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Finite groups all of whose maximal subgroups of even order are PRN-groups

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Abstract

Let G be a finite group. A subgroup H of a group G is called pronormal in G if the subgroups H and H^g are conjugate in $\langle H, H^g \rangle$ for each $g \in G$. A group G is said to be a PRN-group if every minimal subgroup of G or order 4 is pronormal in G. In this paper, we characterize groups G such that G is a non-PRN-group of even order in which every maximal subgroup of even order is a PRN-group, and come to that such groups are solvable, have orders divisible by at most 3 distinct primes. And some additional structural details are provided.

Keywords Finite groups · Pronormal subgroups · Minimal subgroups · Maximal subgroups of even order

Mathematics Subject Classification 20D10 · 20D20

1 Introduction

All groups considered in this paper are finite. The ways in which minimal subgroups can be embedded in finite groups have been investigated by many scholars. For example, a famous result due to Itô [6, III, 5.3 Satz] proved that a group G of odd order is nilpotent if every subgroup of G of prime order lies in the center of G. An extension

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of Itô's result is the following statement [6, IV, 5.5 Satz]: Let p be a prime dividing the order of a group G. If every element of G of order p lies in $Z(G)$ and, when $p = 2$, every element of G of order 4 also lies in $Z(G)$, then G is p-nilpotent. A group is called a PN-group if its minimal subgroups are normal. In 1970, Buckley [3, Theorem 3] proved that a PN-group of odd order is supersolvable. In 1980, the structure of a non-PN-group whose proper subgroups are PN-groups was described by Sastry [9].

Recall that a subgroup H of a group G is called pronormal in G if for each $g \in G$, the subgroups H and H^g are conjugate in $\langle H, H^g \rangle$. This concept was introduced by P. Hall in the 1960's and arose naturally in the investigation of important subgroups of finite solvable groups like Sylow subgroups, Hall subgroups, system normalizers and Carter subgroups. Maximal subgroups, Hall subgroups of solvable normal subgroups and Sylow subgroups of normal subgroups are examples of pronormal subgroups. The pronormality is one of the most significant properties pertaining to subgroups of groups and many researchers have studied it extensively. For example, Peng [7] proved that a finite group G is a solvable T-group (solvable groups for which normality is a transitive relation) if and only if all prime power order subgroups of G are pronormal in G. In 1988, Asaad [1] called that a group G is an (A)-group if every subgroup of G of prime order is pronormal in G and either the Sylow 2-subgroups of G are abelian or every cyclic subgroup of G of order 4 is pronormal in G, and investigated finite groups all of whose maximal subgroups are (A)-groups. In 2011, Shen and Shi [11] defined a group G as a PRN-group if every cyclic subgroup of G of prime order or order 4 is pronormal in G. Then, they classified minimal non-PRN-groups (non-PRN-groups all of whose proper subgroups are PRN-groups).

In this paper, we investigate the finite groups all of whose maximal subgroups of even order are PRN-groups. More precisely, we prove the following theorem:

Main Theorem *Let G be a non-PRN-group of even order. Suppose that all maximal subgroups of even order are PRN-groups. Then G is solvable with $|\pi(G)| \leq 3$, and one of the following is true:*

(a) *G is a minimal non-PRN-group;*
(b) *$G = C_2 \times M$, where M is a minimal non-PRN-group of odd order;*
(c) *$G = (C_2 \times R) \ltimes Q$, where Q is a normal elementary abelian Sylow q-subgroup of G with $C_Q(P) = 1$, R is a cyclic Sylow r-subgroup of G and RQ is a minimal non-PRN-group.*

The notation and terminology used in this paper are standard, as in [4,8]. In addition, $\pi(G)$ denotes the set of primes dividing $|G|$; $A \rtimes B$ or $B \ltimes A$ denotes a semidirect product with a normal subgroup A and a subgroup B. The expression C_n denotes a cyclic group of order n. If G is a p-group, then $\Omega_n(G) = \langle x \in G \,|\, x^{p^n} = 1\rangle$.

2 Preliminaries

In this section we show some lemmas, which are required in the proofs of our main results.

