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The existence or nonexistence of solutions for some equations involving weighted critical exponents on the unit ball

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Abstract
In this paper, we extend the results of Brézis and Willem (J Funct Anal 255:2286–2298, 2008) to the equation with single or double weighted critical exponents, including Hardy–Sobolev, Sobolev and Hénon–Sobolev exponents. More precisely, we establish the existence or nonexistence of equation with different coefficient which has an important impact.

Keywords Nonexistence · Existence · Weighted critical exponents

Mathematics Subject Classification Primary: 35A01 · 35A15 · 35A24 · 35J61

1 Introduction

In this paper we consider the following equation

$$\begin{cases} -\Delta u + h\left(\frac{|x|}{\lambda}\right)\frac{u}{\lambda^2} = |x|^{\alpha_1}|u|^{2^*(\alpha_1)-2}u + \mu|x|^{\alpha_2}|u|^{2^*(\alpha_2)-2}u \text{ in } B, \\ u \in H^1_{0,r}(B), \end{cases} \tag{1.1}$$

where $N \geq 3, \alpha_1 > \alpha_2 > -2, 2^*(\alpha_i) = \frac{2(N+\alpha_i)}{N-2}, i = 1, 2$ $\mu \in \mathbb{R}$, $B := \{x \in \mathbb{R}^N : |x| < 1\}$ is the unit ball in $\mathbb{R}^N$, $H^1_{0,r}(B)$ is the completion of $C^\infty_{0,r}(B)$ with the norm

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$$\|u\| = \left(\int_B |\nabla u|^2 dx \right)^{\frac{1}{2}},$$

where $C^\infty_{0,r}(B)$ is the set of radial functions in $C^\infty_0(B)$. Let

$$L^p(B; |x|^\alpha) = \left\{ u : B \to \mathbb{R} : \quad u \text{ is measurable}, \ \int_B |x|^\alpha |u|^p dx < \infty \right\}$$

be the weighted Lebesgue space with the norm

$$\|u\|_{L^p(B;|x|^\alpha)} := \left(\int_B |x|^\alpha |u|^p dx \right)^{\frac{1}{p}}, 1 \le p < \infty.$$

It holds that

$$H^1_{0,r}(B) \hookrightarrow L^p(B; |x|^\alpha) \tag{1.2}$$

with $\alpha \ge -2$ is continuous for all $1 \le p \le 2^*(\alpha) := \frac{2(N+\alpha)}{N-2}$ and it is compact for all $1 \le p < 2^*(\alpha)$, see [20,21]. The compact embedding of (1.2) for $\alpha > 0$ was first proved in [17]. In [23,24] we have confirmed that $2^*(\alpha)$ is exactly the upper critical exponent of the embedding (1.2) by proving that there is no embedding from $H^1_{0,r}(B)$ into $L^p(B; |x|^\alpha)$ for any $p > 2^*(\alpha)$ and (1.2) is not compact as $p = 2^*(\alpha)$. It is known that $2^*(\alpha)$ is Hardy (resp.,Hardy–Sobolev, Sobolev) critical exponent as $\alpha = -2$(resp., $-2 < \alpha < 0$, $\alpha = 0$), see [11,23]. In [23,24], we named $2^*(\alpha)$ as Hénon–Sobolev critical exponent for $\alpha > 0$ due to Hénon [14] first raised a semilinear elliptic model involving $|x|^\alpha$ with $\alpha > 0$. Therefore there are two critical terms in (1.1).

For $\alpha_1 = 0$ and $\mu = 0$, (1.1) reduces as

$$\begin{cases} -\Delta u + h\left(\frac{|x|}{\lambda}\right)\frac{u}{\lambda^2} = |u|^{2^*(0)-2}u \text{ in } B, \\ u = 0 \qquad\qquad\qquad\qquad\qquad \text{on } \partial B. \end{cases} \tag{1.3}$$

For the case of $h\left(\frac{|x|}{\lambda}\right)\frac{1}{\lambda^2} := a(x) \le 0$, (1.3) has been treated extensively since the great work [5] of Brézis and Nirenberg. In [4], Brézis raised seven open problems and the fourth one read as

Q4. *Assume $a(x) \ge 0$ on B. Find conditions on $a(x)$ (hopefully a necessary and sufficient condition!) which guarantee that* (1.3) *has a solution.*

In [18], Passaseo gave a partial answer to **Q4**. Under some conditions on $a(x)$, Passaseo proved the existence of positive solutions (1.3). In [6,7], Brézis and Peletier studied (1.3) with $N = 3$ and

$$h(|x|) = \frac{K}{(1+|x|^2)^2}, \quad K > 0. \tag{1.4}$$

They proved the existence and non-existence of solutions based on different region of value λ. In [8], Brézis and Willem studied **Q4** for the case of $N \geq 3$ with more general assumptions on h. In the present paper we will extend the results of Brézis and Willem to the equation (1.1) with one or two weighted critical exponents. For other related works we refer to [3] with unbounded domain $\mathbb{R}^N$, to [1] with ball or annular domain, to [16] with p-Laplacian and to [24] with multiple weighted critical exponents.

In Sect. 2 we consider the non-existence of solutions of (1.1) applying the ODE theory. In Sect. 3 we are interested in the existence results of (1.1) with single weighted critical exponent ($\mu = 0$).

2 Nonexistence

In this section, we are interested in (1.1) with multiple Hénon–Sobolev critical exponents as $\alpha_1 > \alpha_2 \geq 0$. We will prove the nonexistence of solutions of (1.1) with different value λ, the methods depend on the ODE theory.

We assume that

$$(\mathrm{h}_1)\quad h \in L^\infty_{\mathrm{loc}} \text{ and } r^2 h(r) \text{ isnondecreasingon } [0, 1].$$

It follows from (h_1) that $\lim\limits_{r \to 1^-} r^2 h(r) = h(1^-)$ exists. The function (1.4) satisfies (h_1).

Theorem 2.1 *Assume that h satisfies* (h_1). *Then* (1.1) *has only the trivial solution in each of the following cases:*

(i) $\lambda \geq 1$ *if* $\mu < 0$*;*

(ii) *there exists* $\lambda^* = \lambda^*(h) \in (0, 1)$ *and* $\lambda > \lambda^*$ *if* $\mu \geq 0$.

Next we consider the following equation

$$\begin{cases} -\Delta u + h(|x|)u = |x|^{\alpha_1}|u|^{2^*(\alpha_1)-2}u + \mu|x|^{\alpha_2}|u|^{2^*(\alpha_2)-2}u & \text{in } B, \\ u \in H^1_{0,r}(B). \end{cases} \tag{2.1}$$

Assume

$$(\mathrm{h}_2)\quad h \in L^\infty(0, 1) \text{ and } r^2 h(r) \text{ isnondecreasingon } (0, \delta) \text{ forsome } \delta \in (0, 1).$$

We remark that the function (1.4) also satisfies (h_2).

Theorem 2.2 *For $\mu \geq 0$ and $\delta \in (0, 1)$, there exists $K_1 = K_1(\delta, \alpha_1, \alpha_2, N) > 0$ such that, if h satisfies* (h_2) *and $\|h\|_\infty \leq K_1$, then* (2.1) *has only trivial solution.*

For $N = 3$, a sharper conclusion will be obtained.

Theorem 2.3 *Assume $N = 3$, $\mu \geq 0$ and $h \in L^\infty(0, 1)$. There exists $K_1 = K_1(\alpha_1, \alpha_2, N) > 0$ such that if $\|h\|_\infty \leq K_1$ then* (2.1) *has only trivial solution.*

Remark 2.4 For the case $\alpha_1 = \mu = 0$ in Theorem 2.3, the conclusion has been proved by Brézis and Willem in [8]. In addition, when $\alpha_1 = \mu = 0$, $N = 3$, $h = -\lambda$ and

$0 < \lambda < \frac{\pi^2}{4}$, Brézis and Nirenberg first prove the solution $u = 0$ of (2.1) in [5]. In [24], we extend the results of Brézis and Nirenberg to the case of $\alpha_1 > 0$.

Now we begin to prove Theorems 2.1–2.3. We follow some arguments in [8] with modifications.

Under (h_1), for $\alpha_1 > \alpha_2 \geq 0$ and $\lambda > 0$, by Brézis-Kato theorem and Sobolev embedding theorem we have a fact that any a solution u of (1.1) must satisfy $u \in C^1(\bar{B})$, furthermore, $u \in L^\infty(B)$.

