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A 2D model for a highly heterogeneous plate

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Abstract
In this paper we investigate the $2d$-model for a thin plate $\Omega_\varepsilon := \omega \times \varepsilon \mathrm{I}$ of $\mathbb{R}^3$ having two components: a circular stiff layer F_ε and its complement the soft matrix M_ε with $\frac{1}{\varepsilon^2}$ as a ratio between their respective elasticity coefficients. We prove that the limit model is associated to a nonlocal system involving Kirchoff-Love displacements in the layer and we exhibit a corrector for the displacements in the initial cylindrical structure of $\mathbb{R}^3$.

Keywords Plate · Thin structure · Corrector · Nonlocal

Mathematics Subject Classification 35B25 · 35B27 · 35B40 · 76M50 · 74K10

1 Introduction, notations and setting of the problem

The aim of this work is the study of the asymptotic behavior of the solutions of the linearized system of elasticity posed in a cylindrical domain $\Omega_\varepsilon := \omega \times \varepsilon \mathrm{I}$ of $\mathbb{R}^3$ which is the configuration domain of a composite material. The material is made up of two components with high contrast: the first one F_ε representing the stiff part of the material has an elasticity tensor with coefficients of order 1. The second component

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M_ε (the soft material) surrounds the first one and the coefficients of its elasticity tensor are of order ε^2.

Under an appropriate assumption on the volumes forces, we aim to approximate the behavior of the displacements and that of the associated tensors as the small parameter ε tends to zero. Hence the present work may be viewed as the $3d - 2d$ version of the study addressed in [14] where the $3d - 1d$ reduction of dimension problem was considered as well as the homogenization of a ε-periodic fibered medium inducing a local $3d - 1d$ reduction of dimension. It was proved in [14] that the homogenized problem in a such setting is a copy of the one-dimensional problem obtained in the $3d - 1d$ study. Although one can also consider here the ε-periodic homogenization problem of a medium containing $\frac{1}{\varepsilon}$ cells which are the translates of Ω_ε in such a way that the homogenization process leads to a local $3d - 2d$ reduction of dimension, for the sake of brevity we restrict ourselves to the $3d - 2d$ reduction of dimension problem arising in the single composite structure Ω_ε. We consider the critical case where the ratio between the elasticity coefficients of the two components is equal to $\frac{1}{\varepsilon^2}$ but other scalings may be considered as pointed out in [14], see also [11,16].

On the other hand, we deal with general elasticity tensors including anisotropic materials, see also [4,8,13,15]. Several studies on composite materials with hight contrast between their components have been performed during the last years, see for instance [3,5–7,10–13]. The founding work studying media with high contrasting properties is the reference [2]. It is known that for this kind of materials, the limit problem has in general a different structure than the starting problem. In particular, nonlocal phenomena can appear at the limit. We show here that the limit problem obtained after reduction of dimension $3d - 2d$ is indeed a nonlocal problem. Theorem 3.2 below, which gives the limit problem, shows that the displacements in the circular layer F are essentially of Kirchov-Love type and the associated equation may be obtained from system (29) by choosing $\bar{z} = 0$; but to determine the limit displacements in the structure, another equation related to the matrix M (the outside of the layer) is necessary. That equation is obtained by choosing $\bar{u} = \bar{v} = 0$ in (29). This nonlocal phenomenon is emphasized in Theorem 3.4 which gives the limit of the average of the displacements $\hat{u}^\varepsilon$ in the three-dimensional structure Ω_ε. In particular convergences (33) and (34) show clearly that the limit displacements are determined after solving the two equations posed in F and in M respectively.

In terms of correctors, our result shows that the transversal displacements u_α^ε behave as $u_\alpha + \varepsilon v_\alpha + z_\alpha$ while the horizontal displacements u_3^ε behave as $u_3 + \varepsilon^2 v_3 + \varepsilon z_3$.

We now make more precise the notations we will use throughout the paper.

A vector x in $\mathbb{R}^3$ is denoted by $x = (x', x_3)$ where x_3 denotes the vertical coordinate. Latin indices will usually range from 1 to 3 and Greek ones take values in $\{1, 2\}$; the summation convention applies whenever indices are repeated. We write $\partial_i := \frac{\partial}{\partial x_i}$ and $\partial_{ij} := \frac{\partial^2}{\partial x_i \partial x_j}$. The gradient with respect to x and x' are denoted by ∇ and ∇' respectively.

Given any $\phi \in (\mathcal{D}'(\Omega))^3$, we denote the strain of ϕ by $\mathrm{E}\phi := \mathrm{sym}\nabla\phi = \frac{1}{2}\left(\nabla\phi + \nabla\phi^{\mathrm{T}}\right)$. We shall also use the following matrices notations

$$\nabla\phi = \left(\begin{array}{c|c} \partial_\alpha\phi_\beta & \partial_3\phi_\beta \\ \hline \partial_\alpha\phi_3 & \partial_3\phi_3 \end{array}\right),$$

$$\mathrm{E}\phi = \left(\begin{array}{c|c} (\mathrm{E}\phi)_{\alpha\beta} & (\mathrm{E}\phi)_{\alpha 3} \\ \hline (\mathrm{E}\phi)_{\alpha 3} & (\mathrm{E}\phi)_{33} \end{array}\right).$$

Let $\omega \subset \mathbb{R}^2$ be an open, bounded, simply connected set with Lipschitz boundary $\partial\omega$. Let $\mathrm{I} := (-\frac{1}{2}, \frac{1}{2})$, $\mathrm{J} := (-\frac{r}{2}, \frac{r}{2})$, $0 < r < \frac{1}{2}$, $\Omega := \omega \times \mathrm{I}$, $F := \omega \times \mathrm{J}$, $M := \omega \times \left(\mathrm{I}\backslash\bar{\mathrm{J}}\right)$, where $\bar{\mathrm{J}}$ denotes the closure of J. According to Remark 1.1 below, physically we can think of Ω as the reference configuration of a rescaled thin plate Ω_ε reinforced by the layer F_ε while its complement, the matrix M_ε, is occupied by a soft material. For every $\varepsilon > 0$ we denote the diagonal matrix whose entries are 1, 1 and ε by $R^\varepsilon := \mathrm{diag}(1, 1, \varepsilon)$, then the scaled gradient $\nabla^\varepsilon\phi$ and the scaled strain $\mathrm{E}^\varepsilon\phi$ are defined respectively by

$$\nabla^\varepsilon\phi := (R^\varepsilon)^{-1}\nabla\phi(R^\varepsilon)^{-1} = \left(\begin{array}{c|c} \partial_\alpha\phi_\beta & \frac{1}{\varepsilon}\partial_3\phi_\beta \\ \hline \frac{1}{\varepsilon}\partial_\alpha\phi_3 & \frac{1}{\varepsilon^2}\partial_3\phi_3 \end{array}\right), \tag{1}$$

and

$$\mathrm{E}^\varepsilon\phi := \mathrm{sym}\nabla^\varepsilon\phi = \left(\begin{array}{c|c} (\mathrm{E}\phi)_{\alpha\beta} & \frac{1}{\varepsilon}(\mathrm{E}\phi)_{\alpha 3} \\ \hline \frac{1}{\varepsilon}(\mathrm{E}\phi)_{\alpha 3} & \frac{1}{\varepsilon^2}(\mathrm{E}\phi)_{33} \end{array}\right). \tag{2}$$

We are now in position to state the problem.