Lemma 2.1 [4, Lemma 6.3] *Let H be a pronormal subgroup of a group G. Then the following statements are true:*

(1) *H is pronormal in K for every subgroup K of G with $H \leq K$.*
(2) *Let N be a normal subgroup of G. Then HN/N is pronormal in G/N.*
(3) *If H is subnormal in G, then H is normal in G.*

Lemma 2.2 [11, Lemma 2.4] *Let A and B be subgroups of a group G. Suppose that $G = AB$, H is a pronormal subgroup of B and H is normalized by A, then H is pronormal in G.*

Lemma 2.3 [2, Theorem 3.2] *Every PRN-group is supersolvable.*

Lemma 2.4 [11, Theorem 3.2] *If G is a minimal non-PRN-group, then G is solvable and $|\pi(G)| \leq 2$.*

Lemma 2.5 [11, Lemma 2.7] *Let G be a PRN-group. If X is a subgroup of G of order q, where q is the largest prime dividing the order of G, then X is normal in G.*

A group is called a minimal simple group if it is a nonabelian simple group and every proper subgroup of it is solvable. Thompson in 1968 classified these groups:

Lemma 2.6 [12, Corollary 1] *Every minimal simple group is isomorphic to one of the following groups:*

(i) $PSL(3, 3)$;
(ii) *The Suzuki group $S_z(2^f)$, where f is an odd prime;*
(iii) *$PSL(2, p)$, where p is a prime with $p > 3$ and $p^2 \not\equiv 1 (mod\ 5)$;*
(iv) *$PSL(2, 2^f)$, where f is a prime;*
(v) *$PSL(2, 3^f)$, where f is an odd prime.*

Lemma 2.7 [10, Lemma 2.10] *Suppose that all cyclic subgroups of a group G of order p are normal in G for a fixed prime p. If $|Z(G)|_p \neq 1$, then all elements of order p of G are in $Z(G)$.*

Lemma 2.8 [5, Theorem 10.1.4] *If a group G has a fixed-point-free automorphism of order 2, then G is abelian.*

3 The proof of main theorem

The proof of the main theorem will be finished by showing the following theorems. We first prove the following result for a group to be solvable.

Theorem 3.1 *Let G be a group of even order. Suppose that all maximal subgroups of G of even order are PRN-groups. Then G is solvable.*

Proof Suppose that G is not solvable. Then G cannot be a minimal non-PRN-group by Lemma 2.4. Thus there exists a maximal subgroup M of odd order in G such that M is a non-PRN-group. Hence $2 \notin \pi(\Phi(G))$, where $\Phi(G)$ is the Frattini subgroup of G. By the Feit-Thompson Theorem on the solvability of a group of odd order, we can see that M is solvable. Then Lemma 2.3 and our hypothesis are combined to give that all proper subgroups of G are solvable and hence $G/\Phi(G)$ is a minimal simple group. We split the proof into the following steps:

(1) $\Phi(G) = Z(G)$
Let M be a maximal subgroup of G containing a Sylow 2-subgroup of G. It is clear from our hypothesis that M is a PRN-group. Let $P \in Syl_p(\Phi(G))$, where $p \in \pi(\Phi(G))$. Then every subgroup H of P of order p is pronormal in M^g for all $g \in G$. Lemma 2.1 and $P \trianglelefteq G$ combine to give that H is normal in G as $\langle M, M^g \big| g \in G\rangle = G$. Let $P_1 \in Syl_p(G)$. Then $H \leq Z(P_1)$ and so $P_1 \leq C_G(H) \trianglelefteq G$. By the simplicity of $G/\Phi(G)$, we conclude that H lies in the center $Z(G)$. Put $K = PT$, where $T \in Syl_2(G)$. By Itô's lemma [5, IV, 5.5 Staz], we can see that K is p-nilpotent, which implies $T \leq C_G(P) \trianglelefteq G$. Again by the simplicity of $G/\Phi(G)$, we get that $P \leq Z(G)$ and so $\Phi(G) \leq Z(G)$. On the other hand, $Z(G) \leq \Phi(G)$. Hence $\Phi(G) = Z(G)$.

(2) $\Phi(G) = 1$, i.e., G is a minimal simple group.
Applying Statement (1), we have that $G/Z(G)$ is a minimal simple group and so a quasisimple group with the center of odd order. By the table on the Schur multipliers of the known simple groups (see [5, p. 302]), we get that the Schur multiplier of each of the minimal simple groups is a 2-group. Hence $\Phi(G) = Z(G) = 1$.

