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Ricci solitons on Riemannian manifolds admitting certain vector field

Devaraja Mallesha Naik[1]

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Abstract
In this paper, we initiate the study of impact of the existence of a unit vector ν, called a concurrent-recurrent vector field, on the geometry of a Riemannian manifold. Some examples of these vector fields are provided on Riemannian manifolds, and basic geometric properties of these vector fields are derived. Next, we characterize Ricci solitons on 3-dimensional Riemannian manifolds and gradient Ricci almost solitons on a Riemannian manifold (of dimension n) admitting a concurrent-recurrent vector field. In particular, it is proved that the Riemannian 3-manifold equipped with a concurrent-recurrent vector field is of constant negative curvature $-\alpha^2$ when its metric is a Ricci soliton. Further, it has been shown that a Riemannian manifold admitting a concurrent-recurrent vector field, whose metric is a gradient Ricci almost soliton, is Einstein.

Keywords Conformal vector field · Ricci soliton · Ricci almost soliton · Gradient Ricci almost soliton

Mathematics Subject Classification 53C25 · 53C44 · 53C21

1 Introduction

Ricci solitons have received a lot of attention by many geometers, mainly due to the intense works of Hamilton (and also Perelman). In the recent years, Ricci solitons are of much interest in the field of differential geometry and geometric analysis as it naturally extends Einstein metric (that is, the Ricci tensor Ric is a constant multiple of the metric tensor g). Thus, it becomes an important issue to investigate Ricci solitons and to classify them geometrically.

✉ Devaraja Mallesha Naik
devaraja.mallesha@christuniversity.in; devarajamaths@gmail.com

1 Department of Mathematics, CHRIST (Deemed to be University), Bengaluru, Karnataka, India

A Riemannian metric g is said to be a *Ricci soliton* if there exist a smooth vector field X (called *soliton vector field*) and a scalar $\lambda \in \mathbb{R}$ (called *soliton constant*) satisfying

$$\pounds_X g + 2Ric + 2\lambda g = 0, \tag{1}$$

where £ denotes the Lie-derivative. Thus, we may regard Ricci soliton as the generalization of the Einstein metric. The metric g satisfying (1) is called *Ricci almost soliton* when λ is a smooth function (i.e., $\lambda \in C^\infty(M)$). We say that the Ricci soliton is *steady*, *expanding* or *shrinking* depending on the value of soliton constant as $\lambda = 0$, $\lambda > 0$ or $\lambda < 0$. Given a Riemannian manifold (M, g), the Hamilton's Ricci flow (see [22]) is given by $\frac{\partial}{\partial t} g(t) = -2S(t)$ with the initial data $g(0) = g$. The study of Ricci solitons is interesting due to the fact that they are self-similar solutions to the Hamilton's Ricci flow. If $X = \text{grad}\, \gamma$, where grad is the *gradient operator* and γ is a differentiable function, then the metric g is called a *gradient Ricci soliton* and so (1) takes the form

$$Hess^\gamma + Ric + \lambda g = 0, \tag{2}$$

where $Hess^\gamma$ is the Hessian of a smooth function γ which is a symmetric bilinear form defined by

$$Hess^\gamma(Y, Z) = g(\nabla_Y \text{grad}\, \gamma, Z).$$

The metric g satisfying (2) is called *gradient Ricci almost soliton* when $\lambda \in C^\infty(M)$. A Ricci soliton (resp. gradient Ricci soliton) is said to be trivial when the soliton vector field X is Killing (resp. γ is constant) or equivalently the metric is Einstein.

Recently there are many interesting results concerning the classification of Ricci solitons on Riemannian manifolds with certain geometric conditions. In particular, Barros and Ribeiro [1] proved that if the metric of a compact Riemannian manifold is a Ricci almost soliton in which the soliton vector field is a nontrivial conformal vector field then the manifold is isometric to Euclidean sphere. In [4], Chen and Deshmukh classified Ricci solitons for which the potential field is a concurrent field. Afterwards, Diógenes et al. [16] investigated gradient Ricci solitons on a complete Riemannian manifold admitting a nonparallel closed conformal vector field. Further, Sharma [28] undertook the study of gradient Ricci solitons on a Riemannian manifold having constant scalar curvature and admitting a non-homothetic conformal vector field which leaves the soliton field invariant. More recently, Silva Filho in [18] (resp. in [19]) investigated Ricci solitons on a Riemannian manifold admitting a nonparallel closed homothetic vector filed (resp. nonparallel closed conformal vector field). For the studies of Ricci solitons on other classes of Riemannian manifolds, we refer the reader to [7–9,21,23–25,29–31].