Let $\mathbb{A} \in L^\infty(\Omega)$ be a symmetric fourth-order tensor field. We assume that $\mathbb{A}$ fulfills the following assumptions:

$$\begin{cases} \mathbb{A}_{ijkl} = \mathbb{A}_{jikl} = \mathbb{A}_{klij}, \text{ a.e. in } \Omega, \\ \exists\, C > 0,\ \mathbb{A}_{ijkl}\xi_{kl}\xi_{ij} \geq C\xi_{ij}\xi_{ij},\ \forall \xi \in \mathbb{R}^9 \text{ s.t. } \xi^T = \xi. \end{cases} \tag{3}$$

We shall assume that the plate is clamped at the lateral boundary of Ω and subjected to body forces $f \in L^2(\Omega; \mathbb{R}^3)$, we thus set

$$H^1_L(\Omega) := \{u \in H^1(\Omega; \mathbb{R}^3) : u = 0 \quad \text{on } \partial\omega \times \mathrm{I}\}.$$

Consider the displacement field u^ε solution of the following system:

$$\begin{cases} u^\varepsilon \in H^1_L(\Omega),\ \forall\phi \in H^1_L(\Omega), \\ \displaystyle\int_\Omega \left(\chi_F + \varepsilon^2\chi_M\right) \mathbb{A}\mathrm{E}^\varepsilon(u^\varepsilon).\mathrm{E}^\varepsilon(\phi)dx = \int_\Omega f\phi dx. \end{cases} \tag{4}$$

By virtue of the assumptions on the tensor $\mathbb{A}$ and the body forces F, for every $\varepsilon > 0$, the problem (4) admits a unique solution by the Lax-Milgram Theorem.

Remark 1.1 As in [9], the homothety along the vertical axis defined by $r^\varepsilon(x) := (x', \varepsilon x_3)$ transforms Ω, F and M respectively into $\Omega_\varepsilon := \omega \times \varepsilon \mathrm{I}$, $F_\varepsilon := \omega \times \varepsilon \mathrm{J}$ and $M_\varepsilon := \omega \times \varepsilon \left(\mathrm{I}\backslash\bar{\mathrm{J}}\right)$. Then, the problem (4) is the variational version in the fixed reference configuration Ω of the elasticity problem (6) below posed in the variable thin domain Ω_ε. Indeed, for any $v : \Omega \longmapsto \mathbb{R}^3$ we define $\hat{v} : \Omega_\varepsilon \longmapsto \mathbb{R}^3$ by

$$\hat{v}(x', \varepsilon x_3) := \left((R^\varepsilon)^{-1} v \circ (r^\varepsilon)^{-1}\right)(x', \varepsilon x_3) = \left(v_\alpha(x), \tfrac{1}{\varepsilon} v_3(x)\right). \tag{5}$$

In addition

$$\nabla \hat{v} = \mathrm{H}^\varepsilon v \circ (r^\varepsilon)^{-1}, \quad \mathrm{E}\hat{v} = \mathrm{E}^\varepsilon v \circ (r^\varepsilon)^{-1}.$$

With these new unknowns, the problem (4) may be rewritten as

$$\begin{cases} \hat{u}^\varepsilon \in H^1_L(\Omega_\varepsilon),\ \forall \phi \in H^1_L(\Omega_\varepsilon), \\ \displaystyle\int_{\Omega_\varepsilon} \left(\chi_{F_\varepsilon} + \varepsilon^2 \chi_{M_\varepsilon}\right) \mathbb{A}^\varepsilon \mathrm{E}(\hat{u}^\varepsilon).\mathrm{E}(\phi) dx = \int_{\Omega_\varepsilon} f^\varepsilon \phi dx \end{cases} \tag{6}$$

where $H^1_L(\Omega_\varepsilon) := \{u \in H^1(\Omega_\varepsilon; \mathbb{R}^3) : u = 0 \quad \text{on } \partial\omega \times \varepsilon \mathrm{I}\}$, $\mathbb{A}^\varepsilon := \mathbb{A} \circ (r^\varepsilon)^{-1}$, $f^\varepsilon = R^\varepsilon f \circ (r^\varepsilon)^{-1}$. Thus, the components of the loads are

$$f^\varepsilon = \left(f_\alpha \circ (r^\varepsilon)^{-1}, \varepsilon f_3 \circ (r^\varepsilon)^{-1})\right). \tag{7}$$

We study the behavior as $\varepsilon \to 0$ of the sequence $\{u^\varepsilon\}$, solution of (4), through the forthcoming steps. That behavior will be described through Theorem 3.2 and then we will deduce the behavior of the sequence $\hat{u}^\varepsilon$ solution of (6) in Theorem 3.4.

2 A priori estimates

First, we shall recall a rescaled Korn's inequality proved in [1]

$$\|\mathrm{E}\hat{v}\|^2_{L^2(F_\varepsilon)} \geq C\varepsilon^2 \left(\|\hat{v}\|^2_{L^2(F_\varepsilon)} + \|\nabla \hat{v}\|^2_{L^2(F_\varepsilon)}\right), \ \forall v \in H^1_L(F), \tag{8}$$

where

$$H^1_L(F) := \{u \in H^1(F; \mathbb{R}^3) : u = 0 \quad \text{on } \partial\omega \times \mathrm{J}\}$$

and $\hat{v} = (R^\varepsilon)^{-1} v \circ (r^\varepsilon)^{-1}$. As a consequence, we deduce

$$\|\mathrm{E}^\varepsilon v\|^2_{L^2(F)} \geq C\varepsilon^2 \left(\|(R^\varepsilon)^{-1} v\|^2_{L^2(F)} + \|\nabla^\varepsilon v\|^2_{L^2(F)}\right), \ \forall v \in H^1_L(F). \tag{9}$$