(3) A contradiction.
Now, we assert that there is no simple group listed in Lemma 2.6 isomorphic to G. Since each of $PSL(2, p)$, $PSL(2, 3^f)$ and $PSL(3, 3)$ contains a subgroup isomorphic to A_4, the alternating group of degree 4. By our hypothesis A_4 is a PRN-group. It follows from Lemma 2.3 that A_4 is supersolvable, it is impossible. Thus G can not be any one of $PSL(2, p)$, $PSL(2, 3^f)$ and $PSL(3, 3)$. If G is isomorphic to $PSL(2, 2^f)$ or $S_z(2^f)$, then G is a Zassenhaus group of odd degree and the stabilizer of a point is a Frobenius group with kernel a 2-group. This means that G has a non-supersolvable subgroup of even order, our hypothesis and Lemma 2.3 provide a contradiction. Therefore G can not be any one of $PSL(2, 2^f)$ or $S_z(2^f)$ as well. □

We are now ready to prove the upper bound of $|\pi(G)|$.

Theorem 3.2 *Let G be a non-PRN-group of even order. If all maximal subgroups of G of even order are PRN-groups, then $|\pi(G)| \leq 3$.*

Proof Since G is solvable by Theorem 3.1, we can see that G has a Sylow system $\{P_1, P_2, \ldots, P_r\}$ with $2 = p_1 < p_2 < \cdots < p_r$, where p_i is the prime dividing $|P_i|$. Suppose that $r \geq 4$. By our hypothesis and Lemma 2.4, there exists a maximal subgroup M of G such that M is not a PRN-group. Furthermore, M is a Hall subgroup of G of odd order. Without loss of generality, we can assume that $M = P_2P_3 \ldots P_r$. Since M is not a PRN-group, there is a minimal subgroup A of M such that A is not a pronomal subgroup of M. If $A \leq P_r$, then $P_1P_iP_r$ is a proper subgroup of G of even order from our assumption for each $i \in \{2, 3, \ldots, r-1\}$. Thus, $P_1P_iP_r$ is a PRN-group. In view of Lemma 2.5, we get that A is normal in $P_1 P_i P_r$ and so is normal in M, a contradiction. Hence we may assume that $A \leq P_k$ for some fixed $k \in \{1, 2, \ldots, r-1\}$.

We claim that $k = 2$. Suppose that $k > 2$. Set

$$U = \prod_{i=2}^{k} P_i, \quad V = \prod_{j=k}^{r} P_j.$$

We have $M = UV$ with $A \leq V < M$. Obviously, both $P_1 U$ and $P_1 V$ are proper subgroups of G of even order and hence are PRN-groups. It follows from Lemmas 2.1 and 2.5 that A is pronormal in V, and is normal in U. Lemma 2.2 yields that A is pronormal in M, this contradiction shows $k = 2$.

It is clear that $P_1 P_2 P_i$ is a PRN-group by hypothesis for each $i \in \{3, .., r\}$. From Lemma 2.3, $P_1 P_2 P_i$ is supersolvable and so P_i is normal in $P_1 P_2 P_i$, which implies that A normalizes P_i. If $A < P_2$, then $B = A \prod_{i=3}^{r} P_i$ is a proper subgroup of M and $M = P_2 B$. On the one hand, A is a Sylow subgroup of B and so A is pronormal in B. On the other hand, since $P_1 P_2$ is a PRN-group, P_2 normalizes A by Lemma 2.5. Thus $M = N_M(A)B$ and we get that A is pronormal in M by Lemma 2.2, a contradiction. We conclude that $A = P_2$, which implies that A is pronormal in G. This final contradiction completes the proof of the theorem. □

It can be verified that either G is a minimal non-PRN-group or G has a normal 2-complement, as stated in the following result.

Theorem 3.3 *Let G be a non-PRN-group of even order. Suppose that all maximal subgroups of G of even order are PRN-groups. Then one of the following holds:*

(1) *G is a minimal non-PRN-group;*
(2) *$G = C_2 H O_{2'}(G)$, where $N_G(C_2) = C_2 \times H$ and $H O_{2'}(G)$ is a 2-complement.*

Proof Suppose that G is not a minimal non-PRN-group. Then there exists a maximal subgroup M of G such that M is not a PRN-group. By hypothesis, M is a non-abelian group of odd order. Furthermore, as G is solvable by Theorem 3.1, M is a Hall $2'$-subgroup of G. Now, let P be a Sylow 2-subgroup of G. It clear that $G = PM$.