Before to proceed, we shall recall some basic well-known vector fields on a Riemannian manifold (M, g). A smooth vector field ν in M is said to be *conformal* if there is a smooth function ψ (called *conformal coefficient*) on M such that

$$\pounds_\nu g = 2\psi g. \tag{3}$$

As a particular case, ν is called *homothetic* (resp. *Killing*) when the conformal coefficient ψ is constant (resp. $\psi = 0$). If ν is *closed* (i.e., it dual 1-form $\nu^\flat$ is closed), then (3) takes the form

$$\nabla_Y \nu = \psi Y, \tag{4}$$

for all vector field Y, where ∇ indicates the Levi–Civita connection on M. If ψ is constant (resp. $\psi = 0$) satisfying the aforementioned equation, then ν is called *closed homothetic vector field* (resp. *parallel*). Particularly, ν is called concurrent when $\psi = 1$ in Eq. (4). For more information regarding this, we recommend [3,6,11,12]. On the other hand, ν is called recurrent vector field if

$$\nabla \nu = \nu^\flat \otimes \nu, \tag{5}$$

where $\nu^\flat$ is the 1-form dual to ν. Details and results for Riemannian manifold carrying recurrent vector field can be found in [5,20]. At this time, one may tempt to consider a vector field satisfying the following equation

$$\nabla_Y \nu = \alpha Y + \beta \nu^\flat(Y)\nu, \quad \alpha, \beta \in \mathbb{R}, \tag{6}$$

which generalizes both closed homothetic vector field (and so concurrent vector field), and recurrent vector field. In such a case, we see that $\pounds_\nu g = 2\alpha\, id + 2\beta \nu^\flat \otimes \nu^\flat$. If ν is a unit vector field, then one may see that $(\pounds_\nu g)(\nu, \nu) = 0 = 2(\alpha + \beta)$, and (6) turns into

$$\nabla_Y \nu = \alpha\{Y - \nu^\flat(Y)\nu\}, \tag{7}$$

for any $Y \in \mathfrak{X}(M)$ and a constant $\alpha \in \mathbb{R}$. In this article, a unit non-parallel vector field ν satisfying the above equation is called a *concurrent-recurrent vector field*. As we shall see later (see Theorem 3), any warped product $I \times_{f(t)} F$ of an open interval $I \subseteq \mathbb{R}$ and a Riemannian manifold F with the warping function $f(t) = e^{\alpha t}$ admits concurrent-recurrent vector field (i.e., a vector field satisfying (7)).

Presence of special vector fields on a Riemannian manifold form a significant portion of the differential geometry of Riemannian manifolds. These vector fields are roughly divided in two classes, one class containing those vector fields whose integral curves are geodesics such as Killing vector fields of constant length, geodesic vector fields (cf. [13]) and other class containing those vector fields whose integral curves are conformal geodesics such as conformal vector fields, generalized geodesic vector fields (cf. [2,10,14,15]). In this sense concurrent-recurrent vector field introduced in this paper belongs to the class of Killing vector fields and therefore it has scope of future developments. It is interesting to remark here that the Riemannian manifold admitting concurrent-recurrent vector field is *not compact* because we have $div\, \nu = (n-1)\alpha$ and by Stokes's theorem we get $\alpha = 0$ for $n > 1$ a contradiction.

Motivated by the study of Ricci solitons and gradient Ricci solitons as mentioned previously, in this paper, we shall examine the geometry of Ricci (almost) solitons on a Riemannian manifold carrying a concurrent-recurrent vector field. Our first result in this direction is:

Theorem 1 *If the metric of a Riemannian manifold admitting a concurrent-recurrent vector field is a Ricci soliton, then the soliton is expanding with soliton constant* $\lambda = (n-1)\alpha^2$.