On the other hand, repeating the same arguments used in [14, Lemma 4.4], the following inequality holds true

$$\left(\|\nabla^{\varepsilon} v\|^2_{L^2(F)} + \|\mathrm{E}^{\varepsilon} v\|^2_{L^2(M)}\right) \geq C\|\nabla^{\varepsilon} v\|^2_{L^2(M)}, \ \forall v \in H^1_L(\Omega). \tag{10}$$

For $v \in H^1_L(\Omega)$ we set $v' := (v_1, v_2)$, $\nabla' v' := \left(\partial_\beta v_\alpha\right)_{\alpha,\beta=1,2}$ and

$$\mathcal{F}_\varepsilon(v) := \int_\Omega \left(\chi_F + \varepsilon^2 \chi_M\right) \mathbb{A}\mathrm{E}^{\varepsilon} v.\mathrm{E}^{\varepsilon} v dx. \tag{11}$$

By using the ellipticity assumption (3) we obtain, for all $v \in H^1_L(\Omega)$

$$\mathcal{F}_\varepsilon(v) \geq C\left(\|\mathrm{E}^{\varepsilon} v\|^2_{L^2(F)} + \varepsilon^2\|\mathrm{E}^{\varepsilon} v\|^2_{L^2(M)}\right) \geq C\|\mathrm{E} v\|^2_{L^2(F)} \geq C\|v\|^2_{H^1(F)}, \tag{12}$$

where the last inequality follows from the Korn's inequality. Furthermore, by virtue of (9) and (10) one has

$$\mathcal{F}_\varepsilon(v) \geq C\varepsilon^2\left(\|\nabla^{\varepsilon} v\|^2_{L^2(F)} + \|\mathrm{E}^{\varepsilon} v\|^2_{L^2(M)}\right) \geq C\varepsilon^2\|\nabla^{\varepsilon} v\|^2_{L^2(M)}. \tag{13}$$

From (13), one has for ε small enough,

$$\mathcal{F}_\varepsilon(v) \geq C\varepsilon^2\left(\tfrac{1}{\varepsilon^2}\|\nabla' v_3\|^2_{L^2(M)} + \tfrac{1}{\varepsilon^4}\|\partial_3 v_3\|^2_{L^2(M)}\right) \geq C\|\nabla v_3\|^2_{L^2(M)}. \tag{14}$$

By means of the Poincaré's inequality, one has

$$\mathcal{F}_\varepsilon(v) \geq C\|v_3\|^2_{H^1(M)}. \tag{15}$$

Likewise, it holds that

$$\mathcal{F}_\varepsilon(v) \geq C\left(\varepsilon^2\|\nabla' v'\|^2_{L^2(M)} + \|\partial_3 v'\|^2_{L^2(M)}\right) \geq C\|\partial_3 v'\|^2_{L^2(M)}. \tag{16}$$

From the following Poincaré's type inequality

$$\|v'\|^2_{L^2(M)} \leq C\left(\|v'\|^2_{L^2(F)} + \|\partial_3 v'\|^2_{L^2(M)}\right),$$

(16) and (12), we get the following inequality

$$\mathcal{F}_\varepsilon(v) \geq C\|v'\|^2_{L^2(M)}. \tag{17}$$

It follows from (15) and (17) that

$$\mathcal{F}_\varepsilon(v) \geq C\left(\|v_3\|^2_{H^1(M)} + \|v'\|^2_{L^2(\omega; H^1(\mathrm{I}\setminus\bar{\mathrm{J}}))}\right). \tag{18}$$

As an immediate consequence of the previous inequalities, the following apriori estimates hold true.

Theorem 2.1 *Let u_ε be the solution of problem* (4). *Then*

$$\sup_\varepsilon \left(\|u^\varepsilon\|_{H^1(F)} + \|\varepsilon\nabla^\varepsilon u^\varepsilon\|_{L^2(F)} + \|\mathrm{E}^\varepsilon u^\varepsilon\|_{L^2(F)}\right) \leq C, \tag{19}$$

$$\sup_\varepsilon \left(\|u_3^\varepsilon\|_{H^1(M)} + \|u_\alpha^\varepsilon\|_{L^2(\omega;H^1(\mathrm{I}\setminus\bar{\mathrm{J}}))} + \|\varepsilon\nabla^\varepsilon u^\varepsilon\|_{L^2(M)}\right) \leq C, \tag{20}$$

in particular

$$\sup_\varepsilon \left(\|u_3^\varepsilon\|_{H^1(\Omega)} + \|u_\alpha^\varepsilon\|_{L^2(\omega;H^1(\mathrm{I}))} + \|\varepsilon\nabla^\varepsilon u^\varepsilon\|_{L^2(\Omega)}\right) \leq C,\ \alpha = 1, 2. \tag{21}$$

Proof
Taking $\phi = u^\varepsilon$ in the problem (4), we derive

$$\mathcal{F}_\varepsilon(u^\varepsilon) \leq \|f\|_{L^2(\Omega)}\|u^\varepsilon\|_{L^2(\Omega)},$$

while from (12), (15) and (17) we find

$$\mathcal{F}_\varepsilon(u^\varepsilon) \geq C\|u^\varepsilon\|_{L^2(\Omega)}.$$

Thus

$$\sup_\varepsilon \mathcal{F}_\varepsilon(u^\varepsilon) \leq C.$$

Therefore, making use of (12)–(13) and then (13)–(18) we deduce (19) and (20) respectively.

3 Convergence results

We first define the following functional spaces:

$$H_{KL}(\Omega) := \left\{u \in H_L^1(\Omega) : (\mathrm{E}u)_{i3} = 0, i = 1, 2, 3\right\}$$

which is the space of Kirchhoff-Love displacements on Ω; it can be characterized also as

$$H_{KL}(\Omega) = \left\{u \in H_L^1(\Omega), \exists(g_\alpha, g_3) \in H_0^1(\omega)\times H_0^2(\omega) : u_\alpha = g_\alpha - x_3\partial_\alpha g_3, u_3 = g_3\right\},$$

for the sake of brevity we set $\mathcal{U} := H_{KL}(\Omega)$,

$$\mathcal{V} := \left\{ v \in L^2\left(\omega; H^1(\mathrm{J})\right)^3 : \fint_{\mathrm{J}} v_i(x', t)dt = 0, \text{a.e. } x' \in \omega, i = 1, 2, 3 \right\},$$

$$\mathcal{Z} := \left\{ z \in L^2\left(\omega; H^1_m(\mathrm{I})\right)^3 : z_i = 0, \text{a.e.in} F, i = 1, 2, 3 \right\},$$

where $H^1_m(\mathrm{I}) := \left\{ \psi \in H^1(\mathrm{I}) : \int_{\mathrm{I}} \psi(t)dt = 0 \right\}$.