Set $u(r) := u(|x|)$ with $r = |x|$. Then (1.1) can be reset as

$$\begin{cases} -u'' - \dfrac{N-1}{r}u' + h\left(\dfrac{r}{\lambda}\right)\dfrac{u}{\lambda^2} = r^{\alpha_1}|u|^{2^*(\alpha_1)-2}u + \mu r^{\alpha_2}|u|^{2^*(\alpha_2)-2}u, \ r \in (0,1), \\ u'(0) = u(1) = 0. \end{cases} \tag{2.2}$$

Applying the classical Emden transformation

$$u(r) = e^{\frac{N-2}{2}t}w(t), \quad t = -\ln r, \tag{2.3}$$

then (2.2) can be reduced as

$$\begin{cases} -w'' + \dfrac{(N-2)^2}{4}w + H_\lambda(t)w = |w|^{2^*(\alpha_1)-2}w + \mu|w|^{2^*(\alpha_2)-2}w, \quad t > 0, \\ w(0) = 0 \end{cases} \tag{2.4}$$

with

$$\begin{aligned} &|w(t)| \leq e^{\frac{2-N}{2}t}\|u\|_{L^\infty(B)}, \quad |w'(t)| \\ &\leq e^{\frac{2-N}{2}t}\left(\frac{N-2}{2}\|u\|_{L^\infty(B)} + e^{-t}\|u'\|_{L^\infty(B)}\right), \end{aligned} \tag{2.5}$$

where

$$H_\lambda(t) = \frac{e^{-2t}}{\lambda^2}h\left(\frac{e^{-t}}{\lambda}\right).$$

By (2.5), we see that

$$\lim_{t\to\infty} w(t) = 0, \qquad \lim_{t\to\infty} w'(t) = 0. \tag{2.6}$$

Let

$$t_\lambda = -\ln \lambda. \tag{2.7}$$

Lemma 2.5 *Let h satisfy* (h_1) *and let $w : [0,\infty) \to \mathbb{R}$ be a solution of* (2.4)*. Then*

$$w'(0)^2 \leq -2\int_0^{t_\lambda} H_\lambda(t)w(t)w'(t)dt + h(1^-)w(t_\lambda)^2, \ \ 0 < \lambda < 1. \tag{2.8}$$

Proof Multiplying (2.4) by w' and integrating on $(0, \infty)$, using (2.6), we get

$$\frac{1}{2}(w'(0))^2 + \int_0^\infty H_\lambda(t)w(t)w'(t)dt = 0. \tag{2.9}$$

Now we decompose the integral interval of second term of (2.9) as

$$\int_0^\infty H_\lambda(t)ww'dt = \int_0^{t_\lambda} H_\lambda(t)ww'dt + \int_{t_\lambda}^\infty H_\lambda(t)ww'dt. \tag{2.10}$$

Integration by parts, we obtain

$$\int_{t_\lambda}^\infty H_\lambda(t)ww'dt = -\frac{1}{2}H_\lambda(t_\lambda^+)w(t_\lambda)^2 - \frac{1}{2}\int_{t_\lambda}^\infty w(t)^2 d(H_\lambda(t)). \tag{2.11}$$

It follows from (h_1) and (2.7) that $H_\lambda(t)$ is non-increasing on (t_λ, ∞) and $H(t_\lambda^+) = h(1^-)$. Thus

$$\int_{t_\lambda}^\infty H_\lambda ww'dt \geq -\frac{1}{2}h(1^-)w(t_\lambda)^2. \tag{2.12}$$

Combining with (2.9), (2.10) and (2.12), we obtain the desired conclusion that

$$w'(0)^2 \leq -2\int_0^{t_\lambda} H_\lambda(t)w(t)w'(t)dt + h(1^-)w(t_\lambda)^2.$$

The proof is complete. □

Lemma 2.6 *([8, Lemma 2.2]) Assume $A \geq 0$, $B > 0$, $L > 0$ and $w \in C^1([0, L])$ satisfies $w(0) = 0$,*

$$w'(t)^2 \leq A^2 + 2B^2\int_0^t |ww'|ds \quad \text{for } 0 \leq t \leq L.$$

Then

$$|w(t)| \leq \frac{A}{B}\left(e^{Bt} - 1\right) \; |w'(t)| \leq Ae^{Bt}, \quad for\ 0 \leq t \leq L.$$

Lemma 2.7 *Assume (h_1) and $\mu \geq 0$. Then for $\frac{1}{2} < \lambda < 1$ and $0 \leq t \leq t_\lambda$, any a solution $w : [0, \infty) \to \mathbb{R}$ of (2.4) satisfies*

$$|w(t)| \leq \frac{1}{c_0}|w'(0)|\left(e^{c_0 t} - 1\right), \tag{2.13}$$

$$|w'(t)| \leq |w'(0)|e^{c_0 t}, \tag{2.14}$$

where

$$c_0=\sup_{1\le r<2}\left(\frac{(N-2)^2}{4}+r^2|h(r)|\right)^{\frac{1}{2}}.$$

Proof By (2.4) we obtain that

$$\begin{aligned}\frac{w'(t)^2}{2}&=\frac{w'(0)^2}{2}+\int_0^t w'w''ds\\&=\frac{w'(0)^2}{2}+\int_0^t\left(\frac{(N-2)^2}{4}ww'+h\left(\frac{e^{-s}}{\lambda}\right)\frac{e^{-2s}ww'}{\lambda^2}\right)ds\\&\quad-\int_0^t|w|^{2^*(\alpha_1)-2}ww'ds-\mu\int_0^t|w|^{2^*(\alpha_2)-2}ww'ds\\&\le\frac{w'(0)^2}{2}+\int_0^t\left[\frac{(N-2)^2}{4}+\frac{e^{-2s}}{\lambda^2}\left|h\left(\frac{e^{-s}}{\lambda}\right)\right|\right]|w(s)w'(s)|ds.\end{aligned}$$

For $\frac{1}{2}<\lambda<1$ and $0\le t\le t_\lambda$, we have

$$\frac{e^{-s}}{\lambda}\in\left[\frac{e^{-t_\lambda}}{\lambda},\frac{e^0}{\lambda}\right]\subset[1,2).$$

It follows that

$$w'(t)^2\le w'(0)^2+2c_0^2\int_0^t|ww'|dx,$$

where

$$c_0=\sup_{1\le r<2}\left(\frac{(N-2)^2}{4}+r^2|h(r)|\right)^{\frac{1}{2}}.$$

Applying Lemma 2.6 with $A=|w'(0)|$ and $B=c_0$, we obtain (2.13) and (2.14). □

Proof of Theorem 2.1 For $\lambda\ge 1,\mu<0$. Multiplying (1.1) by $\sum_{i=1}^N x_i\frac{\partial u}{\partial x_i}$ and integrating on B, we obtain

$$\begin{aligned}0=&\frac{N-2}{2}\int_B|\nabla u|^2dx-\frac{N-2}{2}\int_B\left(|x|^{\alpha_1}|u|^{2^*(\alpha_1)}dx+\mu|x|^{\alpha_2}|u|^{2^*(\alpha_2)}\right)dx\\&+\frac{1}{2}\int_{\partial B}\left|\frac{\partial u}{\partial n}\right|^2d\sigma+\frac{N}{2}\int_B h\left(\frac{|x|}{\lambda}\right)\frac{1}{\lambda^2}u^2dx\\&+\frac{1}{2}\int_B\frac{1}{\lambda^2}\left[\sum_{i=1}^N x_i\frac{\partial}{\partial x_i}h\left(\frac{|x|}{\lambda}\right)\right]u^2dx.\end{aligned}\tag{2.15}$$

Since u satisfies

$$\int_B |\nabla u|^2 + \int_B h\left(\frac{|x|}{\lambda}\right)\frac{u^2}{\lambda^2}dx = \int_B |x|^{\alpha_1}|u|^{2^*(\alpha_1)}dx + \mu\int_B |x|^{\alpha_2}|u|^{2^*(\alpha_2)}dx, \tag{2.16}$$

it follows from (2.15) and (2.16) that

$$-\int_B \left(h\left(\frac{|x|}{\lambda}\right) + \frac{1}{2}\sum_{i=1}^N x_i \frac{\partial}{\partial x_i} h\left(\frac{|x|}{\lambda}\right)\right)\frac{u^2}{\lambda^2}dx = \frac{1}{2}\int_{\partial B}\left|\frac{\partial u}{\partial n}\right|^2 d\sigma.$$

Since

$$h\left(\frac{|x|}{\lambda}\right) + \frac{1}{2}\sum_{i=1}^N x_i \frac{\partial}{\partial x_i} h\left(\frac{|x|}{\lambda}\right) = \frac{1}{s}(s^2 h(s))', \quad s = \frac{|x|}{\lambda},$$

it follows from (h_1) that $u = 0$.

For $\lambda < 1$ and $\mu \geq 0$. We may assume $\frac{1}{2} < \lambda < 1$. The inequality (2.8) from Lemma 2.5 implies that

$$w'(0)^2 \leq 2\int_0^{t_\lambda} |H_\lambda(t)||w(t)||w'(t)|dt + h(1^-)w(t_\lambda)^2. \tag{2.17}$$

Inserting (2.13) and (2.14) from Lemma 2.7 into (2.17), we have

$$w'(0)^2 \leq w'(0)^2 K(\lambda), \tag{2.18}$$

where

$$K(\lambda) = 2c_0\int_0^{t_\lambda}\left|\frac{e^{c_0 t}-1}{c_0}\right|\left|e^{c_0 t}\right|dt + h(1^-)\frac{1}{c_0^2}(e^{c_0 t_\lambda}-1)^2.$$

It is obvious that $t_\lambda \downarrow 0$ and $K(\lambda) \to 0$ as $\lambda \uparrow 1$. It follows that there exists $\lambda^* \in (\frac{1}{2}, 1)$ such that $K(\lambda) < 1$ as $\lambda^* < \lambda < 1$. Using this fact, the inequality (2.18) leads to $w'(0) = 0$ for $\lambda^* < \lambda < 1$. By the uniqueness of the Cauchy problem, we complete the proof. □

Proof of Theorem 2.2 We will argue in the same way as proving Theorem 2.1. Let $u \in H^1_{0,r}(B)$ be a solution of (2.1). Using the same transformation (2.3), (2.1) becomes

$$\begin{cases} -w''(t) + \dfrac{(N-2)^2}{4}w(t) + h\left(e^{-t}\right)e^{-2t}w(t) \\ \qquad = |w(t)|^{2^*(\alpha_1)-2}w(t) + \mu|w(t)|^{2^*(\alpha_1)-2}w(t), \quad t > 0, \\ w(0) = 0, \end{cases} \tag{2.19}$$