Put $\bar{G} = G/O_{2'}(G)$. The solvability of G yields that $O_2(\bar{G}) \neq 1$. We get that $O_2(\bar{G}) = \bar{P}$ by comparing the orders. Thus,

$$\bar{G} = N_{\bar{G}}(\bar{P}) = N_G(P) O_{2'}(G) \big/ O_{2'}(G).$$

It follows that $G = N_G(P) O_{2'}(G)$. Let A be a subgroup of P of order 2. If $G = N_G(P)$, then PC is a proper subgroup of G for every cyclic subgroup C of M. By hypothesis, PC is a PRN-group. Then A is pronormal in PC and so is normal in PC by Lemma 2.1 (3). This implies that M normalizes A. Therefore $G = AM$ since M is a maximal subgroup of G. So $|P| = |A| = 2$ and hence M is the normal 2-complement, as desired. Hence we may assume that $N_G(P) < G$. Then from our hypothesis A is pronormal in $N_G(P)$ and so is normal in $N_G(P)$ by Lemma 2.1 (3) again. Let H be a Hall $2'$-subgroup of $N_G(P)$. Then AH is a subgroup of G. Lemma

2.3 and $G = P(HO_{2'}(G))$ combine to give $M = HO_{2'}(G)$. Moreover, we can see that $AHO_{2'}(G) = A(HO_{2'}(G)) = AM > M$, which implies that $G = AM$. It follows that $P = A$. This completes our proof. $\square$

If $\pi(G) = \{2, q\}$, then we have the following result.

Theorem 3.4 *Let $G = PQ$ be a non-PRN-group, where P is a subgroup of order 2 and Q is a Sylow q-subgroup of G. Suppose that all maximal subgroups of G of even order are PRN-groups. Then one of the following holds:*

(1) *$G = P \times Q$ and Q is a minimal non-PRN-group;*
(2) *Q is a normal abelian subgroup of G.*

Proof If G is nilpotent, then PX is a PRN-group for every maximal subgroup X of Q. Our hypothesis combined with Lemma 2.2 yields that Q is a minimal non-PRN-group. Now we assume that G is not nilpotent.

We first claim that G is supersolvable. By Theorem 3.3, we have that Q is normal in G. We only need to show that every chief factor of G below Q is cyclic in order to conclude that G is supersolvable. Let K/L be a chief factor of G such that $K \leq Q$. Then K/L is an elementary abelian q-group. By [6, VI, 5.4 Satz], we can see that

$$F(G) \leq C_G(K/L).$$

Then the automorphism group induced by G on K/L is of order 2. In view of [13, Chapter 1, Lemma 1.3], we get that K/L is cyclic of order p, as claimed.

It is clear that P acts on Q and $Q/\Phi(Q)$. By the completed reducible theorem, $Q/\Phi(Q) = V_1/\Phi(Q) \times V_2/\Phi(Q) \times \cdots \times V_d/\Phi(Q)$, every $V_i/\Phi(Q)$ is P-invariant and is of order q for any $i \in \{1, 2, \ldots, d\}$. Put

$$Q_i = \prod_{j \neq i} V_j.$$

Then Q_i is a maximal subgroup of Q and P-invariant. By hypothesis, PQ_i is a PRN-group and every subgroup of Q_i of order q is normal in PQ_i by Lemma 2.1.

Suppose that $C_{Q_k}(P) \neq 1$ for some k. Let x be an element of $C_{Q_k}(P)$ of order q, then $x \in Z(PQ_k)$. It follows from Lemma 2.7 all elements of order q of Q_k are in $Z(PQ_k)$. Applying the Itô's lemma [6, IV, 5.5 Staz], we obtain that $PQ_k = P \times Q_k$ since Q_k is P-invariant. In particular, $[P, \Phi(Q)] \leq [P, Q_k] = 1$ and so $\Phi(Q) \leq C_{Q_j}(P)$ for any $j \in \{1, 2, \ldots, d\}$. Again by Lemma 2.7, we conclude that $[P, Q] = 1$ and so G is nilpotent, contrary to our assumption. Hence $C_{Q_i}(P) = 1$ for all i.

If $C_Q(P) \neq 1$, then $C_Q(P)$ has order q as $C_Q(P) \cap Q_i = 1$. Set $V = C_Q(P)\Phi(Q)$. Then V is P-invariant and $V < Q$, which implies that PV is a PRN-group by our hypothesis. By a similar argument, we can see that $[P, \Phi(Q)] = 1$. However, $C_{Q_i}(P) = 1$, a contradiction. Hence $C_Q(P) = 1$. Lemma 2.8 yields that Q is an abelian group. $\square$

Now we suppose that $\pi(G) = \{2, r, q\}$, then we have the following theorem.

