



# Effects of vertical throughflow and variable gravity field on double diffusive convection in a fluid layer

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## Abstract

In this study, the effect of throughflow and variable gravity field on the stability of double diffusive convection in a fluid layer for the system when heating and salting both are done from below, are investigated. The linear and energy techniques are regulated to check the stability characteristics of the system. The numerical results of linear and non-linear techniques are compared to check the subcritical region. The Chebyshev pseudo-spectral technique is applied to find the influences of distinct physical parameters on the stability of the system.

**Keywords** Double diffusive convection · Throughflow · Variable gravity · Linear instability analysis · Nonlinear energy stability analysis

**Mathematics Subject Classification** 76E06 · 76E30 · 76R10

## 1 Introduction

The study of heat and mass transfer problem where two buoyancy driven components (heat and salt) diffuses with different rate is termed as double diffusive convection and the study of this type of problem has captivated much concentration of many researchers due to its comprehensive real life applications, see, e.g. Turner [1], Kaufman [2], Oldenburg and Pruess [3], Bear and Gillman [4], Gilman [5]. Detailed linear analysis for the double diffusive convection problem in a porous medium has been worked out by Nield [6] and in fluid layer by Baines and Gill [7]. Shir and Joseph [8] studied the double diffusive convection problem and found that, for the problem when heating is done from below however, salting is done from above then corresponding

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linearized system is symmetric, and thus, linear and nonlinear bounds coincide. This result was first founded by Shir and Joseph [8]. Furthermore, if both heating and salting of fluid layer is done from below then energy stability theory becomes more intricate to attain accurate results see [9, 10]. Joseph [11] established a new twist into energy stability theory. He found the idea of generalized energy theory to produce sharp nonlinear stability threshold. Mulone [12] achieved a sharp unconditional results for the double diffusive convection in a fluid layer when both heating and salting are done from below. Mulone and Rionero [13], Lombardo et al. [14] derived further sharp nonlinear stability boundaries using modified generalized energies, for a fluid layer, when heating and salting to the system are done from below. Mahajan and Tripathi [15, 16] respectively analyzed the impact of changeable viscosity and non-uniform temperature gradients on the stability of the system where both heat and mass diffusion take place.

The hypothesis of constant gravity in the pure study is not verified for wide ranging convection appearance arising in the airspace, or the mantle of the earth as the earth's gravity field varies with the distance. In order to study the wide-ranging convection flows, it is necessary to consider the gravity field as a function of distance from the centre of earth, as identified by the Pradhan and Samal [17]. Alex et al. [18, 19] studied the impact of internal heating and changeable gravity field on the convective instability in a porous medium in the presence of inclined temperature gradient. The effect of variable gravity on the convective instability was analysed by Straughan [20] by applying linear and nonlinear analysis. The results of linear and nonlinear stability analysis are attained and compared for linearly decreasing type gravity variation function. The different types of gravity fields are chosen to obtain its effect on the onset of Rayleigh Bénard convection problem see, Straughan [21]. Rionero and Straughan [22] also analysed the problem where gravity changes with height in a porous medium. Harfash [23] investigated the impact of gravity field and magnetic field on the convective instability in a porous medium. Later, Mahajan and Sharma [24] made an effort to check the stability by applying linear analysis technique on the convective instability of Magnetic Nano-Fluid layer. An investigation is made using simultaneous impact of variable gravity field and through flow on the stability of the system in a porous medium by Yadav [25, 26]. Recently, a very interesting problem of analyzing the gravity field impact on the stability of double diffusive system in a fluid layer where convective instability occurs due to non-uniformity in temperature and concentration gradients, is explored by Mahajan and Tripathi [27]. For more related knowledge, the researcher may call attention to the Refs. [28–30].

Moreover, the theory of through flow is important to manage the convective operations in geophysics, industries, and in engineering sciences etc. The impact of through flow on the Rayleigh–Bénard convection in a fluid layer was analysed by Nield [31]. The impact of through flow on the convection induced by penetration is studied by Harfash [32]. Shivkumara et al. [33] made an effort to investigate the impact of throughflow with internal heat source on convective operations in a fluid layer. Taking into consideration the work of Shivkumara [33], the combined impact of through flow and constant heat source on the double diffusive convection in anisotropic porous layer are investigated by Capone et al. [34]. In all of the studies mentioned above, the effect of throughflow with variable gravity field on convection problem (where both heat and

mass diffuse) in a fluid layer has not been studied yet. The study of using combined effect of throughflow with variable gravity is very important in controlling the convection mechanism because the direction of throughflow can be taken in the direction of gravity field (throughflow inclined gravity) or in the opposite direction of gravity field (throughflow disinclined gravity). Throughflow inclined and disinclined gravity effects on the convection mechanism plays an important contribution in engineering sciences, geophysics and in hydrothermal vent systems. In the present work, the linear and energy technique are applied for double diffusive convection problem in a fluid layer with the impact of throughflow and gravity field that is non-constant. The linear analysis is applied with the help of normal mode technique while energy method is adopted to conduct the nonlinear analysis. The eigenvalue problems obtained from linear and nonlinear analysis, is solved numerically using Chebyshev pseudospectral method [35] in MATLAB. In the next section, we present a mathematical model to be studied. In Sect. 3, we introduce linear stability analysis for our system. Since the present system is not symmetric so nonlinear stability analysis gives important result thus, in Sect. 4, nonlinear stability analysis is applied using energy method. In Sect. 5, we present the results obtained using linear and nonlinear stability analysis and compare the corresponding results and conclusions are presented in Sect. 6.

## 2 Mathematical formulation of the problem

Let's consider an incompressible fluid layer, heated and salted from below, enclosed between two parallel planes  $z = 0$  and  $z = d$  under the influence of variable gravity field  $\mathbf{g}(z)$ . Let differences of the temperature and concentration are  $\Delta T = T_1 - T_2$  and  $\Delta C = C_1 - C_2$  respectively where  $T_1, C_1$  are the temperature and concentration of lower plates respectively and  $T_2, C_2$  are temperature and concentration of upper plates respectively. The fluid motion can be presented by Navier–Stokes equation.