Then w satisfies (2.5). Set $H(t) = e^{-2t}h(e^{-t})$ and $T_\delta = -\ln\delta$. Multiplying equation (2.19) by w' and integrating on $(0, \infty)$, we deduce that

$$\frac{1}{2}(w'(0))^2 + \int_0^\infty H(t)w(t)w'(t)dt = 0. \tag{2.20}$$

Rewriting

$$\int_0^\infty H(t)w(t)w'(t)dt = \int_0^{T_\delta} H(t)w(t)w'(t)dt + \int_{T_\delta}^\infty H(t)w(t)w'(t)dt. \tag{2.21}$$

Integrating by parts, we obtain

$$\int_{T_\delta}^\infty H(t)w(t)w'(t)dt = -\frac{1}{2}H(T_\delta^+)w(T_\delta)^2 - \frac{1}{2}\int_{T_\delta}^\infty w(t)^2 d(H(t)).$$

By (h_2), we see that $H(t)$ is non-increasing on (T_δ, ∞) and $H(T_\delta^+) = \delta^2 h(\delta^-)$. Hence

$$\int_{T_\delta}^\infty Hww'dt \geq -\frac{1}{2}\delta^2 h(\delta^-)w(T_\delta)^2. \tag{2.22}$$

It follows from (2.20), (2.21), (2.22) that

$$(w'(0))^2 \leq -2\int_0^{T_\delta} H(t)w(t)w'(t)dt + h(\delta^-)\delta^2 w(T_\delta)^2. \tag{2.23}$$

The estimates (2.13) and (2.14) are still valid on $(0, T_\delta)$ with

$$c_0 := \left(\frac{(N-2)^2}{4} + \|h\|_\infty\right)^{\frac{1}{2}}. \tag{2.24}$$

Combining with (2.23) and (2.13), (2.14), we deduce that

$$w'(0)^2 \leq w'(0)^2\|h\|_\infty \left[\frac{2}{c_0}\int_0^{T_\delta}(e^{c_0 t} - 1)e^{c_0 t}dt + \frac{\delta^2}{c_0^2}(e^{c_0 T_\delta} - 1)^2\right]. \tag{2.25}$$

Hence there exists $K_1 > 0$ such that $w'(0) = 0$ as $\|h\|_\infty \leq K_1$. □

Proof of Theorem 2.3 Similar with the proofs of Theorem 2.1 and Theorem 2.2, we have

$$w'(0)^2 \leq 2\int_0^\infty |H(t)||w(t)||w'(t)|dt.$$

Notice that in (2.24) $c_0 = \left(\frac{1}{4} + \|h\|_\infty\right)^{1/2}$ for $N = 3$ so that $c_0 \in (1/2, 1)$ when $\|h\|_\infty$ is small. Therefore

$$|H(t)| \le \|h\|_\infty e^{-2t}, \quad |w(t)| \le \frac{|w'(0)|}{c_0} e^{c_0 t}, \quad |w'(t)| \le |w'(0)| e^{c_0 t}$$

and then

$$w'(0)^2 \le \frac{2}{c_0} |w'(0)|^2 \|h\|_\infty \int_0^\infty e^{2(c_0-1)t} dt.$$

Using the fact that $c_0 < 1$, the conclusion of theorem is proved. □

3 Existence for the case of $\mu = 0$

In this section we prove the existence of nontrivial solutions for the case $\mu = 0$ in (1.1). We reformulate (1.1) as follows by setting $\alpha := \alpha_1$,

$$\begin{cases} -\Delta u + h\left(\frac{|x|}{\lambda}\right)\frac{u}{\lambda^2} = |x|^\alpha |u|^{2^*(\alpha)-2} u \text{ in } B, \\ u \in H^1_{0,r}(B), \end{cases} \tag{3.1}$$

where $N \ge 3$, $\alpha > -2$, $2^*(\alpha) = \frac{2(N+\alpha)}{N-2}$ is the Hardy–Sobolev or Sobolev or Hénon–Sobolev critical exponent. The potential h satisfies

(h_3) $h : [0, \infty) \to [0, \infty)$ is such that $h \not\equiv 0$ on a set of positive measure and

$$h \in L^{N/2}_{\text{loc}}([0, \infty), s^{N-1}), \tag{3.2}$$

$$\lim_{\lambda \to 0} \lambda^{N-2} \int_0^{\frac{1}{\lambda}} h(s) s^{N-1} ds = 0. \tag{3.3}$$

We remark that $h \in L^{N/2}([0, \infty), s^{N-1})$ satisfies (3.2), and (3.3) if $\lim_{s\to\infty} s^2 h(s) = 0$. Hence the function

$$h(s) = \frac{1}{1 + s^3}$$

satisfies (3.2) and (3.3). We will prove the following theorem.

Theorem 3.1 *Assume that h satisfies* (h_3) *and* $\alpha > -2$. *Then there exists* $\lambda_0 > 0$ *such that* (3.1) *has a nonnegative solution for* $0 < \lambda < \lambda_0$.

Remark 3.2 Since $h \ge 0$, it is not possible to prove the existence of solutions of (3.1) by the global minimization as in [5,24], the main difficulty is to estimate the energy level of quotient is less than some number, which guarantee the holds of local $(PS)_c$. For $\alpha = 0$ and $h \in L^{N/2}([0, \infty), s^{N-1} ds)$, the existence of positive solutions was obtained by Passaseo in [18] using the constrained minimization and in [8], Brézis and Willem obtain the nontrivial solution under (h_3). Theorem 3.1 extends the result in [8] to the case $\alpha > -2$.

Remark 3.3 In Theorem 3.1 a positive solution can be obtained via strong maximum principle if (3.2) is replaced by

$$h \in L^{\infty}_{\text{loc}}([0, \infty), s^{N-1}).$$

The approach for proving Theorem 3.1 is from [8] and [16]. Define the manifolds

$$\Gamma(B) := \left\{ u \in H^1_{0,r}(B) : \int_B |x|^\alpha |u|^{2^*(\alpha)} dx = 1 \right\},$$
$$\Gamma(\mathbb{R}^N) := \left\{ u \in D^{1,2}_r(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x|^\alpha |u|^{2^*(\alpha)} dx = 1 \right\},$$

and the functionals for $\lambda > 0$,

$$\varphi_\lambda(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx + h\left(\frac{|x|}{\lambda}\right) \frac{u^2}{\lambda^2} dx, \quad u \in D^{1,2}_r(\mathbb{R}^N),$$
$$\psi_\lambda(u) = \int_{\mathbb{R}^N} a_\lambda(|x|)|x|^\alpha |u|^{2^*(\alpha)} dx, \quad a_\lambda(|x|) = \frac{|x|}{\lambda + |x|}, \quad u \in L^{2^*(\alpha)}(\mathbb{R}^N; |x|^\alpha),$$

where

$$D^{1,2}_r(\mathbb{R}^N) = \left\{ u \in D^{1,2}(\mathbb{R}^N) : u \text{ is radial} \right\}, \quad \|u\|_{D^{1,2}_r} := \|\nabla u\|_{L^2(\mathbb{R}^N)},$$
$$L^{2^*(\alpha)}(\mathbb{R}^N; |x|^\alpha) = \left\{ u : \mathbb{R}^N \to \mathbb{R} : \text{ is measurable, } \int_{\mathbb{R}^N} |x|^\alpha |u|^{2^*(\alpha)} dx < \infty \right\}.$$

Under (h_3), the functional φ_λ is well defined but not necessarily finite on $D^{1,2}_r(\mathbb{R}^N)$. We will prove

$$\Sigma_\lambda = \inf \left\{ \varphi_\lambda(u) : u \in \Gamma(B), \ \psi_\lambda(u) \geq \frac{1}{2} \right\}$$

is a critical value of $\varphi|_{\Gamma(B)}$. We shall estimate the values

$$\Upsilon_\lambda = \inf \left\{ \varphi_\lambda(u) : u \in \Gamma(B), \ \psi_\lambda(u) = \frac{1}{2} \right\},$$
$$\Upsilon = \inf \left\{ \varphi_1(u) : u \in \Gamma(\mathbb{R}^N), \ \psi_1(u) = \frac{1}{2} \right\}.$$

Consider the weighted critical equation

$$\begin{cases} -\Delta u = |x|^\alpha u^{2^*(\alpha)-1}, \ u > 0 \text{ in } \mathbb{R}^N, \\ u \in D^{1,2}_r(\mathbb{R}^N) \end{cases} \tag{3.4}$$

with $\alpha > -2$. By [12,13,15], we have the following key result.

Theorem 3.4 *Let $\alpha > -2$. It holds that*

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq S_\alpha \left(\int_{\mathbb{R}^N} |x|^\alpha |u|^{2^*(\alpha)} dx \right)^{\frac{2}{2^*(\alpha)}}, \quad u \in D_r^{1,2}(\mathbb{R}^N). \tag{3.5}$$

The best constant S_α can be achieved uniquely (up to dilations) by

$$U_\alpha(x) = \frac{C(\alpha, N)}{\left(1 + |x|^{2+\alpha}\right)^{\frac{N-2}{2+\alpha}}}, \quad C(\alpha, N) = \left[(N+\alpha)(N-2)\right]^{\frac{N-2}{2(2+\alpha)}} \tag{3.6}$$

and U_α is the unique (up to dilations) solution of (3.4) and

$$S_\alpha = (N+\alpha)(N-2)\left(\frac{\omega_N}{2+\alpha} \frac{\Gamma^2\left(\frac{N+\alpha}{2+\alpha}\right)}{\Gamma\left(\frac{2(N+\alpha)}{2+\alpha}\right)} \right)^{\frac{2+\alpha}{N+\alpha}}.$$

We give some remarks. For $\alpha = 0$, Theorem 3.4 was proved by Aubin[2], Talenti[22] and S_0 was the best Sobolev constant on $D_r^{1,2}(\mathbb{R}^N)$(see [22]). For $-2 < \alpha < 0$, Theorem 3.4 was established by Ghoussoub and Yuan [11], Lieb[15], and S_α was named as the best Hardy–Sobolev constant on $D_r^{1,2}(\mathbb{R}^N)$(see [10]). As $\alpha > 0$, these results could be found in [12,13,15] and S_α was named in [24] as the best Hénon–Sobolev constant.