Theorem 3.5 *Let $G = PM$ be a non-PRN-group of even order and $M = RQ$, where P is a subgroup of order 2 and R and Q are a Sylow r-subgroup and Sylow q-subgroup of G, respectively. Suppose that all maximal subgroups of G of even order are PRN-groups. Then one of the following holds:*

(1) *G is a minimal non-PRN-group;*
(2) *$G = P \times M$ and M is a minimal non-PRN-group;*
(3) *$G = (P \times R) \ltimes Q$ and M is a minimal non-PRN-group, where R is cyclic and Q is normal elementary abelian with $C_Q(P) = 1$.*

Proof Since P is a Sylow 2-subgroup of G of order 2, we conclude that G is 2-nilpotent and M is the normal 2-complement. Also, we may choose $\{P, R, Q\}$ as a Sylow system from Theorem 3.1.

Suppose that $C_M(P) = 1$. Then P as an automorphism of M of order 2 is fixed-point-free. Lemma 2.8 means that M is abelian and so is a PRN-group. Therefore G is a minimal non-PRN-group.

Hence we may assume that $C_M(P) \neq 1$. Since PR is a PRN-group, we get that every subgroup of R of order r is pronormal in PR and so is normal in PR by Lemma 2.1. If $|C_M(P)|$ has divisor r, then from Lemma 2.7 every element of R of order r lies in $Z(PR)$. The Itô's lemma [6, IV, 5.5 Staz] implies that $PR = P \times R$ and so $C_R(P) = R$. By the same reason, if $q \big| |C_M(P)|$, then $PQ = P \times Q$. Thus, if rq divides $|C_M(P)|$, then we can conclude that $G = P \times M$. Therefore PH is a PRN-group for each proper subgroup H of M. It follows that M is a minimal non-PRN-group, which yields (2).

Now we may assume without loss of generality that $C_R(P) = R$ and $C_Q(P) = 1$. Then

$$N_G(P) = C_G(P) = PR$$

since P is of order 2. Employing the Frattini argument, $G = N_G(P)P^G = PRP^G$ and so $Q \leq P^G$. On the other hand, $P^G = \langle P^g \big| g \in G\rangle = \langle P^x \big| x \in Q\rangle \leq PQ$. This implies that $P^G = PQ$. It follows that Q is normal in G.

Suppose that R is not cyclic. Then R possesses at least two distinct maximal subgroups R_1 and R_2. By hypothesis, both $K_1 = PR_1Q$ and $K_2 = PR_2Q$ are PRN-groups. It follows from Lemma 2.1 that every subgroup of Q of order q is normal in K_1 and K_2 and hence normal in G. Next, for each subgroup A of R of order r, we have that PAQ is a PRN-group by our hypothesis. Since $G = (PAQ)R$ and $A \trianglelefteq R$, we can see that A is pronormal in G by Lemma 2.2. Finally, P is also pronomal in G. Consequently, G would be a PRN-group, which is a contradiction. Hence R is cyclic.

By Lemma 2.8 again, Q is abelian. If $\Omega_1(Q) < Q$, then $PR\Omega_1(Q)$ is a PRN-group by hypothesis. Thus all minimal subgroups of Q are normal in $PR\Omega_1(Q)$ and also normal in G. Since M is not a PRN-group, there exists a subgroup A of R of order r such that A is not pronormal in M and so $|R| > r$. Set $H = PAQ$. By hypotehsis, A is pronormal in H. By Lemma 2.2, A is pronormal in $(PAQ)R = G$, this contradiction forces that Q is elementary abelian.

We claim that M is a minimal non-PRN-group. In fact, let L be a maximal subgroup of M. If $|M : L|$ is a power of r, then $Q \leq L$ and so $L \trianglelefteq G$. Suppose that $|M : L|$ is a power of q. Applying Lemma 2.1 (3), we conclude that P normalizes every subgroup of Q of order q since PQ is a PRN-group and Q is elementary abelian, which means that P normalizes L. In both cases, we have that PL is a PRN-group by hypothesis. Consequently, L is a PRN-group, which completes the proof of the theorem. □

The following corollary follows directly from the above results.

Corollary 3.6 *Let G be a non-PRN-group of even order. Suppose that all maximal subgroups of G of even order are PRN-groups. If $|\pi(G)| \geq 4$, then G is a minimal non-PRN-group.*

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