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} \cdot \nabla \mathbf{q} = -\frac{1}{\rho_0} \nabla p + \nu \nabla^2 \mathbf{q} - [1 - \alpha_T(T - T_0) + \alpha_C(C - C_0)]g(z)\mathbf{k}, \quad (1)$$

The remaining governing equations are the condition of incompressibility, energy equation, and concentration equation, namely

$$\nabla \cdot \mathbf{q} = 0, \quad (2)$$

$$\frac{\partial T}{\partial t} + \mathbf{q} \cdot \nabla T = k_T \nabla^2 T, \quad (3)$$

$$\frac{\partial C}{\partial t} + \mathbf{q} \cdot \nabla C = k_C \nabla^2 C, \quad (4)$$

with boundary conditions

$$\left. \begin{aligned} \mathbf{q} &= (0, 0, w_0), & z &= 0, d \\ T &= T_1, C = C_1 & z &= 0 \\ T &= T_2, C = C_2 & z &= d \end{aligned} \right\} \quad (5)$$

Where  $\mathbf{q} = (u, v, w)$  is the fluid velocity, denoting pressure, temperature, and concentration fields by  $p, T, C$  respectively and  $\alpha_T, \alpha_C, \nu, k_T, k_C$  by coefficient of thermal expansion, coefficient of solute expansion, dynamic viscosity, thermal diffusivity, solutal diffusivity respectively,  $\mathbf{k} = (0, 0, 1)$  is the unit vector, change in  $g(z) = (1 + \varepsilon h(z))g$  denotes change in gravity field.

Let's consider the steady state solution of (1–5) with a through flow in vertical direction of the form  $\mathbf{q} = (0, 0, w_0)$ , where  $w_0$  is the constant velocity. Now, applying the boundary conditions, we obtain the following basic state equations

$$\begin{aligned} T_b(z) &= T_1 + (T_2 - T_1) \frac{\left(1 - \exp\left(\frac{w_0}{k_T}z\right)\right)}{1 - \exp\left(\frac{w_0}{k_T}d\right)}, \\ C_b(z) &= C_1 + (C_2 - C_1) \frac{\left(1 - \exp\left(\frac{w_0}{k_C}z\right)\right)}{1 - \exp\left(\frac{w_0}{k_C}d\right)}, \end{aligned}$$

To assess the stability of the basic state solution, we employed perturbations as of the form

$$\mathbf{q} = \mathbf{q}_b(z) + \mathbf{q}', \quad p = p_b(z) + p', \quad \rho = \rho_b(z) + \rho', \quad T = T_b(z) + \theta, \quad C = C_b(z) + \phi$$

and non-dimensionalize with following scalings

$$\left. \begin{aligned} z &= dz^*, \quad \mathbf{q} = \frac{k_T}{d} \mathbf{q}^*, \quad t = \frac{d^2}{k_T} t^*, \quad p = \frac{\rho_0 \nu k_T}{d^2} p^*, \quad \theta = \sqrt{\frac{\Delta T \nu k_T}{\alpha_T g d^3}} \theta^*, \quad C = \sqrt{\frac{\Delta C \nu k_T^2}{\alpha_C g d^3 k_C}} \phi^*, \\ Le &= \frac{k_T}{k_C}, \quad Pr = \frac{\nu}{k_T}, \quad Ra = R^2 = \frac{g \alpha_T \Delta T d^3}{\nu k_T}, \quad Rc = Rs^2 = \frac{g \alpha_C \Delta C d^3}{\nu k_C}, \quad Q = \frac{w_0 d}{k_T} \end{aligned} \right\} \quad (6)$$

We obtain the following non-dimensional equations (after ignoring prime and asterisks)

$$\frac{1}{Pr} \left( \frac{\partial \mathbf{q}}{\partial t} + Q \frac{\partial \mathbf{q}}{\partial z} + \mathbf{q} \cdot \nabla \mathbf{q} \right) = -\nabla p + \nabla^2 \mathbf{q} + s(z)(R\theta - Rs\phi)\mathbf{k}, \quad (7)$$

$$\nabla \cdot \mathbf{q} = 0, \quad (8)$$

$$\frac{\partial \theta}{\partial t} + Q \frac{\partial \theta}{\partial z} + \mathbf{q} \cdot \nabla \theta = \nabla^2 \theta + Rf_T(z)w, \quad (9)$$

$$Le \left( \frac{\partial \phi}{\partial t} + Q \frac{\partial \phi}{\partial z} + \mathbf{q} \cdot \nabla \phi \right) = \nabla^2 \phi + Rs f_C(z) w, \tag{10}$$

with the perturbed boundary conditions

$$\left. \begin{aligned} \mathbf{q} &= (0, 0, 0) \quad z = 0, 1 \\ \theta = \phi = 0, & \quad z = 0, 1 \end{aligned} \right\} \tag{11}$$

where Pr is Prandtl number, *Le* is Lewis number, *Q* is the Peclet number, *Ra* is the Rayleigh number for heat, *Rc* is the solute Rayleigh number,  $f_T(z) = -\frac{d}{\Delta T} \frac{dT_b}{dz}$ ,  $f_C(z) = -\frac{d}{\Delta C} \frac{dC_b}{dz}$  are the non-dimensional temperature and concentration gradients respectively and  $s(z) = 1 + \varepsilon h(z)$ .

### 3 Linear stability analysis

To proceed with the linear analysis, the nonlinear terms in Eqs. (7–10) are ignored and assuming the amplitude of the perturbations are infinitely small we take the normal mode solution in the form

$$(w, \theta, \phi) = [W(z), \Theta(z), \Phi(z)] \exp [i(a_x x + a_y y) + \sigma t] \tag{12}$$

where  $a_x$  and  $a_y$  are horizontal wave numbers and  $\sigma$  is growth rate. Now the pressure term from Eq. (7), is removed by applying  $\mathbf{k} \cdot \nabla \times \nabla \times$  Eq. (7) thus, obtain

$$\begin{aligned} \frac{-1}{Pr} \sigma (D^2 - a^2) W &= - (D^2 - a^2)^2 W + \frac{Q}{Pr} D (D^2 - a^2) W \\ &+ a^2 s(z) R \Theta - a^2 s(z) R s \Phi, \end{aligned} \tag{13}$$

$$\sigma \Theta = R f_T(z) W - Q D \Theta + (D^2 - a^2) \Theta, \tag{14}$$

$$Le(\sigma \Phi) = R s f_C(z) W - Le Q D \Phi + (D^2 - a^2) \Phi \tag{15}$$

with boundary conditions

$$W = DW = \Theta = \Phi = 0 \text{ at } z = 0, 1 \tag{16}$$

Now, to solve the eigenvalue problem (13–16), we applied the Chebyshev pseudo-spectral method [35]. We first converted the domain of present problem [0, 1] into [-1, 1] to match the present domain with that of Chebyshev pseudo-spectral method, applying the transformation  $z \rightarrow 2z_1 - 1$ . Thus, we obtain a system of linearized

equations

$$\frac{-1}{\text{Pr}}\sigma(4D^2 - a^2)W = -(4D^2 - a^2)^2W + \frac{Q}{\text{Pr}}D(8D^2 - 2a^2)W + a^2s(z_1)R\Theta - a^2s(z_1)Rs\Phi \quad (17)$$