The corresponding energy functional of (3.1) is defined as

$$\Phi(u) = \frac{1}{2}\int_B |\nabla u|^2 dx + h\left(\frac{|x|}{\lambda}\right)\frac{|u|^2}{\lambda^2}dx - \frac{1}{2^*(\alpha)}\int_B |x|^\alpha |u|^{2^*(\alpha)} dx, \quad u \in H_{0,r}^1(B).$$

We define

$$\Psi(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2^*(\alpha)}\int_{\mathbb{R}^N} |x|^\alpha |u|^{2^*(\alpha)} dx.$$

Lemma 3.5 *If $u_n \rightharpoonup u$ in $D_r^{1,2}(\mathbb{R}^N)$, then*

$$|x|^\alpha |u_n|^{2^*(\alpha)-2} u_n - |x|^\alpha |u_n - u|^{2^*(\alpha)-2}(u_n - u) \to |x|^\alpha |u|^{2^*(\alpha)-2} u \text{ in } (D_r^{1,2}(\mathbb{R}^N))^*.$$

Proof The proof is similar with the argument in [9, Lemma 3.3]. Denote $w_n = u_n - u$. We have

$$\left| |x|^\alpha |u_n|^{2^*(\alpha)-2} u_n - |x|^\alpha |w_n|^{2^*(\alpha)-2} w_n \right| \leq (2^*(\alpha) - 1)\left(|u_n| + |u|\right)^{2^*(\alpha)-2} |x|^\alpha |u|.$$

For $T > 0$ and $\varphi \in C_{0,r}^\infty(\mathbb{R}^N)$, applying the Hölder inequality and (3.5),

$$\left| \int_{|x|>T} \left(|x|^\alpha |u_n|^{2^*(\alpha)-2} u_n - |x|^\alpha |w_n|^{2^*(\alpha)-2} w_n \right) \varphi dx \right|$$

$$\leq C\left[\left(\int_{|x|>T}|x|^{\alpha}|u_n|^{2^*(\alpha)}dx\right)^{\frac{2^*(\alpha)-2}{2^*(\alpha)}}+\left(\int_{|x|>T}|x|^{\alpha}|u|^{2^*(\alpha)}dx\right)^{\frac{2^*(\alpha)-2}{2^*(\alpha)}}\right]$$
$$\times\left(\int_{|x|>T}|x|^{\alpha}|u|^{2^*(\alpha)}dx\right)^{\frac{1}{2^*(\alpha)}}\left(\int_{|x|>T}|x|^{\alpha}|\varphi|^{2^*(\alpha)}dx\right)^{\frac{1}{2^*(\alpha)}}$$
$$\leq C\|\varphi\|_{D_r^{1,2}}\left(\int_{|x|>T}|x|^{\alpha}|u|^{2^*(\alpha)}dx\right)^{\frac{1}{2^*(\alpha)}}.$$

Similarly, we get that

$$\left|\int_{|x|>T}\left(|x|^{\alpha}|u|^{2^*(\alpha)-2}u\right)\varphi dx\right|\leq C\|\varphi\|_{D_r^{1,2}}\left(\int_{|x|>T}|x|^{\alpha}|u|^{2^*(\alpha)}dx\right)^{\frac{1}{2^*(\alpha)}}.$$

Therefore, for any $\epsilon>0$, there exists $T>0$ such that, for any $\varphi\in C_{0,r}^{\infty}(\mathbb{R}^N)$, it holds

$$\left|\int_{|x|>T}\left(|x|^{\alpha}|u_n|^{2^*(\alpha)-2}u_n-|x|^{\alpha}|w_n|^{2^*(\alpha)-2}w_n-|x|^{\alpha}|u|^{2^*(\alpha)-2}u\right)\varphi dx\right|\leq\epsilon\|\varphi\|_{D^{1,2}}$$

Applying [26, Proposition 5.4.7]. We obtain on $\bar{B}_T$ with $B_T:=\{x\in\mathbb{R}^N,|x|<T\}$ that

$$\int_{|x|\leq T}|x|^{\alpha}|w_n|^{2^*(\alpha)-2}w_n\varphi dx\to 0,$$
$$\int_{|x|\leq T}|x|^{\alpha}|u_n|^{2^*(\alpha)-2}u_n\varphi dx\to\int_{|x|\leq T}|x|^{\alpha}|u|^{2^*(\alpha)-2}u\varphi dx.$$

Hence

$$\int_{|x|\leq T}\left(|x|^{\alpha}|u_n|^{2^*(\alpha)-2}u_n\varphi dx-|x|^{\alpha}|w_n|^{2^*(\alpha)-2}w_n\right)\varphi dx$$
$$\to\int_{|x|\leq T}|x|^{\alpha}|u|^{2^*(\alpha)-2}u\varphi dx.$$

The proof is complete. □

Theorem 3.6 *Let* $\{v_n\}\subset D_r^{1,2}(\mathbb{R}^N)$ *be a* $(\mathrm{PS})_c$ *sequence of* Ψ, *i.e.*

$$\Psi(v_n)\to c,\quad \Psi'(v_n)\to 0\ \text{ in } (D_r^{1,2}(\mathbb{R}^N))^*.$$

Then, passing subsequence if necessary, there exist a finite sequence $\{u^0,u^1,u^2,\ldots,u^k\}\subset D_r^{1,2}(\mathbb{R}^N)$ *of solutions for*

$$-\Delta u=|x|^{\alpha}|u|^{2^*(\alpha)-2}u\ \text{ on }\mathbb{R}^N$$

and k sequences $\{\lambda_n^i\} \subset \mathbb{R}_+$ *such that* $\lambda_n^i \to 0$ *or* ∞ *and*

$$\left\| v_n - u^0 - \sum_{i=1}^{k} (\lambda_n^i)^{\frac{2-N}{2}} u^i \left(\frac{\cdot}{\lambda_n^i} \right) \right\|_{D_r^{1,2}} \to 0,$$

$$\|v_n\|_{D_r^{1,2}}^2 \to \sum_{i=0}^{k} \|u^i\|_{D_r^{1,2}}^2,$$

$$\Psi(v_n) = \sum_{i=0}^{k} \Psi(u^i) + o(1).$$

Proof It is easy to see that $\{v_n\}$ is bounded in $D_r^{1,2}(\mathbb{R}^N)$. Passing if necessary to a subsequence, we assume that $v_n \rightharpoonup u^0$ in $D_r^{1,2}(\mathbb{R}^N)$ and $v_n(x) \to u^0(x)$ a.e. on $\mathbb{R}^N$. By Lemma 3.5, we have that $\Psi'(u^0) = 0$. Set $v_n^1 := v_n - u^0$. Then $\{v_n^1\}$ satisfies

$$\begin{aligned} &\text{i)}\ \lim_{n\to\infty} \left(\|v_n\|_{D_r^{1,2}}^2 - \|v_n^1\|_{D_r^{1,2}}^2 \right) = \|u^0\|_{D_r^{1,2}}^2, \\ &\text{ii)}\ \Psi(v_n^1) \to c - \Psi(u^0), \\ &\text{iii)}\ \Psi'(v_n^1) \to 0 \ \text{ in } \left(D_r^{1,2}(\mathbb{R}^N) \right)^*. \end{aligned} \tag{3.7}$$

If $v_n^1 \to 0$ in $L^{2^*(\alpha)}(\mathbb{R}^N, |x|^\alpha)$, then it follows from $\Psi'(v_n^1) \to 0$ in $(D_r^{1,2}(\mathbb{R}^N))^*$ that $v_n^1 \to 0$ in $D_r^{1,2}(\mathbb{R}^N)$ and the proof is complete. Assume that there exists $0 < \delta < \left(\frac{S_\alpha}{2}\right)^{\frac{N+\alpha}{2+\alpha}}$ such that for all n large,

$$\int_{\mathbb{R}^N} |x|^\alpha \left| v_n^1 \right|^{2^*(\alpha)} dx > \delta.$$

Defining the Levy concentration function

$$Q_n(r) := \int_{B_r(0)} |x|^\alpha \left| v_n^1 \right|^{2^*(\alpha)} dx.$$

It follows from $Q_n(0) = 0$ and $Q_n(\infty) > \delta$ that there exists a sequence $\{\lambda_n^1\} \subset (0, \infty)$ such that

$$\delta = \int_{B_{\lambda_n^1}(0)} |x|^\alpha |v_n^1|^{2^*(\alpha)} dx.$$

We denote $u_n^1(x) := (\lambda_n^1)^{\frac{N-2}{2}} v_n^1(\lambda_n^1 x)$ and assume that u_n^1 converges weakly to u^1 in $D_r^{1,2}(\mathbb{R}^N)$ and converges u^1 a.e. on $\mathbb{R}^N$. We claim that $u^1 \neq 0$. Otherwise, suppose that $u^1 = 0$. We note that

$$\delta = \int_{B_{\lambda_n^1}(0)} |x|^\alpha |v_n^1|^{2^*(\alpha)} dx = \int_B |x|^\alpha |u_n^1|^{2^*(\alpha)} dx. \tag{3.8}$$