In the sequel, we characterize the Riemannian 3-manifolds admitting a concurrent-recurrent vector field when its metric is a Ricci soliton.

Corollary 1 *If the metric of a Riemannian 3-manifold admitting a concurrent-recurrent vector field is a Ricci soliton, then the manifold is of constant negative curvature* $-\alpha^2$.

On the other hand, we also characterize Riemannian manifolds admitting a concurrent-recurrent vector field whose metric is a gradient Ricci almost soliton.

Theorem 2 *If the metric of a Riemannian manifold admitting a concurrent-recurrent vector field is a gradient Ricci almost soliton, then it is Einstein.*

As a consequence, we prove:

Corollary 2 *If the metric of a Riemannian 3-manifold* M *endowed with a concurrent-recurrent vector field is a gradient Ricci almost soliton, then* M *is of constant negative curvature* $-\alpha^2$.

Corollary 3 *Let* $M = I \times_f F$ *with the warping function* $f(t) = e^{\alpha t}$, *where* $\alpha \in \mathbb{R}$, I *is an open interval in* $\mathbb{R}$ *and* F *is a Riemannian* 2*-manifold. If the metric of* M *is a Ricci soliton (or gradient Ricci almost soliton), then the manifold is of constant negative curvature* $-\alpha^2$.

2 Background and key results

Let ν be a concurrent-recurrent vector field. Thus, from (7) we find $\nabla_\nu \nu = 0$. Thus, we may see that the integral curves of ν are geodesics in M and so the distribution $D = span\{\nu\}$ is a totally geodesic foliation, i.e., D is an integrable distribution whose leaves are totally geodesic in M.

Now, we extend $\nu = e_1$ to an orthonormal frame $e_1, e_2, \dots, e_n$ on M. Let us define the connection forms ω_i^j, $(1 \leq i, j \leq n)$ as given below

$$\nabla_Y e_i = \sum_{j=1}^{n} \omega_i^j(Y) e_j, \quad 1 \leq i \leq n.$$

The above equation together with (7) shows that

$$\omega_1^j(e_k) = \alpha \delta_{kj}, \quad 2 \leq k, j \leq n. \tag{8}$$

It follows from (8) that the distribution $D^\perp = span\{e_2, \dots, e_n\}$ is integrable whose leaves are totally umbilical hypersurfaces of M with constant mean curvature, i.e., $D^\perp$

is a spherical foliation. Using Corollary 1 of [27], we see that M is isometric to the warped product $I \times_f F$, where F is a Riemannian $(n-1)$-manifold and $\nu = \frac{\partial}{\partial t}$, $t \in I$. Conversely, suppose that $M = I \times_f F$ with the metric given by

$$g = dt^2 + f(t)^2 g_F,$$

where g_F denotes the metric of F. Let $f(t) = e^{\alpha t}$ and $\nu = \frac{\partial}{\partial t}$. By Proposition 7.35 of [26], one can easily verify that (7) holds and so ν is a concurrent-recurrent vector field. The above discussion can be summarized in the following way:

Theorem 3 *A Riemannian n-manifold admitting a concurrent-recurrent vector field is locally isometric to the warped product $I \times_f F$, where $I \subseteq \mathbb{R}$ is an open interval and F is a Riemannian $(n-1)$-manifold. Conversely, the warped product $I \times_f F$ with the warping function $f(t) = e^{\alpha t}$ admits a concurrent-recurrent vector field.*

2.1 Some examples

Now, we present few examples of Riemannian manifolds admitting a concurrent-recurrent vector field. Also, we provide Ricci soliton on a Riemannian manifold admitting a concurrent-recurrent vector field.