$$\sigma\Theta = Rf_T(z_1)W - 2QD\Theta + (4D^2 - a^2)\Theta, \quad (18)$$

$$Le(\sigma\Phi) = Rsf_C(z_1)W - 2LeQD\Phi + (4D^2 - a^2)\Phi \quad (19)$$

The boundary conditions are

$$W = DW = \Theta = \Phi = 0 \text{ at } z = -1, 1 \quad (20)$$

Now, to solve the above eigenvalue problem Eqs. (17–19), we discretize the set of Eqs. (17–19) into matrix form as

$$\sigma BX = AX, \quad (21)$$

$$\text{where, } X = [W \ \Theta \ \Phi]^T, \ B = \begin{bmatrix} \frac{-1}{\text{Pr}}(4D^2 - a^2I) & O & O \\ O & I & O \\ O & O & LeI \end{bmatrix},$$

$$A = \begin{bmatrix} -(4D^2 - a^2I)^2 + \frac{Q}{\text{Pr}}D(8D^2 - 2a^2I) & a^2s(z_1)RI & -a^2s(z_1)RsI \\ Rf_T(z_1)I & (4D^2 - a^2I) - 2QD & O \\ Rsf_C(z_1)RsI & O & (4D^2 - a^2I) - 2LeQD \end{bmatrix}.$$

denoting 'O' and 'I' by the zero and identity matrices respectively. Using the boundary conditions (20) the equation (21) is now solved by applying QZ-algorithm in MATLAB. Now, the critical Rayleigh number  $Ra_L$  for linear theory is calculated as,

$$Ra_L = \min_{a>0} R^2(a) \quad (22)$$

The matrix  $A$  of Eq. (21) above, in the absence of throughflow and gravity variation can be easily compared with Eq. (6.15) of Sect. 6.2 in Rionero [36]. So the results and conditions for the principle of exchange of stability given in Rionero [36] for the problem heated and salted from below also holds here when throughflow and gravity fields are absent. In the current problem, the inclusion of throughflow and gravity variation makes it difficult to find condition for principle of exchange of stability, however, it has been seen in the literature [see, Hill et al. [37]] that the  $\sigma$  remains real with the inclusion of throughflow; so one can argue that with the inclusion of throughflow the condition for principle of exchange of stability will not change and so the stationary convection will occur for  $Le < 1$  and convection will be oscillatory for  $1 \leq Le$ .

Moreover, in order to obtain the critical Rayleigh number explicitly, we take the trial function satisfying the rigid-rigid boundary condition (16) in the form, as below

$$W = A_1 z^2(1 - z)^2, \Theta = A_2 z(1 - z), \Phi = A_3 z(1 - z) \tag{23}$$

Substituting Eq. (23) into the Eqs. (13–15) and integrating over [0,1] results in a homogeneous system, which is solved to find the Rayleigh number given by

$$Ra = \frac{\left(\frac{\sigma a^2}{30Pr} + \frac{a^4}{30} + 24\right)\left(2 + \frac{a^2 + \sigma}{6}\right)}{a^2 F_1 F_2} + \frac{Rc F_3}{F_2} \frac{\left(2 + \frac{a^2 + \sigma}{6}\right)}{\left(2 + \frac{a^2 + Le\sigma}{6}\right)} \tag{24}$$

where  $F_1 = \int_0^1 s(z)(z - z^2)dz$ ,  $F_2 = \int_0^1 f_T(z)z^2(1 - z)^2 dz$ ,  $F_3 = \int_0^1 f_C(z)z^2(1 - z)^2 dz$ .

### 3.1 Marginal stationary state

We put the value of  $\sigma = \sigma_r + i\sigma_i$  equal to zero, to obtain the Rayleigh number for stationary mode as

$$Ra = R^2 = \frac{\left(\frac{a^4}{30} + 24\right)\left(2 + \frac{a^2}{6}\right)}{a^2 F_1 F_2} + \frac{Rc F_3}{F_2} \tag{25}$$

In the limiting case, when  $Rc = 0$ ,  $Q = 0$ ,  $\varepsilon = 0$  above reduces to

$$Ra = R^2 = \frac{180\left(\frac{a^4}{30} + 24\right)\left(2 + \frac{a^2}{6}\right)}{a^2} \tag{26}$$

Now, by calculating the critical wave numbers using  $\frac{dR^2}{da^2} = 0$ , the critical Rayleigh number comes out to be  $Ra_L = 1707.76$  which is the result of standard Bénard problem.

### 3.2 Marginal oscillatory state

In order to obtain the Rayleigh number for oscillatory mode, we apply the process as applied by Mahajan and Tripathi [38] to write

$$Ra^{osc} = \frac{\left(2 + \frac{a^2}{6}\right)\left(\frac{a^4}{30} + 24\right) - \frac{\sigma_i^2 a^2}{180Pr}}{a^2 F_1 F_2} + \frac{\frac{Rc F_3}{F_2} \left(2 + \frac{a^2}{6}\right)^2 + \frac{Le\sigma_i^2}{36}}{\left(2 + \frac{a^2}{6}\right)^2 + \frac{Le^2\sigma_i^2}{36}} \tag{27}$$

where oscillatory frequency is given by

$$\sigma_i^2 = \frac{36}{Le^2} \left[ \frac{\frac{a^2 R_c F_3 F_1}{6} \left(2 + \frac{a^2}{6}\right) (Le - 1)}{\frac{a^2}{30 \text{Pr}} \left(2 + \frac{a^2}{6}\right) + \frac{1}{6} \left(\frac{a^4}{30} + 24\right)} - \left(2 + \frac{a^2}{6}\right)^2 \right] \quad (28)$$

#### 4 Nonlinear stability analysis

To solve the present problem using nonlinear stability analysis, we applied energy method [21]. Generally, the linear stability analysis gives little information about the nature of the nonlinear system since any potential growth in nonlinear terms are not taken into consideration. To analyze the nature of the nonlinear system, finite perturbations are considered and quadratic terms are not ignored. Now, to initiate the nonlinear stability analysis, Eq. (7) is multiplied by  $\mathbf{q}$ , Eq. (9) is multiplied by  $\theta$  and Eq. (10) is multiplied by  $\phi$ , and on integrating the resultant over the periodic cell  $V$  to obtain

$$\frac{1}{2 \text{Pr}} \frac{d}{dt} \|\mathbf{q}\|^2 = -\|\nabla \mathbf{q}\|^2 + R \langle s(z)\theta w \rangle - R s \langle s(z)\phi w \rangle \quad (29)$$

$$\frac{1}{2} \frac{d}{dt} \|\theta\|^2 = -\|\nabla \theta\|^2 + R \langle f_T(z)\theta w \rangle, \quad (30)$$

$$\frac{Le}{2} \frac{d}{dt} \|\phi\|^2 = -\|\nabla \phi\|^2 + R s \langle f_C(z)w\phi \rangle, \quad (31)$$

where the symbols  $\|\cdot\|$  is norm and  $\langle \cdot \rangle$  is inner product on  $L^2(V)$ . Now, adding Eq. (29–31), with the help of coupling parameters ( $\lambda_1 > 0$ ,  $\lambda_2 > 0$ ), a functional in terms of energy  $E(t)$  is constructed as

$$E(t) = \frac{1}{2 \text{Pr}} \|\mathbf{q}\|^2 + \frac{\lambda_1}{2} \|\theta\|^2 + \frac{\lambda_2 Le}{2} \|\phi\|^2, \quad (32)$$