By the Riesz-Fréchet representation theorem, there exists $f_n \in D_r^{1,2}(\mathbb{R}^N)$ such that

$$\left\langle \Psi'(v_n^1), w \right\rangle = \int_{\mathbb{R}^N} \nabla f_n \nabla w dx, \quad \forall\, w \in D_r^{1,2}(\mathbb{R}^N).$$

Then $g_n(x) := (\lambda_n^1)^{\frac{N-2}{2}} f_n(\lambda_n^1 x)$ satisfies

$$\langle \Psi'(u_n^1), w \rangle = \int_{\mathbb{R}^N} \nabla g_n \nabla w dx, \quad \forall\, w \in D_r^{1,2}(\mathbb{R}^N), \tag{3.9}$$

$$\int_{\mathbb{R}^N} |\nabla g_n|^2 dx = \int_{\mathbb{R}^N} |\nabla f_n|^2 dx = o(1). \tag{3.10}$$

Taking $\nu \in C_{0,r}^\infty(\mathbb{R}^N)$ such that supp $\nu \in B$ and the measure of supp ν is small enough. By Hölder inequality and (3.5), we get

$$\begin{aligned}
&\int_{\text{supp }\nu} |\nu|^2 |x|^\alpha \left|u_n^1\right|^{2^*(\alpha)} dx \\
&\leq S_\alpha^{-1} \int_{\text{supp }\nu} \left|\nabla(\nu u_n^1)\right|^2 dx \left(\int_{\text{supp }\nu} |x|^\alpha \left|u_n^1\right|^{2^*(\alpha)} dx \right)^{\frac{2^*(\alpha)-2}{2^*(\alpha)}}.
\end{aligned}$$

Hence, combining with $u_n^1 \to 0$ in $L^2(B)$, (3.9) and (3.10), we have

$$\begin{aligned}
\int_{\text{supp }\nu} \left|\nabla(\nu u_n^1)\right|^2 dx &= \int_{\text{supp }\nu} |\nu|^2 \left|\nabla u_n^1\right|^2 dx + o(1) \\
&= \int_{\text{supp }\nu} \nabla u_n^1 \nabla \left(|\nu|^2 u_n^1\right) dx + o(1) \\
&= \int_{\text{supp }\nu} \nabla g_n \nabla \left(|\nu|^2 u_n^1\right) dx \\
&\quad + \int_{\text{supp }\nu} |x|^\alpha \left|u_n^1\right|^{2^*(\alpha)} |\nu|^2 dx + o(1) \\
&\leq S_\alpha^{-1} \delta^{\frac{2+\alpha}{N+\alpha}} \int_{\text{supp }\nu} \left|\nabla\left(\nu u_n^1\right)\right|^2 dx + o(1) \\
&\leq \frac{1}{2} \int_{\text{supp }\nu} \left|\nabla\left(\nu u_n^1\right)\right|^2 dx + o(1).
\end{aligned}$$

Thus we get $\nabla u_n^1 \to 0$ in $L^2(B_r)$ with $0 < r < 1$ and by (3.5) we obtain $u_n^1 \to 0$ in $L^{2^*(\alpha)}(B_r; |x|^\alpha)$. Using the radial lemma(see [19]), it is easy to see that $u_n^1 \to 0$ in $L^{2^*(\alpha)}(B_{r,1}; |x|^\alpha)$, where $B_{r,1} := \{x \in \mathbb{R}^N : 0 < r < |x| < 1\}$. Furthermore $u_n^1 \to 0$ in $L^{2^*(\alpha)}(B, |x|^\alpha)$, this contradicts to (3.8). Therefore $u^1 \neq 0$.

We claim that $\lambda_n^1 \to 0$ or ∞. Assume that $\lambda_n^1 \to \kappa_\infty$ with $0 < \kappa_\infty < \infty$. Since $u^1 \neq 0$, then there exists a ball B_R such that $u^1 \neq 0$ in B_R. On one hand, by locally compact embedding, we deduce that

$$\int_{B_R} |u_n^1|^2 dx \to \int_{B_R} |u^1|^2 dx > 0. \tag{3.11}$$

On the other hand, using the facts that $0 < \kappa_\infty < \infty$ and $v_n^1 \rightharpoonup 0$ in $D^{1,2}(\mathbb{R}^N)$, we have

$$\int_{B_R} |u_n^1|^2 dx = (\lambda_n^1)^{-2} \int_{B_{R\lambda_n^1}} |v_n^1|^2 dx \to 0,$$

a contradiction with (3.11). Thus $\lambda_n^1 \to 0$ or ∞. It follows from (3.7) that $\Psi'(u^1) = 0$. Combining with Lemma 3.5, the sequence

$$v_n^2(x) = v_n^1(x) - (\lambda_n^1)^{\frac{2-N}{2}} u^1 \left(\frac{x}{\lambda_n^1} \right)$$

satisfies

$$\begin{aligned} &\text{i)} \ \left\| v_n^2 \right\|_{D_r^{1,2}}^2 = \|v_n\|_{D_r^{1,2}}^2 - \|u^0\|_{D_r^{1,2}}^2 - \|u^1\|_{D_r^{1,2}}^2 + o(1), \\ &\text{ii)} \ \Psi(v_n^2) = c - \Psi(u^0) - \Psi(u^1) + o(1), \\ &\text{iii)} \ \Psi'(v_n^2) \to 0 \text{ in } \left(D_r^{1,2}(\mathbb{R}^N) \right)^*. \end{aligned}$$

For any a nontrivial critical point u of Ψ, using (3.5), we have

$$S_\alpha \left(\int_{\mathbb{R}^N} |x|^\alpha |u|^{2^*(\alpha)} dx \right)^{\frac{2}{2^*(\alpha)}} \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx = \int_{\mathbb{R}^N} |x|^\alpha |u|^{2^*(\alpha)} dx.$$

Thus

$$\Psi(u) \geq \frac{2+\alpha}{2(N+\alpha)} S_\alpha^{\frac{N+\alpha}{2+\alpha}}. \tag{3.12}$$

Iterating the above procedure, we can construct the sequence $\{u^i\}$, $\{\lambda_n^i\}$, $\{u_n^i\}$. But the inequality (3.12) implies that only a finite number of iterations is allowed. □

Lemma 3.7 *Under the assumption* (h$_3$)*, for any* $\lambda > 0$*, we have* $S_\alpha < \Upsilon \leq \Upsilon_\lambda$.

Proof It is obvious that $S_\alpha \leq \Upsilon$. Assume that $S_\alpha = \Upsilon$. Then there exists a sequence $\{u_n\} \subset \Gamma(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + h(|x|)|u_n|^2 dx \to S_\alpha \ \text{ as } \ n \to \infty,$$

$$\int_{\mathbb{R}^N} a_1(|x|)|x|^\alpha |u_n|^{2^*(\alpha)} dx = \frac{1}{2}, \tag{3.13}$$

$$\int_{\mathbb{R}^N} |x|^\alpha |u_n|^{2^*(\alpha)} dx = 1. \tag{3.14}$$

By the definition of S_α in Theorem 3.4, the nonnegativity of h implies

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \to S_\alpha, \quad \int_{\mathbb{R}^N} |x|^\alpha |u_n|^{2^*(\alpha)} dx = 1.$$

Define

$$S(u_n) := \int_{\mathbb{R}^N} |\nabla u_n|^2 dx.$$

Applying the Ekeland principle(see [25, Theorem 8.5]), there exists Palas-Smale sequence for $S\big|_{\Gamma(\mathbb{R}^N)}$ at the level S_α, i.e., there exist $\{\beta_n\} \subset \mathbb{R}_+$ and $\tilde{u}_n \subset D_r^{1,2}(\mathbb{R}^N)$ such that as $n \to \infty$

$$\|\tilde{u}_n - u_n\|_{D_r^{1,2}} \to 0, \tag{3.15}$$

$$S(\tilde{u}_n) \to S_\alpha, \quad -\Delta \tilde{u}_n - \beta_n |x|^\alpha |\tilde{u}_n|^{2^*(\alpha)-2} \tilde{u}_n \to 0 \text{ in } (D_r^{1,2}(\mathbb{R}^N))^*. \tag{3.16}$$

It follows that

$$\beta_n \to S_\alpha, \tag{3.17}$$

$$\Psi(v_n) \to \frac{2+\alpha}{2(N+\alpha)} S_\alpha^{\frac{N+\alpha}{2+\alpha}}, \quad \Psi'(v_n) \to 0, \tag{3.18}$$

where $v_n := \beta_n^{\frac{1}{2^*(\alpha)-2}} \tilde{u}_n$. From (3.18), it is easy to see that $\{v_n\}$ is bounded in $D_r^{1,2}(\mathbb{R}^N)$, passing to a subsequence such that $v_n \rightharpoonup v^0$ in $D_r^{1,2}(\mathbb{R}^N)$. It follows from Theorem 3.6 that there exist k functions $v^1, v^2, \ldots, v^k \in D_r^{1,2}(\mathbb{R}^N)$ and k sequences $\{\lambda_n^i\} \subset \mathbb{R}_+$ satisfying

$$-\Delta v^i = |x|^\alpha |v^i|^{2^*(\alpha)-2} v^i \quad \text{in } \mathbb{R}^N \tag{3.19}$$

for $i = 0, 1, \ldots, k$, $\lambda_n^i \to 0$ or ∞ and

$$\Psi(v_n) = \frac{2+\alpha}{2(N+\alpha)} \sum_{i=0}^{k} \int_{\mathbb{R}^N} |x|^\alpha |v^i|^{2^*(\alpha)} dx + o(1), \tag{3.20}$$

$$\|v_n\|_{D_r^{1,2}}^2 \to \sum_{i=0}^{k} \|v^i\|_{D_r^{1,2}}^2, \tag{3.21}$$

$$\left\| v_n - v^0 - \sum_{i=1}^{k} \left(\lambda_n^i\right)^{\frac{2-N}{2}} v^i \left(\frac{\cdot}{\lambda_n^i}\right) \right\|_{D_r^{1,2}} \to 0. \tag{3.22}$$