Example 1 Consider the manifold $M = \mathbb{R}^{n-1} \times \mathbb{R}_+$ with coordinates (x^i, z) where $i = 1, \ldots, n-1$. Let us define the Riemannian metric g on M by

$$g = \frac{1}{(\alpha z)^2} \sum_{i=1}^{n-1} (dx^i)^2 + \frac{1}{(\alpha z)^2} dz^2, \tag{9}$$

where $\alpha = const. \neq 0$. Now using Koszul's formula (or Christoffel symbols), the non-zero components of Levi–Civita connection is given by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial z} = -\frac{1}{z} \frac{\partial}{\partial x^i},$$
$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^i} = \frac{1}{z} \frac{\partial}{\partial z},$$
$$\nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial z} = -\frac{1}{z} \frac{\partial}{\partial z}.$$

Let $\nu = -\alpha z \frac{\partial}{\partial z}$. Then from above we can easily verify

$$\nabla_Y \nu = \alpha\{Y - \nu^\flat(Y)\nu\},$$

for any $Y \in \mathfrak{X}(M)$. Thus, the vector field $\nu = -\alpha z \frac{\partial}{\partial z}$ is a concurrent-recurrent vector field.

Example 2 Let g be the Riemannian metric on $M = \mathbb{R}^2 \times R_+ \subset \mathbb{R}^3$ given by

$$(g_{ij}) = \begin{pmatrix} e^{2\alpha z} + x^2 & 0 & x \\ 0 & e^{2\alpha z} & 0 \\ x & 0 & 1 \end{pmatrix},$$

where $\alpha = const. \neq 0$. Using Koszul's formula, the components of Levi–Civita connection is given by

$$\begin{aligned}
&\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = \alpha x \frac{\partial}{\partial x} - (\alpha x^2 + \alpha e^{2\alpha z} + 1)\frac{\partial}{\partial z}, \quad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = 0, \quad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial z} = \alpha \frac{\partial}{\partial x} - \alpha x \frac{\partial}{\partial z},\\
&\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} = 0, \quad \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = \alpha x \frac{\partial}{\partial x} - \alpha(x^2 + e^{2\alpha z})\frac{\partial}{\partial z}, \quad \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial z} = \alpha \frac{\partial}{\partial y},\\
&\nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial x} = \alpha \frac{\partial}{\partial x} - \alpha x \frac{\partial}{\partial z}, \quad \nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial y} = \alpha \frac{\partial}{\partial y}, \quad \nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial z} = 0.
\end{aligned}$$

Using these we can verify that

$$\nabla_{\partial_i} \frac{\partial}{\partial z} = \alpha \left\{ \partial_i - \nu^\flat(\partial_i) \frac{\partial}{\partial z} \right\},$$

for all $1 \leq i \leq 3$, where $\partial_1 = \frac{\partial}{\partial x}$, $\partial_2 = \frac{\partial}{\partial y}$ and $\partial_3 = \frac{\partial}{\partial z}$. Thus, the vector field $\nu = \frac{\partial}{\partial z}$ is a concurrent-recurrent vector field.

Example 3 Let $M = \mathbb{R}^2 \times R_+ \subset \mathbb{R}^3$ and we denote the Cartesian coordinates by (x, y, z). Let g be the Riemannian metric given by

$$g = e^{2az} dx^2 + e^{2az} dy^2 + dz^2,$$

where $\alpha = const. \neq 0$. Then, we have

$$\begin{aligned}
&\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = -\alpha e^{2az} \frac{\partial}{\partial z}, \quad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = 0, \quad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial z} = \alpha \frac{\partial}{\partial x},\\
&\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} = 0, \quad \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = -\alpha e^{2az} \frac{\partial}{\partial z}, \quad \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial z} = \alpha \frac{\partial}{\partial y},\\
&\nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial x} = \alpha \frac{\partial}{\partial x}, \quad \nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial y} = \alpha \frac{\partial}{\partial y}, \quad \nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial z} = 0.
\end{aligned} \tag{10}$$