The following lemma is concerned with some convergence results.

Lemma 3.1 *There exists $(u, v, z) \in \mathcal{U} \times \mathcal{V} \times \mathcal{Z}$, such that up to a subsequence, we have*

$$u^\varepsilon_\alpha \rightharpoonup u_\alpha + z_\alpha \text{ in } L^2(\omega; H^1(\mathrm{I})),\ u^\varepsilon_3 \rightharpoonup u_3 \text{ in } H^1(\Omega),$$

$$\mathrm{E}^\varepsilon u^\varepsilon \chi_F \rightharpoonup \left(\begin{array}{c|c} (\mathrm{E}u)_{\alpha\beta} & \partial_3 v_\alpha \\ \hline \partial_3 v_\alpha & \partial_3 v_3 \end{array} \right) \chi_F \text{ in } L^2(\Omega)^{3\times 3},$$

$$\varepsilon \mathrm{E}^\varepsilon u^\varepsilon \chi_M \rightharpoonup \left(\begin{array}{c|c} 0 & \frac{1}{2}\partial_3 z_\alpha \\ \hline \frac{1}{2}\partial_3 z_\alpha & \partial_3 z_3 \end{array} \right) \chi_M \text{ in } L^2(\Omega)^{3\times 3}.$$

Proof

Step 1: convergence of $(u^\varepsilon)_\varepsilon$.

From estimate (19) which implies that u^ε_α is bounded in $H^1(F)$ and from estimate (20) which implies that u^ε_α is bounded in $L^2(\omega; H^1(I\setminus\overline{J}))$ for $\alpha = 1, 2$, we conclude that there exist $u_\alpha \in H^1(\Omega)$ such that for a subsequence

$$u^\varepsilon_\alpha \rightharpoonup u_\alpha \text{ in } H^1(F), \quad (\mathrm{E}^\varepsilon u^\varepsilon)_{\alpha\beta}\chi_F \rightharpoonup (\mathrm{E}u)_{\alpha\beta}\chi_F \text{ in } L^2(F). \tag{22}$$

On the other hand, from (21) we get that up to a subsequence,

$$\begin{cases} u^\varepsilon_\alpha \rightharpoonup \tilde{u}_\alpha & \text{in } L^2(\omega; H^1(\mathrm{I})), \\ u^\varepsilon_3 \rightharpoonup u_3 & \text{in } H^1(\Omega). \end{cases} \tag{23}$$

Moreover, due to the estimate $\|\frac{1}{\varepsilon}\partial_3 u^\varepsilon_3\|_{L^2(\Omega)} \leq C$ which is a consequence of (21), we have

$$\partial_3 u^\varepsilon_3 \to 0 \text{ in } L^2(\Omega)$$

in such a way that

$$\partial_3 u_3 = 0.$$

Thus, there exists $g_3 \in H^1_0(\omega)$ such that

$$u_3(x) = g_3(x') \text{ for a.e. } x \in \Omega. \tag{24}$$

Comparing the first convergences in (22) and (23), we derive the equality

$$\tilde{u}_\alpha = u_\alpha \quad \text{in } F. \tag{25}$$

On the other hand, using (19) we infer

$$(\mathrm{E}u^\varepsilon)_{\alpha 3} \rightharpoonup (\mathrm{E}\tilde{u})_{\alpha 3} \text{ in } L^2(F).$$

Since

$$(\mathrm{E}u^\varepsilon)_{i3} \to 0 \text{ in } L^2(F),$$

we are led to

$$(\tilde{u}_\alpha, u_3) \in H_{KL}(F).$$

By definition of the space $H_{KL}(F)$, the component u_3 belongs to $H_0^2(\omega)$, so that equality (24) implies that $g_3 \in H_0^2(\omega)$. Hence, there exist $g_\alpha \in H_0^1(\omega)$ for $\alpha = 1, 2$, such that

$$\tilde{u}_\alpha(x) = g_\alpha(x') - x_3\partial_\alpha g_3(x'), \quad u_3(x) = g_3(x') \text{ for a. e. } x \in F.$$

For a. e. $x \in \Omega$, we set

$$u_\alpha(x) := g_\alpha(x') - x_3\partial_\alpha g_3(x') \in H_L^1(\Omega),$$
$$z_\alpha(x) := \tilde{u}_\alpha(x) - u_\alpha(x) \in L^2(\omega; H^1(\mathrm{I})),$$

so that $z_\alpha = 0$, a. e. in F as seen from (25) and $u = (u_\alpha, u_3) \in \mathcal{U}$ and

$$u_\alpha^\varepsilon \rightharpoonup u_\alpha + z_\alpha, \text{ in } L^2(\omega; H^1(\mathrm{I})).$$

Step 2: convergence of $(\mathrm{E}^\varepsilon u^\varepsilon)_\varepsilon$.

To identify the limit of the sequence $(\mathrm{E}^\varepsilon u^\varepsilon)_{i3}\chi_F$, we introduce the following sequences:

$$\hat{u}_\alpha^\varepsilon := \int_{-\frac{1}{2}}^{x_3} \tfrac{1}{\varepsilon}\partial_\alpha u_3^\varepsilon(x', t)dt, \tag{26}$$

$$\begin{aligned} v_\alpha^\varepsilon &:= \tfrac{1}{\varepsilon}u_\alpha^\varepsilon + \hat{u}_\alpha^\varepsilon - \fint_{\mathrm{J}} \left(\tfrac{1}{\varepsilon}u_\alpha^\varepsilon + \hat{u}_\alpha^\varepsilon\right) dx_3, \\ v_3^\varepsilon &:= \tfrac{1}{\varepsilon^2}u_3^\varepsilon - \fint_{\mathrm{J}} \tfrac{1}{\varepsilon^2}u_3^\varepsilon dx_3. \end{aligned} \tag{27}$$

Obviously, on has

$$\partial_3 v_\alpha^\varepsilon = \tfrac{2}{\varepsilon}(\mathrm{E}u^\varepsilon)_{\alpha 3},$$
$$\partial_3 v_3^\varepsilon = \tfrac{1}{\varepsilon^2}(\mathrm{E}u^\varepsilon)_{33}.$$

Since $\frac{2}{\varepsilon}(\mathrm{E}u^\varepsilon)_{\alpha 3}$ and $\frac{1}{\varepsilon^2}(\mathrm{E}u^\varepsilon)_{33}$ are bounded in $L^2(F)$, due to (19), and $v^\varepsilon := (v^\varepsilon_\alpha, v^\varepsilon_3)$ has mean-value zero with respect to x_3, the sequence v^ε is bounded in $L^2(\omega; H^1_m(\mathrm{J}))^3$, where $H^1_m(\mathrm{J}) := \left\{\psi \in H^1(\mathrm{J}) : \fint_{\mathrm{J}} \psi(t)dt = 0\right\}$.