The progress of  $E(t)$  when time grows is given as

$$\frac{dE(t)}{dt} = \frac{1}{2 \text{Pr}} \frac{d}{dt} \|\mathbf{q}\|^2 + \frac{\lambda_1}{2} \frac{d}{dt} \|\theta\|^2 + \frac{\lambda_2 Le}{2} \frac{d}{dt} \|\phi\|^2 \quad (33)$$

Substituting Eqs. (29–31) into Eq. (33), to write

$$\frac{d(E)}{dt} = I_1 - D_1, \quad (34)$$

now we have

$$I_1 = R \langle s(z)\theta w \rangle - R s \langle s(z)\phi w \rangle + \lambda_1 R \langle f_T(z)\theta w \rangle + \lambda_2 R s \langle f_C(z)w\phi \rangle$$

$$D_1 = \|\nabla \mathbf{q}\|^2 + \lambda_1 \|\nabla \theta\|^2 + \lambda_2 \|\nabla \phi\|^2,$$



we define now  $R_E$  in such a way

$$\frac{1}{R_E} = \max_{\Omega} \frac{I_1}{D_1}, \tag{35}$$

where  $\Omega = \{q, \theta, \phi | q, \theta, \phi \in L^2(V)\}$  and  $q, \theta, \phi$  satisfy the boundary conditions (Eq. 11). Now, the Eqs. (34) and (35), may be written as

$$\frac{dE(t)}{dt} \leq -D_1 \left(1 - \frac{1}{R_E}\right), \tag{36}$$

If  $R_E > 1$ , then we may write  $b_1 = \left(1 - \frac{1}{R_E}\right) > 0$ , and further using Poincare inequality in Eq.(36), we may write the following energy inequality

$$E(t) \leq E(0) \exp\left(-2\pi^2 k' b_1 t\right) \tag{37}$$

where  $k' = \min\{1, Pr, \frac{1}{Le}\}$ . Now, Eq. (37) shows exponential decay of energy function with time for all values of  $E(0)$ . Now, using calculus of variation we will solve the Euler–Lagrange equation (35) using the transformation  $\theta = \frac{\hat{\theta}}{\sqrt{\lambda_1}}$ ,  $\phi = \frac{\hat{\phi}}{\sqrt{\lambda_2}}$ , to obtain the following system of equations

$$2 Pr \nabla^2 q + Rk \left(\frac{Pr s(z) + \lambda_1 f_T(z)}{\sqrt{\lambda_1}}\right) \theta - Rsk \left(\frac{Pr s(z) - \lambda_2 Le^{-1} f_C(z)}{\sqrt{\lambda_2}}\right) \phi = -\nabla p \tag{38}$$

$$2\nabla^2 \theta + R \left(\frac{s(z) Pr + \lambda_1 f_T(z)}{\sqrt{\lambda_1}}\right) w = 0, \tag{39}$$

$$2\nabla^2 \phi - LeRs \left(\frac{s(z) Pr - \lambda_2 Le^{-1} f_C(z)}{\sqrt{\lambda_2}}\right) w = 0, \tag{40}$$

denoting  $p$  as Lagrange’s multiplier and applying the *curl curl* on Eq. (38) and adopting the vertical component of resulting equation, to acquire

$$2 Pr \nabla^4 w + \nabla_1^2 R \left(\frac{Pr s(z) + \lambda_1 f_T(z)}{\sqrt{\lambda_1}}\right) \theta - \nabla_1^2 Rs \left(\frac{Pr s(z) - \lambda_2 Le^{-1} f_C(z)}{\sqrt{\lambda_2}}\right) \phi = 0, \tag{41}$$

Supposing a solution showing a plane tiling in the form  $(w, \theta, \phi) = (W, \Theta, \Phi)\psi(x, y)$  that satisfy  $(\nabla_1^2 + a^2)\psi = 0$ , where  $\psi$  denote plane tiling function. Thus, Eqs. (39–41) can be written as

$$2 \operatorname{Pr} \left( D^2 - a^2 \right)^2 W - a^2 R \left( \frac{s(z) \operatorname{Pr} + \lambda_1 f_T(z)}{\sqrt{\lambda_1}} \right) \Theta + a^2 R s \left( \frac{\operatorname{Pr} s(z) - \lambda_2 L e^{-1} f_C(z)}{\sqrt{\lambda_2}} \right) \Phi = 0, \tag{42}$$

$$2 \left( D^2 - a^2 \right) \Theta + R \left( \frac{s(z) \operatorname{Pr} + \lambda_1 f_T(z)}{\sqrt{\lambda_1}} \right) W = 0, \tag{43}$$

$$2 \left( D^2 - a^2 \right) \Phi - L e R s \left( \frac{s(z) \operatorname{Pr} - \lambda_2 L e^{-1} f_C(z)}{\sqrt{\lambda_2}} \right) W = 0, \tag{44}$$

with Eq. (16) represent an eigenvalue problem. To solve the above eigenvalue problem Eqs. (42–44) with boundary conditions (16), Chebyshev pseudospectral method is applied in MATLAB (discussed in Sect. 3). Now the critical Rayleigh number of non-linear theory is obtained on minimizing  $R^2$  for fixed wave number  $a$  and maximizing for  $(\lambda_1 > 0, \lambda_2 > 0)$  thus the critical Rayleigh number  $Ra_E$

$$Ra_E = \max_{\lambda_1 \lambda_2} \min_a R^2(\lambda_1, \lambda_2, a) \tag{45}$$

Now, in order to solve the eigenvalue problem (42–44) analytically, substitute Eq. (23) into the Eqs. (42–44) and integrating over  $[0, 1]$  to obtain a homogeneous system, which yields to following Rayleigh number

$$R^2 = \frac{8 \operatorname{Pr} \left( \frac{a^4}{30} + 24 \right) \left( \frac{a^2}{6} + 2 \right)^2 - 2a^2 L e \left( \frac{a^2}{6} + 2 \right) R c N_3 N_4}{2a^2 N_1 N_2 \left( 2 + \frac{a^2}{6} \right)} \tag{46}$$

where,  $N_1 = \int_0^1 L_1(z) z(1-z) dz$ ,  $N_2 = \int_0^1 L_1(z) z^2(1-z)^2 dz$ ,  $N_3 = \int_0^1 L_2(z) z(1-z) dz$ ,  $N_4 = \int_0^1 L_2(z) z^2(1-z)^2 dz$  and  $L_1(z) = \left( \frac{s(z) \operatorname{Pr} + \lambda_1 f_T(z)}{\sqrt{\lambda_1}} \right)$ ,  $L_2(z) = \left( \frac{s(z) \operatorname{Pr} - \lambda_2 L e^{-1} f_C(z)}{\sqrt{\lambda_2}} \right)$ .

