra-Krull domains were introduced by M. Martin and M. Zafrullah in [15] and it is easily seen that they are weakly-Krull. Conversely, weakly-Krull domains are not necessarily infra-Krull: for instance, consider $D = \mathbb{Q} + X\mathbb{R}[[X]]$. In this case D is a local, Mori non-Noetherian domain of dimension one ([6, Corollary 3.5]). Since Mori domains have the t-finite character and D is one-dimensional, then D is weakly-Krull. But D is not infra-Krull because it is not Noetherian.

By definition, an integrally closed infra-Krull domain is Krull (since a one-dimensional, local, Noetherian, integrally closed domain is DVR). Then, any non integrally closed, one-dimensional Noetherian domain is infra-Krull and not Krull.

We recall that an integral domain D is *Strong Mori* if it satisfies the ascending chain condition on integral w-ideals (see [12]). Note that the class of Strong Mori domains includes Noetherian domains and Krull domains and it is well-known that an infra-Krull domain is exactly a Strong Mori domain of t-dimension one.

Corollary 1.11 *Let D be an integral domain. Then* $\mathrm{Int}(D)$ *is infra-Krull if and only if D is infra-Krull and* $\mathrm{Int}(D) = D[X]$.

Proof Assume that $\mathrm{Int}(D)$ is infra-Krull. Then, by Theorem 1.7, $\mathrm{Int}(D) = D[X]$ because any infra-Krull domain is weakly-Krull. Since infra-Krull is Strong Mori of t-dimension one, it follows from [11, Theorem 2.2] and Lemma 1.5 that D is a Strong Mori domain of t-dimension one, i.e., D is infra-Krull.

The converse follows from [17, Theorem 4.3]. □

We remark that the notions "infra-Krull" and "weakly-Krull" for $\mathrm{Int}(D)$ coincide when D is an infra-Krull domain.

Corollary 1.12 *Let D be an infra-Krull domain. Then the following statements are equivalent.*

(i) $\mathrm{Int}(D)$ *is an infra-Krull domain;*
(ii) $\mathrm{Int}(D)$ *is a weakly-Krull domain;*
(iii) $\mathrm{Int}(D) = D[X]$.

Another interesting class of weakly-Krull domains are the *generalized Krull* domains, described by R. Gilmer in [13, Section 43]. These are weakly-Krull domains such that their localizatons $D_{\mathfrak{p}}$ are valuation domains for each $\mathfrak{p} \in X^1(D)$, or equivalently, weakly-Krull PvMDs (recall that a PvMD is a domain whose localizations at t-maximal ideals are valuation domains).

The following result that characterizes when $\mathrm{Int}(D)$ is generalized Krull was obtained by the current authors in [18] in the context of Krull-type domains. Here we get it as a corollary of Theorem 1.7.

Corollary 1.13 *Let D be an integral domain. Then* $\mathrm{Int}(D)$ *is generalized Krull if and only if D is generalized Krull and* $\mathrm{Int}(D) = D[X]$.

Proof Suppose that $\mathrm{Int}(D)$ is a generalized Krull domain. Then, by Theorem 1.7, $\mathrm{Int}(D) = D[X]$ and hence $D[X]$ is generalized Krull. Thus, as cited above, $D[X]$ is a weakly-Krull PvMD and therefore it follows from Lemma 1.6 and [21, Corollary 4] that D is a weakly-Krull PvMD, i.e., D is generalized Krull. The converse follows from [13, Theorem 43.11(3)]. □

In the following Corollary we give a description of when an infra-Krull or generalized Krull domain D is such that $\mathrm{Int}(D) = D[X]$.

Corollary 1.14 *Let D be an integral domain.*

(a) If D is infra-Krull, then $\mathrm{Int}(D) = D[X]$ *if and only if each height-one prime ideal of D has infinite residue field.*
(b) If D is generalized-Krull, then $\mathrm{Int}(D) = D[X]$ *if and only if* $D_{\mathfrak{p}}$ *has infinite residue field or non-principal maximal ideal, for each* $\mathfrak{p} \in X^1(D)$.

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