Therefore there exists some $v \in L^2(\omega; H^1_m(\mathrm{J}))^3$ such that

$$v^\varepsilon \rightharpoonup v \text{ in } L^2(\omega; H^1_m(\mathrm{J}))^3.$$

In particular, one has $v \in \mathcal{V}$,

$$(\mathrm{E}^\varepsilon u^\varepsilon)_{\alpha 3}\chi_F \rightharpoonup \tfrac{1}{2}\partial_3 v_\alpha \chi_F \text{ in } L^2(\Omega),$$

and

$$(\mathrm{E}^\varepsilon u^\varepsilon)_{33}\chi_F \rightharpoonup \partial_3 v_3 \chi_F \text{ in } L^2(\Omega).$$

As a consequence, one obtains the following convergence

$$\mathrm{E}^\varepsilon u^\varepsilon \chi_F \rightharpoonup \left(\begin{array}{c|c} (\mathrm{E}u)_{\alpha\beta} & \frac{1}{2}\partial_3 v_\alpha \\ \hline \frac{1}{2}\partial_3 v_\alpha & \partial_3 v_3 \end{array}\right)\chi_F \text{ in } L^2(\Omega)^{3\times 3},$$

where $(u, v) \in \mathcal{U} \times \mathcal{V}$.

We now seek for the limit of the sequence $\varepsilon \mathrm{E}^\varepsilon u^\varepsilon \chi_M$. To that aim, we consider the following sequence

$$\begin{aligned} z^\varepsilon_\alpha &:= u^\varepsilon_\alpha + \hat{u}^\varepsilon_\alpha - \int_{\mathrm{I}} \left(u^\varepsilon_\alpha + \hat{u}^\varepsilon_\alpha\right) dx_3, \\ z^\varepsilon_3 &:= \tfrac{1}{\varepsilon}u^\varepsilon_3 - \int_{\mathrm{I}} \tfrac{1}{\varepsilon}u^\varepsilon_3 dx_3. \end{aligned} \tag{28}$$

where $\hat{u}^\varepsilon$ is defined by (26). Since

$$\partial_3 z^\varepsilon_\alpha = 2(\mathrm{E}u^\varepsilon)_{\alpha 3}, \tag{29}$$
$$\partial_3 z^\varepsilon_3 = \tfrac{1}{\varepsilon}(\mathrm{E}u^\varepsilon)_{33}. \tag{30}$$

Due to (21), $2(\mathrm{E}u^\varepsilon)_{\alpha 3}$ and $\frac{1}{\varepsilon}(\mathrm{E}u^\varepsilon)_{33}$ are bounded in $L^2(\Omega)$ with mean-value zero with respect to x_3; hence the sequence z^ε is bounded in $L^2(\omega; H^1_m(\mathrm{I}))^3$. Therefore there exists some $\hat{z} \in L^2(\omega; H^1_m(\mathrm{I}))^3$, such that

$$z^\varepsilon \rightharpoonup \hat{z} \text{ in } L^2(\omega; H^1_m(\mathrm{I}))^3.$$

We claim that $\hat{z}_\alpha = z_\alpha$. Indeed,

$$\partial_3 u^\varepsilon_\alpha \rightharpoonup \partial_3 u_\alpha + \partial_3 z_\alpha \text{ in } L^2(\Omega)$$

and

$$\partial_3 z_\alpha^\varepsilon \rightharpoonup \partial_3 \hat{z}_\alpha \text{ in } L^2(\Omega).$$

But

$$\partial_3 z_\alpha^\varepsilon = \partial_3 u_\alpha^\varepsilon + \partial_\alpha u_3^\varepsilon \rightharpoonup \partial_3 u_\alpha + \partial_3 z_\alpha + \partial_\alpha u_3 \text{ in } L^2(\Omega).$$

Thus

$$\partial_3 \hat{z}_\alpha = \partial_3 u_\alpha + \partial_3 z_\alpha + \partial_\alpha u_3.$$

Since $u \in \mathcal{U}$, we have

$$\partial_3 u_\alpha + \partial_\alpha u_3 = 0,$$

and therefore

$$\partial_3 \hat{z}_\alpha = \partial_3 z_\alpha \text{ i.e. } \hat{z}_\alpha(x) = z_\alpha(x) + c(x'), \text{ a.e. } x \in \Omega.$$

In particular

$$\hat{z}_\alpha(x) = c(x'), \text{ a.e. } x \in F$$

since

$$z_\alpha(x) = 0, \text{ a.e. } x \in F.$$

On the other hand one has by virtue of (20),

$$z^\varepsilon \to 0 \text{ in } L^2(F).$$

Hence

$$c(x') = 0, \text{ a.e. } x \in F$$

and therefore

$$\hat{z}_\alpha(x) = z_\alpha(x), \text{ a.e. } x \in \Omega.$$

This proves our claim.

Finally, from (29) and (30) we deduce that

$$\varepsilon(\mathrm{E}^\varepsilon u^\varepsilon)_{\alpha 3}\chi_M \rightharpoonup \tfrac{1}{2}\partial_3 z_\alpha \chi_M, \; \varepsilon(\mathrm{E}^\varepsilon u^\varepsilon)_{33}\chi_M \rightharpoonup (\mathrm{E}z)_{33}\chi_M \text{ in } L^2(\Omega),$$

where $z_3 := \hat{z}_3$.

Moreover, since $\varepsilon u_\alpha^\varepsilon \longrightarrow 0$ in $L^2(M)$, we get $\varepsilon(\mathrm{E}^\varepsilon u^\varepsilon)_{\alpha\beta}\chi_M \rightharpoonup 0$ in $L^2(\Omega)$. We conclude that

$$\varepsilon \mathrm{E}^\varepsilon u^\varepsilon \chi_M \rightharpoonup \left(\begin{array}{c|c} 0 & \frac{1}{2}\partial_3 z_\alpha \\ \hline \frac{1}{2}\partial_3 z_\alpha & \partial_3 z_3 \end{array}\right) \chi_M \text{ in } L^2(\Omega)^{3\times 3}.$$

This ends the proof of Lemma 3.1.