Now, in the limiting case, when  $Rc = 0, Q = 0, \varepsilon = 0$ ,  $N_1$  tends to  $\frac{1}{6} \left( \frac{\operatorname{Pr} + \lambda_1}{\sqrt{\lambda_1}} \right)$  and  $N_2$  tends to  $\frac{1}{30} \left( \frac{\operatorname{Pr} + \lambda_1}{\sqrt{\lambda_1}} \right)$  thus, Eq. (46) reduces in the following form

$$R^2 = \frac{720 \operatorname{Pr} \lambda_1 \left( \frac{a^4}{30} + 24 \right) \left( \frac{a^2}{6} + 2 \right)}{a^2 (\operatorname{Pr} + \lambda_1)^2} \tag{47}$$

Now, to find the optimum value of  $R^2$  corresponding to  $\lambda_1$ , we solve  $\frac{dR^2}{d\lambda_1} = 0$  which gives  $\lambda_1 = \text{Pr}$  and thus Eq.(47) reduces to

$$R^2 = \frac{180\left(\frac{a^4}{30} + 24\right)\left(\frac{a^2}{6} + 2\right)}{a^2} \quad (48)$$

The Rayleigh number in Eq. (48), shows the results of nonlinear stability analysis which is equal to the Rayleigh number obtained from linear analysis [see, Eq. (26)]. Thus, it is also clear that for a constant gravity, in absence of solute and throughflow, linear analysis accurately encapsulates the physics of convection.

## 5 Results and discussion section

In the present paper, the effect of throughflow on the double diffusive convection in a fluid layer with variable gravity field for heated from below and salted from below system is discussed using linear and nonlinear stability analysis. The linear analysis is applied with the help of normal mode technique however; the energy method is conducted to apply nonlinear analysis. The eigenvalue problems obtained from the linear and nonlinear stability analyses are solved using Chebyshev pseudo-spectral method. The numerical technique is checked, after solving the present problem where solute and throughflow are absent [see Table 1]. The present results show a close concurrence with the previous results.

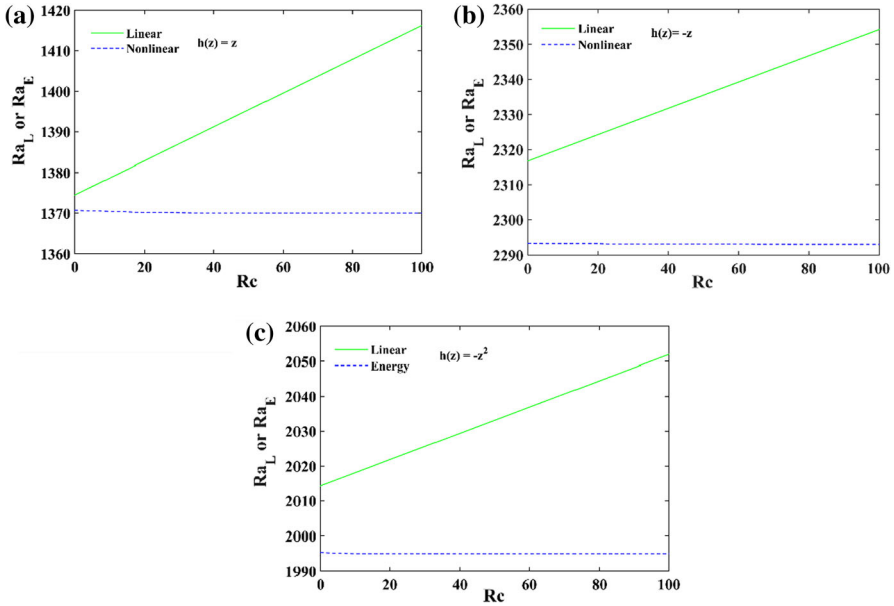
In the Fig. 1, the variation of critical Rayleigh numbers ( $Ra_L$ ,  $Ra_E$ ) against solute Rayleigh number  $Rc$  is shown for the different cases of gravity variation function  $h(z)$  when the fixed parameters are  $Le = 10$ ,  $\text{Pr} = 10$ ,  $Q = 0.5$ ,  $\varepsilon = 0.5$ . It is clear that an increase in value of the solute Rayleigh number  $Rc$  increases the critical Rayleigh number of linear theory ( $Ra_L$ ) however, the critical Rayleigh number of nonlinear theory ( $Ra_E$ ) remains approximately unchanged for all the assumed cases of gravity variation functions  $h(z)$ . Thus, an advancement in  $Rc$  values widens the region of subcritical instability.

Figure 2 express the variation of ( $Ra_L$ ,  $Ra_E$ ) with gravity variation parameter  $\varepsilon$ . It is deduced that the region of subcritical instability widens with the advancement in gravity variation parameter  $\varepsilon$  for the case of  $h(z) = -z$ ,  $-z^2$  however, there is unnoticeable changes in the region of subcritical instability on advancing in gravity variation parameter  $\varepsilon$  for  $h(z) = z$ . It is also very clear that for  $h(z) = z$ , increase in the values of  $\varepsilon$  destabilizes the system, since an increase in the  $\varepsilon$  value reveals an increase in the gravity field. On the other hand, on increasing the value of  $\varepsilon$  increases the ( $Ra_L$ ,  $Ra_E$ ) values for  $h(z) = -z$ ,  $-z^2$  and an increase in the critical Rayleigh number values are sufficiently large for  $h(z) = -z$  when compared with the other form of  $h(z)$ . Thus,  $h(z) = -z$  is the most stable profile of gravity among the other gravity profiles that we have considered.