Multiplying the equation (3.19) by $(v^i)^+$ and $(v^i)^-$, combining with (3.5) and the uniqueness of solution of (3.4), for any $i = 0, 1, \ldots, k$, one of the following cases holds:

$$\int_{\mathbb{R}^N} |\nabla v^i|^2 dx = \int_{\mathbb{R}^N} |x|^\alpha |v^i|^{2^*(\alpha)} dx = 0,$$
$$\int_{\mathbb{R}^N} |\nabla v^i|^2 dx = \int_{\mathbb{R}^N} |x|^\alpha |v^i|^{2^*(\alpha)} dx = S_\alpha^{\frac{N+\alpha}{2+\alpha}},$$
$$\int_{\mathbb{R}^N} |\nabla v^i|^2 dx = \int_{\mathbb{R}^N} |x|^\alpha |v^i|^{2^*(\alpha)} dx \geq 2S_\alpha^{\frac{N+\alpha}{2+\alpha}}.$$

If $v^0 \neq 0$, then it follows from (3.18) and (3.20) that $k = 0$. By (3.21), we get that $v_n \to v^0$ in $D_r^{1,2}(\mathbb{R}^N)$ and then $u_n \to u := S_\alpha^{\frac{1}{2-2^*(\alpha)}} v^0 \neq 0$. By (3.15) and the first limit in (3.16), we get that S_α is arrived at u. The key Theorem 3.4 implies that u is positive. Combining with assumption $h \neq 0$, we obtain a contradiction as

$$S_\alpha < \int_{\mathbb{R}^N} |\nabla u|^2 + h(|x|)|u|^2 dx = S_\alpha.$$

If $v^0 = 0$, then it follows from (3.18) and (3.20) that $k = 0, 1$.

(i) The case of $k = 0$. By (3.20), (3.21) and (3.15), we have $u_n \to 0$ in $D_r^{1,2}(\mathbb{R}^N)$ and this contradicts to the fact that

$$\int_{\mathbb{R}^N} |x|^\alpha |u_n|^{2^*(\alpha)} dx = 1.$$

(ii) The case of $k = 1$. We distinguish the cases $\lambda_n^1 \to 0$ and $\lambda_n^1 \to \infty$. Define

$$w_n(x) := (\lambda_n^1)^{\frac{2-N}{2}} v^1 \left(\frac{x}{\lambda_n^1} \right).$$

(ii-1) As $\lambda_n^1 \to 0$. It follows from (3.13), (3.15) and (3.17) that

$$\int_{\mathbb{R}^N} a_1(|x|)|x|^\alpha |v_n|^{2^*(\alpha)} dx = S_\alpha^{\frac{N+\alpha}{2+\alpha}} \left(\frac{1}{2} + o(1) \right).$$

However

$$\begin{aligned}\int_{\mathbb{R}^N} a_1(|x|)|x|^\alpha |v_n|^{2^*(\alpha)} dx &= \int_{\mathbb{R}^N} a_1(|x|)|x|^\alpha |w_n|^{2^*(\alpha)} dx + o(1) \text{ (by (3.22))} \\ &= \int_{\mathbb{R}^N} a_1(\lambda_n^1 |x|)|x|^\alpha |v^1(x)|^{2^*(\alpha)} dx + o(1) \\ &= o(1),\end{aligned}$$

where using the fact that $\lim_{n\to\infty} a_1(\lambda_n^1 |x|) = 0$ a.e. on $\mathbb{R}^N$ and Lebesgue Theorem. Hence we get a contradiction.

(ii-2) As $\lambda_n^1 \to \infty$. We have

$$
\begin{aligned}
S_\alpha^{\frac{N+\alpha}{2+\alpha}}\left(\frac{1}{2}+o(1)\right) &= \int_{\mathbb{R}^N} a_1(|x|)|x|^\alpha |v_n|^{2^*(\alpha)}dx \text{ (by (3.13), (3.15) and (3.17))}\\
&= \int_{\mathbb{R}^N} a_1(|x|)|x|^\alpha |w_n|^{2^*(\alpha)}dx + o(1) \text{ (by (3.22))}\\
&= \int_{\mathbb{R}^N} a_1(|x|)|x|^\alpha \left|(\lambda_n^1)^{\frac{2-N}{2}} v^1\left(\frac{x}{\lambda_n^1}\right)\right|^{2^*(\alpha)} dx + o(1)\\
&= \int_{\mathbb{R}^N} a_1(\lambda_n^1|x|)|x|^\alpha |v^1(x)|^{2^*(\alpha)}dx + o(1)\\
&= \int_{\mathbb{R}^N} |x|^\alpha |v^1(x)|^{2^*(\alpha)}dx + o(1) \text{ (byLebesgueTheorem)}\\
&= \int_{\mathbb{R}^N} |x|^\alpha |w_n(x)|^{2^*(\alpha)}dx + o(1) \text{ (by (3.13), (3.15) and (3.17))}\\
&= \int_{\mathbb{R}^N} |x|^\alpha |v_n(x)|^{2^*(\alpha)}dx + o(1)\\
&= S_\alpha^{\frac{N+\alpha}{2+\alpha}}(1+o(1)) \text{ (by(3.14)., (3.15))}
\end{aligned}
$$

where using the fact that $\lim_{n\to\infty} a_\lambda(\lambda_n^1|x|) = 1$ a.e. on $\mathbb{R}^N$. This leads to a contradiction. Therefore $S_\alpha < \Upsilon$.

Finally, taking $u \in \Gamma(B)$ such that $\psi_\lambda(u) = \frac{1}{2}$ and set $v_\lambda(x) := \lambda^{\frac{N-2}{2}} u(\lambda x)$ if $|x| \le \frac{1}{\lambda}$ and $v_\lambda(x) = 0$ for $|x| > \frac{1}{\lambda}$. Since

$$
\psi_1(v_\lambda) = \psi_\lambda(u) = \frac{1}{2}, \quad \varphi_1(v_\lambda) = \varphi_\lambda(u),
$$
$$
\int_{\mathbb{R}^N} |x|^\alpha |v_\lambda|^{2^*(\alpha)}dx = \int_{\{|y|\le 1\}} |y|^\alpha |u|^{2^*(\alpha)}dy = 1,
$$

it follows from the definitions of Υ and Υ_λ that $\Upsilon \le \Upsilon_\lambda$. □

Theorem 3.8 *Assume* (h_3) *and* $\lambda > 0$. *Let* $\{v_n\} \subset H_{0,r}^1(B)$ *be a* $(\mathrm{PS})_\mathrm{c}$ *sequence of* Φ, *i.e.*

$$
\Phi(v_n) \to c, \quad \Phi'(v_n) \to 0 \text{ in } (H_{0,r}^1(B))^*.
$$

Then, passing subsequence if necessary, there exist a solution $v^0 \in H_{0,r}^1(B)$ *of* (3.1), *a finite sequence* $\{u^1, u^2, \ldots, u^k\} \subset D_r^{1,2}(\mathbb{R}^N)$ *of solutions for*

$$
-\Delta u = |x|^\alpha |u|^{2^*(\alpha)-2}u \text{ on } \mathbb{R}^N
$$

and k sequences $\{\lambda_n^i\} \subset \mathbb{R}_+$ *such that* $\lambda_n^i \to 0$ *and*

$$\left\| v_n - v^0 - \sum_{i=1}^{k} (\lambda_n^i)^{\frac{2-N}{2}} u^i \left(\frac{\cdot}{\lambda_n^i} \right) \right\|_{D_r^{1,2}} \to 0,$$

$$\|v_n\|^2 \to \|v^0\| + \sum_{i=1}^{k} \|u^i\|^2_{D_r^{1,2}},$$

$$\Phi(v_n) = \Phi(v^0) + \sum_{i=1}^{k} \Psi(u^i) + o(1).$$

Proof The proof is similar with Theorem 3.6, but there need to make modify and we give a sketch proof. The boundedness of $\{v_n\}$ in $H^1_{0,r}(B)$ is obvious and which implies there exists a subsequence such that $v_n \rightharpoonup v^0$ in $H^1_{0,r}(B)$ and $v_n(x) \to v^0(x)$ a.e. on B. Combining with (h_3) and Lemma 3.5, it is obvious that $\Phi'(v_0) = 0$ and $v_n^1 := v_n - v^0$ satisfies

$$\begin{aligned} &\text{i)}\ \lim_{n\to\infty} \left(\|v_n\|^2 - \|v_n^1\|^2 \right) = \|v^0\|^2, \\ &\text{ii)}\ \Psi(v_n^1) \to c - \Phi(v^0), \\ &\text{iii)}\ \Psi'(v_n^1) \to 0 \ \text{ in } (H^1_{0,r}(B))^*. \end{aligned} \tag{3.23}$$