In the following theorem, we state our main result.

Theorem 3.2 *There exists $(u, v, z) \in \mathcal{U} \times \mathcal{V} \times \mathcal{Z}$, such that the sequence of solution u^ε of the problem (4) fulfills the following strong convergences*

$$\begin{aligned} u_\alpha^\varepsilon &\longrightarrow u_\alpha + z_\alpha \text{ strongly in } L^2(\omega; H^1(\mathrm{I})),\ u_3^\varepsilon \longrightarrow u_3 \text{ strongly in } H^1(\Omega), \\ \mathrm{E}^\varepsilon u^\varepsilon \chi_F &\longrightarrow \left(\begin{array}{c|c} (\mathrm{E}u)_{\alpha\beta} & \frac{1}{2}\partial_3 v_\alpha \\ \hline \frac{1}{2}\partial_3 v_\alpha & \partial_3 v_3 \end{array}\right) \chi_F \text{ strongly in } L^2(\Omega)^{3\times 3}, \\ \varepsilon \mathrm{E}^\varepsilon u^\varepsilon \chi_M &\longrightarrow \left(\begin{array}{c|c} 0 & \frac{1}{2}\partial_3 z_\alpha \\ \hline \frac{1}{2}\partial_3 z_\alpha & \partial_3 z_3 \end{array}\right) \chi_M \text{ strongly in } L^2(\Omega)^{3\times 3}, \end{aligned}$$

where the limit (u, v, z) is the unique solution of the problem

$$\begin{cases} (u, v, z) \in \mathcal{U} \times \mathcal{V} \times \mathcal{Z},\ \forall (\bar{u}, \bar{v}, \bar{z}) \in \mathcal{U} \times \mathcal{V} \times \mathcal{Z}, \\ \displaystyle\int_\Omega \mathbb{A}(x) \left(\begin{array}{c|c} (\mathrm{E}u)_{\alpha\beta} & \frac{1}{2}\partial_3 v_\alpha \\ \hline \frac{1}{2}\partial_3 v_\alpha & \partial_3 v_3 \end{array}\right) . \left(\begin{array}{c|c} (\mathrm{E}\bar{u})_{\alpha\beta} & \frac{1}{2}\partial_3 \bar{v}_\alpha \\ \hline \frac{1}{2}\partial_3 \bar{v}_\alpha & \partial_3 \bar{v}_3 \end{array}\right) \chi_F dx \\ + \displaystyle\int_\Omega \mathbb{A}(x) \left(\begin{array}{c|c} 0 & \frac{1}{2}\partial_3 z_\alpha \\ \hline \frac{1}{2}\partial_3 z_\alpha & \partial_3 z_3 \end{array}\right) . \left(\begin{array}{c|c} 0 & \frac{1}{2}\partial_3 \bar{z}_\alpha \\ \hline \frac{1}{2}\partial_3 \bar{z}_\alpha & \partial_3 \bar{z}_3 \end{array}\right) \chi_M dx \\ = \displaystyle\int_\Omega \Big(f_\alpha\, (\bar{u}_\alpha + \bar{z}_\alpha) + f_3 \bar{u}_3 \Big) dx. \end{cases} \tag{31}$$

Proof We first prove the convergence of the sequence of energies to the energy associated to the limit problem, assuming that the limit problem is the system (31). To that aim we will use the weak convergences proved in Lemma 3.1. We follow the argument already used in [14, Proof of Theorem 3.1].

Choosing $(\bar{u}, \bar{v}, \bar{z}) = (u, v, z)$ in (31) and passing to the limit thanks to Lemma 3.1, we get

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_\Omega \Big(\chi_F + \varepsilon^2 \chi_M\Big)\, \mathbb{A}\mathrm{E}^\varepsilon(u^\varepsilon).\mathrm{E}^\varepsilon(u^\varepsilon) dx &= \lim_{\varepsilon\to 0} \int_\Omega f u^\varepsilon dx \\ &= \int_\Omega \Big(f_\alpha\, (u_\alpha + z_\alpha) + f_3 u_3 \Big) dx \\ &= \int_\Omega (AE_F.E_F\chi_F + AE_M.E_M\chi_M)\, dx, \end{aligned}$$

where

$$E_F := \left(\begin{array}{c|c} (\mathrm{E}u)_{\alpha\beta} & \frac{1}{2}\partial_3 v_\alpha \\ \hline \frac{1}{2}\partial_3 v_\alpha & \partial_3 v_3 \end{array} \right), \ E_M := \left(\begin{array}{c|c} 0 & \frac{1}{2}\partial_3 z_\alpha \\ \hline \frac{1}{2}\partial_3 z_\alpha & \partial_3 z_3 \end{array} \right).$$

Using the equivalence of the norms

$$L^2(\Omega) \ni E \longmapsto \int_\Omega AE.Edx, \, L^2(\Omega) \ni E \longmapsto \int_\Omega E.Edx$$

we deduce that

$$\lim_{\varepsilon\to 0} \left\| (\chi_F + \varepsilon\chi_M)\, \mathrm{E}^\varepsilon(u^\varepsilon) \right\|_{L^2(\Omega)} = \|\chi_F E_F + \chi_M E_M\|_{L^2(\Omega)}\,.$$

Hence the strong convergence of the tensor sequences is proved. Now, by using the last strong convergence together with the following Korn's inequality

$$\left\| \mathrm{E}^\varepsilon(u^\varepsilon) \right\|_{L^2(F)} \geq \left\| \mathrm{E}(u^\varepsilon) \right\|_{L^2(F)} \geq C \left\| u^\varepsilon \right\|_{L^2(F)},$$

we deduce that u^ε is a Cauchy sequence in $H^1(F)$.