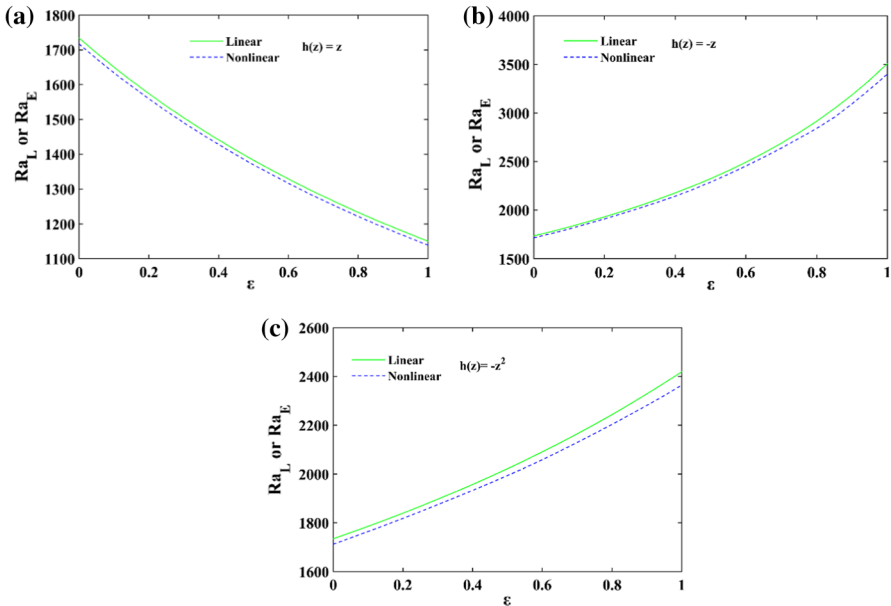
Figure 3 gives a visual representation of linear and nonlinear stability thresholds against  $Q$  for the different cases of  $h(z)$  and it is observed that direction of throughflow has considerable effect on the stability of the system. The positive value of  $Q$

**Table 1** Comparison of present work with the study of Straughan [20], [Table 1], Straughan [21], [Table 4.4, 4.5]. The fixed parameters are  $Le = 10$ ,  $Pr = 10$

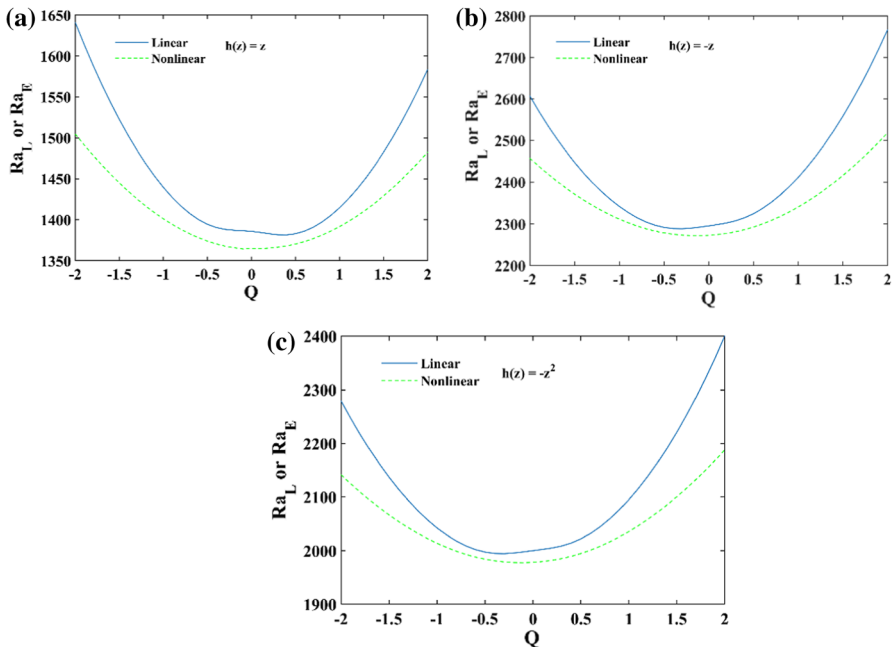
$h(z)$	$\varepsilon$	Existing work				Present study					
		$Ra_L$	$Ra_E$	$a_L^2$	$a_E^2$	$\lambda_1$	$Ra_L$	$Ra_E$	$a_L^2$	$a_E^2$	$\lambda_1$
$z$	0.2	1552.41	1552.26	9.712	9.712	1.101	1552.4151	1552.2890	9.7344	9.7131	1.10
	0.8	1219.07	1217.95	9.717	9.720	1.404	1219.0781	1217.9471	9.7344	9.7200	1.40
	1	1137.55	1136.12	9.719	9.723	1.507	1137.5510	1136.1146	9.7344	9.7223	1.50
$-z$	0.2	1897.338	1897.072	9.712	9.712	0.90	1897.3429	1897.0815	9.7344	9.7131	0.90
	0.8	2836.711	2822.691	9.740	9.756	0.60	2836.7128	2822.6906	9.7344	9.7556	0.61
	1	3389.902	3353.267	9.852	9.919	0.51	3389.5216	3353.4409	9.8596	9.9187	0.55
$-z^2$	0.2	1807.39	1807.15	9.713	9.713	0.856	1807.413	1807.3738	9.7344	9.7138	0.86
	0.8	2187.35	2180.86	9.775	9.808	0.444	2187.452	2180.7308	9.7969	9.8087	0.44
	1	2350.35	2337.92	10.84	10.90	0.212	2350.387	2337.1361	10.8241	10.9045	0.21



**Fig. 1** Critical Rayleigh numbers ( $Ra_L$ ,  $Ra_E$ ) are displayed against  $Rc$ . The fixed parameters are  $Le = 10$ ,  $Pr = 10$ ,  $Q = 0.5$ ,  $\varepsilon = 0.5$ , for **a**  $h(z) = z$ , **b**  $h(z) = -z$ , **c**  $h(z) = -z^2$