If $v_n^1 \to 0$ in $L^{2^*(\alpha)}(B, |x|^\alpha)$, then the proof is complete. Otherwise we assume that

$$\int_B |x|^\alpha |v_n^1|^{2^*(\alpha)} dx > \delta$$

for some $0 < \delta < \left(\frac{S_\alpha}{2} \right)^{\frac{N+\alpha}{2+\alpha}}$. Defining the Levy concentration function

$$Q_n(r) := \int_{B_r(0)} |x|^\alpha |v_n^1|^{2^*(\alpha)} dx.$$

Since $Q_n(0) = 0$ and $Q_n(1) > \delta$, there exists a sequence $\{\lambda_n^1\} \subset (0, 1)$ such that

$$\delta = \int_{B_{\lambda_n^1}(0)} |x|^\alpha |v_n^1|^{2^*(\alpha)} dx.$$

We assume that $u_n^1(x) := (\lambda_n^1)^{\frac{N-2}{2}} v_n^1(\lambda_n^1 x)$ converges weakly to u^1 in $D_r^{1,2}(\mathbb{R}^N)$ and a.e. on $\mathbb{R}^N$. Using the Riesz-Fréchet representation theorem, Hölder inequality, inequality (3.5) and the radial lemma(see [19]), we can prove that $u^1 \neq 0$. Set $\lambda_n^1 \to \lambda_0^1$. If $\lambda_0^1 > 0$, since $v_n^1 \rightharpoonup 0$ in $H^1_{0,r}(B)$, we get $u_n^1 \rightharpoonup 0$ in $D_r^{1,2}(\mathbb{R}^N)$, this is impossible. If $\lambda_n^1 \to 0$, from (3.23), then we have $\Psi'(u^1) = 0$. The sequence

$$v_n^2(x) = v_n^1(x) - (\lambda_n^1)^{\frac{2-N}{2}} u^1\left(\frac{x}{\lambda_n^1}\right)$$

satisfies

i) $\|v_n^2\|^2_{D_r^{1,2}} = \|v_n\|^2 - \|v^0\|^2 - \|u^1\|^2_{D_r^{1,2}} + o(1)$

ii) $\Psi(v_n^2) = c - \phi(v^0) - \Psi(u^1) + o(1)$,

iii) $\Psi'(v_n^2) \to 0$ in $(D_r^{1,2}(\mathbb{R}^N))^*$.

Similar with Theorem 3.6, there exists a finite number of sequence such that the conclusions of theorem hold. □

Lemma 3.9 *Assume* (h$_3$). *Then for any* $\lambda > 0$, *we have* $S_\alpha < \Sigma_\lambda$ *and*

$$\lim_{\lambda \to 0^+} \Sigma_\lambda = S_\alpha. \tag{3.24}$$

Proof **(1)** We first prove $S_\alpha < \Sigma_\lambda$ using a similar argument as in Lemma 3.7. Assume that $S_\alpha = \Sigma_\lambda$, then there exists a sequence $\{u_n\} \subset \Gamma(B)$ satisfying

$$\int_B |\nabla u_n|^2 + h\left(\frac{|x|}{\lambda}\right)\frac{1}{\lambda^2}|u_n|^2 dx \to S_\alpha,$$
$$\int_B a_\lambda(|x|)|x|^\alpha |u_n|^{2^*(\alpha)} dx \geq \frac{1}{2},$$
$$\int_B |x|^\alpha |u_n|^{2^*(\alpha)} dx = 1.$$

Since h is nonnegative and $\lambda > 0$, we get

$$\int_B |\nabla u_n|^2 dx \to S_\alpha, \quad \int_B |x|^\alpha |u_n|^{2^*(\alpha)} dx = 1.$$

Let

$$\phi(u) = \frac{1}{2}\int_B |\nabla u|^2 dx - \frac{1}{2^*(\alpha)}\int_B |x|^\alpha |u|^{2^*(\alpha)} dx$$

and

$$\bar{S}(u_n) := \int_B |\nabla u_n|^2 dx.$$

Applying the Ekeland principle(see [25, Theorem 8.5]), there exists a (PS) sequence for $\bar{S}\big|_{\Gamma(B)}$ at the level S_α, i.e. there exist $\{\beta_n\} \subset \mathbb{R}_+$ and $\tilde{u}_n \subset H^1_{0,r}(B)$ such that

$\|\tilde{u}_n - u_n\| \to 0$ as $n \to \infty$,

$$\bar{S}(\tilde{u}_n) \to S_\alpha, \quad -\Delta \tilde{u}_n - \beta_n |x|^\alpha |\tilde{u}_n|^{2^*(\alpha)-2} \tilde{u}_n \to 0 \text{ in } (H^1_{0,r}(B))^*.$$

Set $v_n := \beta_n^{\frac{1}{2^*(\alpha)-2}} \tilde{u}_n$, then

$$\begin{aligned} &\beta_n \to S_\alpha, \\ &\phi(v_n) \to \frac{2+\alpha}{2(N+\alpha)} S_\alpha^{\frac{N+\alpha}{2+\alpha}}, \quad \phi'(v_n) \to 0. \end{aligned}$$

It is easy to see that $\{v_n\}$ is bounded in $H^1_{0,r}(B)$, passing to a subsequence, that $v_n \rightharpoonup v^0$ in $H^1_{0,r}(B)$. Using Theorem 3.8 with $h = 0$, there exist k functions $v^1, v^2, \ldots, v^k \in D^{1,2}_r(\mathbb{R}^N)$ such that

$$-\Delta v^i = |x|^\alpha |v^i|^{2^*(\alpha)-2} v^i \ \text{ in } \mathbb{R}^N,$$

for $i = 0, 1, \ldots, k$,

$$\phi(v_n) = \phi(v^0) + \sum_{i=1}^k \Psi(v^i) + o(1) = \frac{2+\alpha}{2(N+\alpha)} \sum_{i=0}^k \int_{\mathbb{R}^N} |x|^\alpha |v^i|^{2^*(\alpha)} dx. \quad (3.25)$$

Multiplying the equation by $(v^i)^+$, $(v^i)^-$ and using (3.5), for any $i = 0, 1, \ldots, k$, one of the following cases holds:

$$\begin{aligned} &\int_{\mathbb{R}^N} |x|^\alpha |v^i|^{2^*(\alpha)} dx = 0, \\ &\int_{\mathbb{R}^N} |x|^\alpha |v^i|^{2^*(\alpha)} dx = S_\alpha^{\frac{N+\alpha}{2+\alpha}}, \\ &\int_{\mathbb{R}^N} |x|^\alpha |v^i|^{2^*(\alpha)} dx \geq 2 S_\alpha^{\frac{N+\alpha}{2+\alpha}}. \end{aligned}$$

Similar with the arguments of Lemma 3.7, the case of $v^0 \neq 0$ is impossible, so $v^0 = 0$ and $k = 0, 1$. When $k = 0$, we have $u_n \to 0$ in $H^1_{0,r}(B)$ and this is impossible since $\int_B |x|^\alpha |u_n|^{2^*(\alpha)} dx = 1$. If $k = 1$ and $\lambda_n^1 \to 0$. Then

$$\begin{aligned} S_\alpha^{\frac{N+\alpha}{2+\alpha}} \left(\frac{1}{2} + o(1)\right) &\leq \int_B a_\lambda(|x|)|x|^\alpha |v_n|^{2^*(\alpha)} dx \\ &= \int_B a_\lambda(|x|)|x|^\alpha |(\lambda_n^1)^{\frac{2-N}{2}} v^1(\frac{x}{\lambda_n^1})|^{2^*(\alpha)} dx + o(1) \\ &\leq C\lambda_n^1 + o(1) \to 0, \end{aligned}$$

where $C > 0$ is a constant. This leads to a contradiction. Therefore $S_\alpha < \Sigma_\lambda$.

(2) Now we prove the limit (3.24). Let $\epsilon > 0$ and $u \in \Gamma(B) \cap C^{\infty}_{0,r}(B)$ be such that

$$\int_B |\nabla u|^2 dx < S_\alpha + \epsilon.$$

By (h_3) we have

$$\lim_{\lambda \to 0^+} \int_B h\left(\frac{|x|}{\lambda}\right) \frac{u^2}{\lambda^2} dx = 0.$$

Hence we obtain

$$\lim_{\lambda \to 0^+} \varphi_\lambda(u) = \int_B |\nabla u|^2 dx < S_\alpha + \epsilon.$$

Since $\lim_{\lambda \to 0^+} \psi_\lambda(u) = 1 > \frac{1}{2}$ there exists $\delta > 0$ such that for $0 < \lambda < \delta$,

$$S_\alpha < \Sigma_\lambda < S_\alpha + \epsilon.$$

Therefore $\lim_{\lambda \to 0^+} \Sigma_\lambda = S_\alpha$. □

Proof of Theorem 3.1 By Lemma 3.7 and Lemma 3.9, there exists $\delta > 0$ such that

$$S_\alpha < \Sigma_\lambda < \min\left\{\Upsilon, 2^{\frac{2+\alpha}{N+\alpha}} S_\alpha\right\} \leq \Upsilon_\lambda, \quad \forall\, 0 < \lambda < \delta. \tag{3.26}$$