On the other hand, using (9) and (10) one has

$$\|\mathrm{E}^\varepsilon u^\varepsilon\|^2_{L^2(F)} \geq C\varepsilon^2 \left(\|(R^\varepsilon)^{-1} u^\varepsilon\|^2_{L^2(F)} + \|\nabla^\varepsilon u^\varepsilon\|^2_{L^2(F)} \right)$$

and

$$\left(\|\nabla^\varepsilon u^\varepsilon\|^2_{L^2(F)} + \|\mathrm{E}^\varepsilon u^\varepsilon\|^2_{L^2(M)} \right) \geq C \|\nabla^\varepsilon u^\varepsilon\|^2_{L^2(M)}.$$

Hence, $\varepsilon\nabla^\varepsilon u^\varepsilon$ is also a Cauchy sequence in $L^2(\Omega)$. It remains to prove that u^ε_α and u^ε_3 are two Cauchy sequences in $L^2(\omega; H^1(\mathrm{I}))$ and $H^1(M)$ respectively. To that aim, it suffices to use (18) to obtain

$$\left\| (\chi_F + \varepsilon\chi_M)\, \mathrm{E}^\varepsilon(u^\varepsilon) \right\|^2_{L^2(\Omega)} = \mathcal{F}_\varepsilon(u^\varepsilon) \geq C \left(\|u^\varepsilon_3\|^2_{H^1(M)} + \|u^\varepsilon_\alpha\|^2_{L^2(\omega; H^1(\mathrm{I}\setminus\bar{\mathrm{J}}))} \right)$$

then we can argue as previously to complete the proof. □

We now prove that the limit problem is nothing but the system (31).

We choose a test function ϕ^ε in (4) in the following form

$$\begin{cases} \phi^\varepsilon_\alpha = \bar{u}_\alpha + \varepsilon\bar{v}_\alpha + \bar{z}_\alpha \\ \phi^\varepsilon_3 = \bar{u}_3 + \varepsilon^2\bar{v}_3 + \varepsilon\bar{z}_3 \end{cases} \tag{32}$$

where $(\bar{u}, \bar{v}, \bar{z}) \in \mathcal{U} \times \mathcal{D}\left(\omega; H^1_m(J)\right)^3 \times \left\{ \mathcal{D}\left(\omega; H^1_m(I)\right)^3 : z = 0, \text{ a.e. } x \in F \right\}$. Since, from the definition of $\mathcal{U}$, one has $(E\bar{u})_{i3} = 0$, for all $i = 1, 2, 3$, an elementary calculation shows that

$$\begin{cases} (\mathrm{E}^\varepsilon \phi^\varepsilon)_{\alpha\beta} = (\mathrm{E}\bar{u})_{\alpha\beta} + \varepsilon(\mathrm{E}\bar{v})_{\alpha\beta} + (\mathrm{E}\bar{z})_{\alpha\beta}, \\ 2(\mathrm{E}^\varepsilon \phi^\varepsilon)_{\alpha 3} = \partial_3 \bar{v}_\alpha + \varepsilon \partial_\alpha \bar{v}_3 + \frac{1}{\varepsilon}\partial_3 \bar{z}_\alpha + \partial_\alpha \bar{z}_3, \\ (\mathrm{E}^\varepsilon \phi^\varepsilon)_{33} = (\mathrm{E}\bar{v})_{33} + \frac{1}{\varepsilon}(\mathrm{E}\bar{z})_{33}. \end{cases}$$

Using the following strong convergences

$$\mathrm{E}^\varepsilon \phi^\varepsilon \chi_F \longrightarrow \left(\begin{array}{c|c} (\mathrm{E}\bar{u})_{\alpha\beta} & \frac{1}{2}\partial_3 \bar{v}_\alpha \\ \hline \frac{1}{2}\partial_3 \bar{v}_\alpha & \partial_3 \bar{v}_3 \end{array}\right) \chi_F \text{ in } L^2(\Omega)^{3\times 3},$$

$$\varepsilon \mathrm{E}^\varepsilon \phi^\varepsilon \chi_M \longrightarrow \left(\begin{array}{c|c} 0 & \frac{1}{2}\partial_3 \bar{z}_\alpha \\ \hline \frac{1}{2}\partial_3 \bar{z}_\alpha & \partial_3 \bar{z}_3 \end{array}\right) \chi_M \text{ in } L^2(\Omega)^{3\times 3},$$

we can pass to the limit in (4) as ε goes to zero to get (31) thanks to the density of $\mathcal{D}\left(\omega; H^1_m(J)\right)^3$ and $\left\{z \in \mathcal{D}\left(\omega; H^1_m(I)\right)^3 : z_i = 0, \text{ a.e. } x \in F, \ i = 1, 2, 3\right\}$ in $\mathcal{V}$ and $\mathcal{Z}$ respectively.

Remark 3.3 The space $\mathcal{U} \times \mathcal{V} \times \mathcal{Z}$ is a Hilbert space for the following norm

$$\|(u, v, z)\|^2 := \sum_{\alpha,\beta=1,2} \left\|(\mathrm{E}u)_{\alpha\beta}\right\|^2_{L^2(\Omega)} + \sum_{i=1}^{3} \left[\|\partial_3 v_i\|^2_{L^2(\Omega)} + \|\partial_3 z_i\|^2_{L^2(\Omega)}\right].$$

Therefore, using the Lax-Milgram Theorem we obtain the well-posedness of the limit problem (31) since the tensor A is coercive by virtue of (3) and $f \in L^2(\Omega; \mathbb{R}^3)$.

We now turn back to the sequence of solutions of problem (6) to derive the following theorem from Theorem 3.2.

Theorem 3.4 *Let* $(u, v, z) \in \mathcal{U} \times \mathcal{V} \times \mathcal{Z}$ *be the solution of the problem* (31). *Let*

$$\mathrm{E}_F := \left(\begin{array}{c|c} (\mathrm{E}u)_{\alpha\beta} & \frac{1}{2}\partial_3 v_\alpha \\ \hline \frac{1}{2}\partial_3 v_\alpha & \partial_3 v_3 \end{array}\right), \quad \mathrm{E}_M := \left(\begin{array}{c|c} 0 & \frac{1}{2}\partial_3 z_\alpha \\ \hline \frac{1}{2}\partial_3 z_\alpha & \partial_3 z_3 \end{array}\right).$$

Then the sequence of solutions $\hat{u}^\varepsilon$ *of the problem* (6) *fulfills the following convergences*

$$\begin{cases} ⨍_{\varepsilon \mathrm{J}} \mathrm{E}\hat{u}^\varepsilon dx_3 \longrightarrow ⨍_{\mathrm{J}} \mathrm{E}_F dx_3, \text{ in } L^2(\omega; \mathbb{R}^{3\times 3}), \\ ⨍_{\varepsilon(\mathrm{I}\setminus\mathrm{J})} \mathrm{E}\hat{u}^\varepsilon dx_3 \longrightarrow ⨍_{\mathrm{I}\setminus\mathrm{J}} \mathrm{E}_M dx_3 \text{ in } L^2(\omega; \mathbb{R}^{3\times 3}), \\ ⨍_{\varepsilon \mathrm{I}} \hat{u}^\varepsilon_\alpha dx_3 \longrightarrow U_\alpha(x') := ⨍_{\mathrm{I}} (u_\alpha + z_\alpha)\, dx_3, \ \text{ in } L^2(\omega), \\ ⨍_{\varepsilon \mathrm{I}} \varepsilon \hat{u}^\varepsilon_3 dx_3 \longrightarrow u_3(x') \text{ in } L^2(\omega), \end{cases} \tag{33}$$