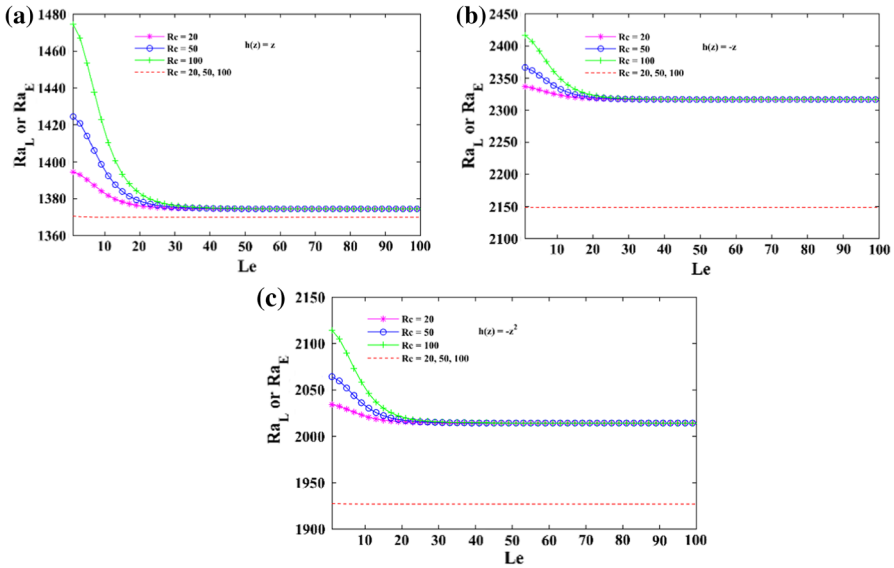


**Fig. 2** Critical Rayleigh numbers ( $Ra_L$ ,  $Ra_E$ ) are displayed against  $\varepsilon$  for the different cases of gravity variation function  $h(z)$ , for **a**  $h(z) = z$ , **b**  $h(z) = -z$ , **c**  $h(z) = -z^2$ . The fixed parameters are  $Le = 10$ ,  $Pr = 10$ ,  $Q = 0.5$ ,  $Rc = 20$



**Fig. 3** Critical Rayleigh numbers ( $Ra_L$ ,  $Ra_E$ ) are displayed against throughflow parameter  $Q$ . The fixed parameters are  $Le = 10$ ,  $Pr = 10$ ,  $Rc = 20$ ,  $\varepsilon = 0.5$ , for **a**  $h(z) = z$ , **b**  $h(z) = -z$ , **c**  $h(z) = -z^2$

corresponds to the throughflow in the upward direction while, the negative value of  $Q$  corresponds the throughflow in downward direction. And, for  $h(z) = z$ , the convection occurs fast with increasing in  $Q$  value in the range  $Q \in [-2, 0]$ , this is because of the presence of downward throughflow, the highest value of temperature difference stagnate in that part of medium where the vertical perturbation velocity takes its largest value and thus, the rate at which energy is supplied to the disturbances is increased, and this leads to the destabilization of the system however, advancing in  $Q$  values delay the occurrence of convection in the range  $Q \in [0.5, 2]$ , this is because the upward throughflow shifts the thermal gradients to the temperature boundary layer at the boundary towards which throughflow is directed and thus, large value of critical Rayleigh number is required to initiate the convection. On the other hand, for  $h(z) = -z$ ,  $-z^2$  the onset of convection hastens with the increase in  $Q$  in the range  $Q \in [-2, -0.5]$  however, increasing in  $Q$  values delaying the onset of convection in the range  $Q \in [0, 2]$ . From the Fig. 3, it is also noticed that for  $h(z) = z$ , the linear and nonlinear stability thresholds have less agreement for negative value of throughflow parameter  $Q$  in comparison of positive value of throughflow parameter  $Q$ . On the other hand, for the function  $h(z) = -z$ ,  $-z^2$ , the thresholds have less agreement for positive values of throughflow parameter  $Q$  in comparison of negative values of throughflow parameter  $Q$ . This pattern of results is approximately similar to the nature of throughflow in a porous medium [39]. As we have explained above that



**Fig. 4** Critical Rayleigh numbers ( $Ra_L$ ,  $Ra_E$ ) are displayed against Lewis number  $Le$ . The fixed parameters are  $Q = 0.5$ ,  $Pr = 10$ ,  $\varepsilon = 0.5$ , for **a**  $h(z) = z$ , **b**  $h(z) = -z$ , **c**  $h(z) = -z^2$

for increasing upward throughflow the convection is delayed and for increasing downward throughflow the onset of convection hastens but it is very interesting to notice that the effect of gravity field on the system is dominant to the effect of throughflow, which is visible in Fig. 3 i.e. on advancing the gravity field, leads to hastens the onset of convection for the small upward throughflow  $Q \in [0, 0.5]$ . However, for the case when gravity field decreases with height, leads to delay the convection in the range  $Q \in [-0.5, 0]$ (small downward throughflow). The reader may refer the work [40, 41], for more studies related to throughflow.

Figure 4 displays the variation of critical Rayleigh numbers against Lewis number  $Le$  for the different cases of gravity variation function  $h(z)$ . For each examined cases of  $h(z)$ , increasing the Lewis number rapid the onset of convection however, the energy stability threshold remains constant with increasing Lewis number. It is very interesting to note that increasing the  $Rc$  values delays the onset of convection for  $Le \leq 20$  but, for a value of Lewis number  $Le$  larger than a certain value, the linear instability thresholds coincide for the different values of  $Rc$ .

### 6 Conclusions

In the fluid layer, our proposal is to apply the linear and nonlinear stability analysis for the double diffusive convection problem with the effect of through flow and variable gravity field. The linear theory is performed using normal mode technique, however nonlinear stability analysis is performed using energy method. The eigenvalue

problems obtained from the linear and nonlinear theories are solved using Chebyshev pseudo-spectral method in MATLAB. A comparison is made between linear and nonlinear results. The following observations are found from the investigation of the present problem. The solute Rayleigh number  $Rc$  is found to have stabilizing impact on the system for  $Le \leq 20$ . The system is most stable for gravity variation function  $h(z) = -z$  while the most unstable for  $h(z) = z$ . For  $h(z) = z$ , the thresholds have good agreement in the neighborhood of  $Q = 0.5$ , however, for  $h(z) = -z, -z^2$ , the thresholds have good agreement in the neighborhood of  $Q = -0.5$ . The linear instability thresholds coincide for the different values of solute Rayleigh number  $Rc$  at Lewis number  $Le$  values greater than a fixed value however, the nonlinear stability thresholds remains constant with the increase in Lewis number.