Since $\Sigma_\lambda < \Upsilon_\lambda$, by Ekeland variational principle(see [25, Theorem 8.5]), there exists a Palais-Smale sequence for $\varphi_\lambda|_{\Gamma(B)}$ at the level Σ_λ. Namely, there exists a sequence $\{u_n\} \subset \Gamma(B)$ and $\{\theta_n\} \subset \mathbb{R}$ such that

$$\varphi_\lambda(u_n) \to \Sigma_\lambda, \quad -\Delta u_n + h\left(\frac{|x|}{\lambda}\right) \frac{u_n}{\lambda^2} - \theta_n |x|^\alpha |u_n|^{2^*(\alpha)-2} u_n \to 0 \text{ in } (H^1_{0,r}(B))^*.$$

It follows from $u_n \in \Gamma(B)$ that $\varphi(u_n) - \theta_n \to 0$ and $\theta_n \to \Sigma_\lambda$. Now define

$$v_n = \theta_n^{\frac{1}{2^*(\alpha)-2}} u_n.$$

Then

$$\begin{aligned} \Phi(v_n) &= \frac{1}{2} \int_B |\nabla v_n|^2 + h\left(\frac{|x|}{\lambda}\right) \frac{|v_n|^2}{\lambda^2} dx - \frac{1}{2^*(\alpha)} \int_B |x|^\alpha |v_n|^{2^*(\alpha)} dx \\ &\to \frac{2+\alpha}{2(N+\alpha)} \Sigma_\lambda^{\frac{N+\alpha}{2+\alpha}}, \end{aligned} \tag{3.27}$$

$$-\Delta v_n + h\left(\frac{|x|}{\lambda}\right) \frac{v_n}{\lambda^2} - |x|^\alpha |v_n|^{2^*(\alpha)-2} v_n \to 0 \text{ in } (H^1_{0,r}(B))^*. \tag{3.28}$$

The relation (3.26) implies that

$$\frac{2+\alpha}{2(N+\alpha)} S_\alpha^{\frac{N+\alpha}{2+\alpha}} < \frac{2+\alpha}{2(N+\alpha)} \Sigma_\lambda^{\frac{N+\alpha}{2+\alpha}} < \frac{2+\alpha}{N+\alpha} S_\alpha^{\frac{N+\alpha}{2+\alpha}}. \tag{3.29}$$

According to Theorem 3.8, we get the following decomposition:

$$\Phi(v_n) = \Phi(v) + \frac{2+\alpha}{2(N+\alpha)} \sum_{i=1}^{k} \int_{\mathbb{R}^N} |x|^\alpha |w_i|^{2^*(\alpha)} dx + o(1),$$

$$\|v_n\|^2 \to \|v\|^2 + \sum_{i=1}^{k} \|w_i\|^2_{D_r^{1,2}}, \tag{3.30}$$

where $w_i \in D_r^{1,2}(\mathbb{R}^N)$ is the solutions of

$$-\Delta w = |x|^\alpha |w|^{2^*(\alpha)-2} w \ \text{ in } \mathbb{R}^N \tag{3.31}$$

and $v \in H^1_{0,r}(B)$ satisfies

$$-\Delta v + h\left(\frac{|x|}{\lambda}\right) \frac{v}{\lambda^2} = |x|^\alpha |v|^{2^*(\alpha)-2} v. \tag{3.32}$$

Hence

$$\Phi(v) + \frac{2+\alpha}{2(N+\alpha)} \sum_{i=1}^{k} \int_{\mathbb{R}^N} |x|^\alpha |w_i|^{2^*(\alpha)} dx = \frac{2+\alpha}{2(N+\alpha)} \Sigma_\lambda^{\frac{N+\alpha}{2+\alpha}}. \tag{3.33}$$

Multiplying the equation (3.31) by w_i^+, w_i^- and using (3.5), for any $i = 0, 1, \ldots, k$, one of the following cases holds:

$$w_i = 0 \ \Rightarrow \ \int_{\mathbb{R}^N} |x|^\alpha |w_i|^{2^*(\alpha)} dx = 0, \tag{3.34}$$

$$w_i \text{ has a constant sign and } \int_{\mathbb{R}^N} |x|^\alpha |w_i|^{2^*(\alpha)} dx = S_\alpha^{\frac{N+\alpha}{2+\alpha}}, \tag{3.35}$$

$$w_i \text{ changes a sign and } \int_{\mathbb{R}^N} |x|^\alpha |w_i|^{2^*(\alpha)} dx \ge 2 S_\alpha^{\frac{N+\alpha}{2+\alpha}}. \tag{3.36}$$

Similarly we have

$$v = 0 \ \Rightarrow \ \Phi(v) = 0, \tag{3.37}$$

$$v \text{ has a constant sign and } \Phi(v) \ge \frac{2+\alpha}{2(N+\alpha)} S_\alpha^{\frac{N+\alpha}{2+\alpha}}, \tag{3.38}$$

$$v \text{ changes a sign and } \Phi(v) \ge \frac{2+\alpha}{N+\alpha} S_\alpha^{\frac{N+\alpha}{2+\alpha}}. \tag{3.39}$$

It follows from (3.29) and (3.33) that the only possible case is (3.34) together (3.38). Combining with (3.30), we know $v_n \to v$ in $H^1_{0,r}(B)$. By (3.27) and (3.38), v is a constant sign solution and $\Phi(v) = \frac{2+\alpha}{2(N+\alpha)} \Sigma_\lambda^{\frac{N+\alpha}{2+\alpha}}$, moreover by structure of (3.32), we can obtain the nonnegative solution v. □

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Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. Ambrosetti, A., Malchiodi, A., Ni, W.-M.: Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres. II. Indiana Univ. Math. J. **53**, 297–329 (2004)
2. Aubin, T.: Problèmes isopérimétriques et espaces de Sobolev. (French) J. Differ. Geom. **11**, 573–598 (1976)
3. Benci, V., Cerami, G.: Existence of positive solutions of the equation $-\Delta u + a(x)u = u^{(N+2)/(N-2)}$ in $\mathbb{R}^N$. J. Funct. Anal. **88**, 90–117 (1990)
4. Brézis, H.: Elliptic equations with limiting Sobolev exponents–the impact of topology. Commun. Pure Appl. Math. **39**, S17–S39 (1986)
5. Brézis, H., Nirenberg, L.: Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. Commun. Pure Appl. Math. **36**, 437–477 (1983)
6. Brézis, H., Peletier, L.A.: Elliptic equations with critical exponent on S^3: new non-minimising solutions. C. R. Math. Acad. Sci. Paris **339**, 391–394 (2004)
7. Brézis, H., Peletier, L.A.: Elliptic equations with critical exponent on spherical caps of S^3. J. Anal. Math. **98**, 279–316 (2006)
8. Brézis, H., Willem, M.: On some nonlinear equations with critical exponents. J. Funct. Anal. **255**, 2286–2298 (2008)
9. Ghoussoub, N., Kang, X.S.: Hardy–Sobolev critical elliptic equations with boundary singularities. Ann. Inst. H. Poincaré Anal. Non Linéaire, **21**, 767–793 (2004)
10. Ghoussoub, N., Robert, F.: Sobolev inequalities for the Hardy–Schrödinger operator: extremals and critical dimensions. Bull. Math. Sci. **6**, 89–144 (2016)
11. Ghoussoub, N., Yuan, C.: Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents. Trans. Am. Math. Soc. **352**, 5703–5743 (2000)
12. Gidas, B., Spruck, J.: Global and local behavior of positive solutions of nonlinear elliptic equations. Commun. Pure Appl. Math. **24**, 525–598 (1981)
13. Gladiali, F., Grossi, M., Neves, S.L.N.: Nonradial solutions for the Hénon equation in $\mathbb{R}^N$. Adv. Math. **249**, 1–36 (2013)
14. Hénon, M.: Numerical experiments on the stability oh spherical stellar systems. Astronom. Astrophys. **24**, 229–238 (1973)
15. Lieb, E.: Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities. Ann. Math. **118**, 349–374 (1983)
16. Mercuri, C., Willem, M.: A global compactness result for the p-Laplacian involving critical nonlinearities. Discrete Contin. Dyn. Syst. **28**, 469–493 (2010)
17. Ni, W.M.: A nonlinear Dirichlet problem on the unit ball and its applications. Indiana Univ. Math. J. **31**, 801–807 (1982)
18. Passaseo, D.: Some sufficient conditions for the existence of positive solutions to the equation $-\Delta u + a(x)u = u^{2^*-1}$ in bounded domians. Ann. Inst. H. Poincaré Anal. Non Linéaire, **13**, 185–227 (1996)

19. Strauss, W.A.: Existence of solitary waves in higher dimensions. Commun. Math. Phys. **55**, 149–162 (1977)
20. Su, J., Wang, Z.-Q., Willem, M.: Nonlinear Schrödinger equations with unbounded and decaying radial potentials. Commun. Contemp. Math. **9**, 571–583 (2007)
21. Su, J., Wang, Z.-Q., Willem, M.: Weighted Sobolev embedding with unbounded and decaying radial potential. J. Differ. Equ. **238**, 201–219 (2007)
22. Talenti, G.: Best constant in Sobolev inequality. Ann. Mat. Pura Appl. **110**, 353–372 (1976)
23. Wang, C., Su, J.: Critical exponents of weighted Sobolev embeddings for radial functions. Appl. Math. Lett. **107**, 106484 (2020)
24. Wang, C., Su, J.: Positive radial solutions of critical Hénon equations on the unit ball in $\mathbb{R}^N$. Preprint
25. Willem, M.: Minimax Theorems. Birkhuser Boston. Inc., Boston (1996)
26. Willem, M.: Functional Analysis, Fundamentals and Applications. Birkhäuser/Springer, New York (2013)