Moreover, U_α may be written as

$$U_\alpha(x') = ⨍_{\mathrm{I}} u_\alpha dx_3 + m^0_\alpha(x') ⨍_{\mathrm{I}\setminus\mathrm{J}} f_\alpha(x', x_3) dx_3 + m^{00}_\alpha(x'), \tag{34}$$

where m^0_α and m^{00}_α are given by

$$m^0_\alpha(x') := \int_{\mathrm{I}\setminus\mathrm{J}} z^0_\alpha(x', x_3) dx_3, \ m^{00}_\alpha(x') := \int_{\mathrm{I}\setminus\mathrm{J}} z^{00}_\alpha(x', x_3) dx_3,$$

and z^0, z^{00} are respectively the solutions of the following problems

$$\begin{cases} z^0 \in L^\infty\left(\omega, \mathcal{Z}^0\right), \mathcal{Z}^0 := \left\{z \in (H^1(\mathrm{I}\backslash\mathrm{J})^3 : z = 0 \text{ on } \partial J\right\} \\ \displaystyle\int_{\mathrm{I}\backslash\mathrm{J}} A(x) \left(\begin{array}{c|c} 0 & \frac{1}{2}\partial_3 z^0_\alpha \\ \hline \frac{1}{2}\partial_3 z^0_\alpha & \partial_3 z^0_3 \end{array}\right)\left(\begin{array}{c|c} 0 & \frac{1}{2}\partial_3 \bar{z}_\alpha \\ \hline \frac{1}{2}\partial_3 \bar{z}_\alpha & \partial_3 \bar{z}_3 \end{array}\right) dx_3 = \int_{\mathrm{I}\backslash\mathrm{J}} \bar{z}_\alpha(x_3)dx_3, \\ \forall \bar{z} \in \mathcal{Z}^0, \quad \text{a.e. in } \omega, \end{cases} \tag{35}$$

$$\begin{cases} z^{00} \in \mathcal{Z} \\ \displaystyle\int_M A(x) \left(\begin{array}{c|c} 0 & \frac{1}{2}\partial_3 z^{00}_\alpha \\ \hline \frac{1}{2}\partial_3 z^{00}_\alpha & \partial_3 z^{00}_3 \end{array}\right)\left(\begin{array}{c|c} 0 & \frac{1}{2}\partial_3 \bar{z}_\alpha \\ \hline \frac{1}{2}\partial_3 \bar{z}_\alpha & \partial_3 \bar{z}_3 \end{array}\right) dx_3 = \\ \displaystyle\int_M \left(f_\alpha(x) - \fint_{\mathrm{I}\backslash\mathrm{J}} f_\alpha(x)dx_3\right) \bar{z}_\alpha(x)dx, \\ \forall \bar{z} \in \mathcal{Z}. \end{cases} \tag{36}$$

Proof

The convergences (33) follow from the corresponding convergences stated in Theorem 3.2 by the change of variables (5).

Let us prove for instance the third convergence arising in (33) by the use of the first convergence arising in Theorem 3.2. Setting $y_3 = \varepsilon x_3$ for $x_3 \in I$ and bearing in mind that $\hat{u}^\varepsilon$ is defined according to (5) by $u^\varepsilon_\alpha(x', x_3) = \hat{u}^\varepsilon(x', \varepsilon x_3)$ for $(x', x_3) \in \omega \times I$ and that the length of the interval I is equal to 1, we get thanks to the Cauchy-Schwarz inequality (with respect to x_3)

$$\begin{cases} \displaystyle\int_\omega |\fint_{\varepsilon \mathrm{I}} \hat{u}^\varepsilon_\alpha(x', y_3)dy_3 - \fint_I (u_\alpha(x', x_3) + z_\alpha(x', x_3)dx_3|^2 dx' \\ \displaystyle= \int_\omega \left|\int_I \Big(u^\varepsilon_\alpha(x', x_3) - (u_\alpha(x', x_3) + z_\alpha(x', x_3))\Big)dx_3\right|^2 dx' \\ \displaystyle\leq \int_\omega \int_I |u^\varepsilon_\alpha(x', x_3) - (u_\alpha(x', x_3) + z_\alpha(x', x_3))|^2 dx_3 dx' \longrightarrow 0. \end{cases} \tag{37}$$

To prove (34), we first notice that one can derive the following equation satisfied by z_α by choosing $\bar{u} = \bar{v} = 0$ in (31),

$$\begin{cases} z \in \mathcal{Z}, \quad \forall \bar{z} \in \mathcal{Z}, \\ \displaystyle\int_\Omega \mathbb{A}(x) \left(\begin{array}{c|c} 0 & \frac{1}{2}\partial_3 z_\alpha \\ \hline \frac{1}{2}\partial_3 z_\alpha & \partial_3 z_3 \end{array}\right) \cdot \left(\begin{array}{c|c} 0 & \frac{1}{2}\partial_3 \bar{z}_\alpha \\ \hline \frac{1}{2}\partial_3 \bar{z}_\alpha & \partial_3 \bar{z}_3 \end{array}\right) \chi_M dx \\ \displaystyle= \int_\Omega f_\alpha \bar{z}_\alpha dx. \end{cases} \tag{38}$$

Taking advantage from the linearity of (38), one can check easily by a uniqueness argument that z_α may be identified as $z_\alpha = z^1_\alpha + z^2_\alpha$ where z^1_α and z^2_α are solutions of

(38) but with right hand sides defined respectively by

$$\fint_{\mathrm{I}\setminus\mathrm{J}} f_\alpha(x)dx_3 \text{ and } \left(f_\alpha(x) - \fint_{\mathrm{I}\setminus\mathrm{J}} f_\alpha(x)dx_3\right).$$

By linearity and using once again a uniqueness argument, we conclude that

$$z^1_\alpha = \left(\fint_{\mathrm{I}\setminus\mathrm{J}} f_\alpha(x)dx_3\right) z^0_\alpha \quad \text{and} \quad z^2_\alpha = z^{00}_\alpha$$

where z^0 and z^{00} are the solutions of (35) and (36) respectively. Hence, U_α defined in the third convergence (33) may be written as (34). This completes the proof of Theorem 3.4.

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