## Declarations

**Conflict of interest** The authors have no conflict of interest.

## References

1. Turner, J.S.: Double-diffusive phenomema. *Annu. Rev. Fluid Mech.* **6**(14), 37–54 (1974)
2. Kaufman, J.: Numerical models of fluid flow in carbonate platforms: implications for dolomitization. *J. Sediment. Res. A Sediment. Petrol. Process.* **64A**(1), 128–139 (1994)
3. Oldenburg, C.M., Pruess, K.: Layered thermohaline convection in hypersaline geothermal systems. *Transp. Porous Media* **33**(1), 29–63 (1998)
4. Bear, J., Gilman, A.: Migration of salts in the unsaturated zone caused by heating. *Transp. Porous Media* **19**(2), 139–156 (1995)
5. Gilman, A.: The influence of free convection on soil salinization in arid regions. *Transp. Porous Media* **23**(3), 275–301 (1994)
6. Nield, D.A.: Onset of thermohaline convection in a porous medium. *Water Resour. Res.* **4**(3), 553–560 (1968)
7. Baines, P.G., Gill, A.E.: On thermohaline convection with linear gradients. *J. Fluid Mech.* **37**(2), 289–306 (1969)
8. Shir, C.C., Joseph, D.D.: Convective instability in a temperature and concentration field. *Arch. Ration. Mech. Anal.* **30**(1), 38–80 (1968)
9. Proctor, M.R.E.: Steady subcritical thermohaline convection. *J. Fluid Mech.* **105**, 507–521 (1981)
10. Hansen, U., Yuen, D.A.: Geophysical & astrophysical fluid dynamics subcritical double-diffusive convection at infinite prandtl number. *Geophys. Astrophys. fluid Dyn.* **47**, 199–224 (1989)
11. Joseph, D.D.: Global stability of the conduction-diffusion solution. *Arch. Ration. Mech. Anal.* **36**(4), 285–292 (1970)
12. Mulone, G.: On the nonlinear stability of a fluid layer of a mixture heated and salted from below. *Contin. Mech. Thermodyn.* **6**(3), 161–184 (1994)
13. Mulone, G., Rionero, S.: Unconditional nonlinear exponential stability in the Benard problem for a mixture: Necessary and sufficient conditions. *Circ. Mat. di palermo* **57**, 347–356 (1998)
14. Lombardo, S., Mulone, G., Rionero, S.: Global nonlinear exponential stability of the conduction-diffusion solution for schmidt numbers greater than prandtl numbers. *J. Math. Anal. Appl.* **262**(1), 191–207 (2001)
15. Mahajan, A., Tripathi, V.K.: Unconditional nonlinear stability for double-diffusive convection with temperature-and pressure-dependent viscosity. *Heat Transf.* **50**(2), 1523–1542 (2020)
16. Mahajan, A., Tripathi, V.K.: Effect of nonlinear temperature and concentration profiles on the stability of a layer of fluid with chemical reaction. *Can. J. Phys.* **99**(5), 367–377 (2020)
17. Pradhan, G.K., Samal, P.C.: Thermal stability of a fluid layer under variable body forces. *J. Math. Anal. Appl.* **122**(2), 487–495 (1987)



18. Alex, S.M., Patil, P.R., Venkatakrishnan, K.S.: Variable gravity effects on thermal instability in a porous medium with internal heat source and inclined temperature gradient. *Fluid Dyn. Res.* **29**(2), 1–6 (2001)
19. Alex, S.M., Patil, P.R.: Effect of a variable gravity field on convection in an anisotropic porous medium with internal heat source and inclined temperature gradient. *J. Heat Transfer* **124**(1), 144–150 (2002)
20. Straughan, B.: Convection in a variable gravity field. *J. Math. Anal. Appl.* **140**(2), 467–475 (1989)
21. Straughan, B.: *The Energy Method, Stability, and Nonlinear Convection*. Springer-Verlag, New York, Verlag New York (2004)
22. Rionero, S., Straughan, B.: Convection in a porous medium with variable internal heat source and variable gravity. *Int. J. Eng. Sci.* **28**(6), 497–503 (1990)
23. Harfash, A.J., Alshara, A.K.: Chemical reaction effect on double diffusive convection in porous media with magnetic and variable gravity effects. *Korean J. Chem. Eng.* **32**(6), 1046–1059 (2015)
24. Mahajan, A., Sharma, M.K.: The onset of convection in a magnetic nanofluid layer with variable gravity effects. *Appl. Math. Comput.* **339**, 622–635 (2018)
25. Yadav, D.: Numerical investigation of the combined impact of variable gravity field and throughflow on the onset of convective motion in a porous medium layer. *Int. Commun. Heat Mass Transf.* **108**, 104274 (2019)
26. Yadav, D.: The onset of Darcy-Brinkman convection in a porous medium layer with vertical throughflow and variable gravity field effects. *Heat Transf.* **49**(5), 3161–3173 (2020)
27. Mahajan, A., Tripathi, V.K.: Effects of spatially varying gravity, temperature and concentration fields on the stability of a chemically reacting fluid layer. *J. Eng. Math.* **125**(1), 23–45 (2020)
28. Kaloni, P.N., Qiao, Z.: Non-linear convection in a porous medium with inclined temperature gradient and variable gravity effects. *Int. J. Heat Mass Transf.* **44**(8), 1585–1591 (2001)
29. Herron, I.H.: Onset of convection in a porous medium with internal heat source and variable gravity. *Int. J. Eng. Sci.* **39**(2), 201–208 (2001)
30. Harfash, A.J.: Three-dimensional simulations for convection in a porous medium with internal heat source and variable gravity effects. *Transp. Porous Media* **101**(2), 281–297 (2014)
31. Nield, D.A.: Convective instability in porous media with throughflow. *AIChE J.* **33**(7), 1222–1224 (1987)
32. Harfash, A.J., Challob, H.A.: Slip boundary conditions and through flow effects on double-diffusive convection in internally heated heterogeneous Brinkman porous media. *Chinese J. Phys.* **56**(1), 10–22 (2018)
33. Shivakumara, I.S., Suma, S.P.: “Effects of throughflow and internal heat generation on the onset of convection in a fluid layer. *Acta Mech.* **140**, 207–217 (2000)
34. Capone, F., Gentile, M., Hill, A.A.: Double-diffusive penetrative convection simulated via internal heating in an anisotropic porous layer with throughflow. *Int. J. Heat Mass Transf.* **54**(7), 1622–1626 (2011)
35. Canuto, C., Hussaini, M.Y., Quarteroni, A.M., Zang, T.A.: *Spectral Methods in Fluid Dynamics*. Springer-Verlag, Berlin Heidelberg, Verlag Berlin Heidelberg (1988)
36. Rionero, S.: Heat and mass transfer by convection in multicomponent navier-stokes mixtures: absence of subcritical instabilities and global nonlinear stability via the auxiliary system method. *Rend. Lincei-Mat. e Appl.* **25**(4), 369–412 (2014)
37. Hill, A.A., Rionero, S., Straughan, B.: Global stability for penetrative convection with throughflow in a porous material. *IMA J. Appl. Math.* **72**, 635–643 (2007)
38. Mahajan, A., Tripathi, V.K.: Stability of a chemically reacting double-diffusive fluid layer in a porous medium. *Heat Transf.* **50**(6), 6148–6163 (2021)
39. Capone, F., Gentile, M., Hill, A.A.: Penetrative convection in a fluid layer with throughflow. *Ric. di Mat.* **57**(2), 251–260 (2008)
40. Nield, D.A.: Throughflow effects in the Rayleigh-B6nard convective instability problem. *J Fluid Mech.* **185**(2), 353–360 (1987)
41. Chen, F., Lu, J.W.: Variable viscosity effects on convective instability in superposed fluid and porous layers. *Phys. Fluids A* **4**(9), 1936–1944 (1992)