From (10), one easily verifies:

$$\nabla_{\partial_i} \frac{\partial}{\partial z} = \alpha \left\{ \partial_i - \nu^\flat(\partial_i) \frac{\partial}{\partial z} \right\}, \tag{11}$$

for all $1 \le i \le 3$, where $\partial_1 = \frac{\partial}{\partial x}$, $\partial_2 = \frac{\partial}{\partial y}$ and $\partial_3 = \frac{\partial}{\partial z}$. Thus, the vector field $\nu = \frac{\partial}{\partial z}$ is a concurrent-recurrent vector field. With the help of (10), we find the following:

$$
\begin{gathered}
R(\frac{\partial}{\partial y}, \frac{\partial}{\partial x})\frac{\partial}{\partial z} = R(\frac{\partial}{\partial z}, \frac{\partial}{\partial y})\frac{\partial}{\partial x} = R(\frac{\partial}{\partial z}, \frac{\partial}{\partial x})\frac{\partial}{\partial y} = 0,\\
R(\frac{\partial}{\partial z}, \frac{\partial}{\partial x})\frac{\partial}{\partial x} = R(\frac{\partial}{\partial z}, \frac{\partial}{\partial y})\frac{\partial}{\partial y} = -\alpha^2 e^{2az}\frac{\partial}{\partial z},\\
R(\frac{\partial}{\partial y}, \frac{\partial}{\partial x})\frac{\partial}{\partial x} = -\alpha^2 e^{2az}\frac{\partial}{\partial y}, \qquad R(\frac{\partial}{\partial y}, \frac{\partial}{\partial z})\frac{\partial}{\partial z} = -\alpha^2\frac{\partial}{\partial y},\\
R(\frac{\partial}{\partial z}, \frac{\partial}{\partial x})\frac{\partial}{\partial z} = \alpha^2\frac{\partial}{\partial x}, \qquad R(\frac{\partial}{\partial y}, \frac{\partial}{\partial x})\frac{\partial}{\partial y} = \alpha^2 e^{2\alpha z}\frac{\partial}{\partial x}.
\end{gathered} \tag{12}
$$

From (12), we see that

$$
R(\partial_i, \partial_j)\partial_k = -\alpha^2\{g(\partial_j, \partial_k)\partial_i - g(\partial_i, \partial_k)\partial_j\},
$$

for all $1 \le i, j, k \le 3$. Thus, M is of constant curvature $-\alpha^2$ and so M is Einstein. If we choose $e_1 = e^{-\alpha z}\frac{\partial}{\partial x}$, $e_2 = e^{-\alpha z}\frac{\partial}{\partial y}$ and $e_3 = \nu = \frac{\partial}{\partial z}$, then we see that $\{e_1, e_2, e_3\}$ is an orthonormal frame. Hence, we have

$$
Ric(\partial_i, \partial_j) = \sum_{k=1}^{3} g(R(e_k, \partial_i)\partial_j, e_k) = -2\alpha^2 g(\partial_i, \partial_j). \tag{13}
$$

Let $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. Then, we see that

$$
(\pounds_X g)(\partial_i, \partial_j) + 2Ric(\partial_i, \partial_j) + 4\alpha^2 g(\partial_i, \partial_j) = 0,
$$

for all $1 \le i, j \le 3$. Hence the metric g is a Ricci soliton having the potential field $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ and the soliton constant $\lambda = 2\alpha^2$.

Example 4 Here we shall consider a particular case of Example 1, and we show that the metric is a Ricci soliton. Let g be the Riemannian metric on $M = \mathbb{R}^2 \times R_+ \subset \mathbb{R}^3$ defined by

$$
g = \frac{1}{(\alpha z)^2}(dx^2 + dy^2 + dz^2),
$$

where $\alpha = const. \neq 0$. From Koszul's formula, we have

$$
\begin{gathered}
\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = \frac{1}{z}\frac{\partial}{\partial z}, \qquad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = 0, \qquad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial z} = -\frac{1}{z}\frac{\partial}{\partial x},\\
\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x} = 0, \qquad \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = \frac{1}{z}\frac{\partial}{\partial z}, \qquad \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial z} = -\frac{1}{z}\frac{\partial}{\partial y},\\
\nabla_{\frac{\partial}{\partial z}}\frac{\partial}{\partial x} = -\frac{1}{z}\frac{\partial}{\partial x}, \qquad \nabla_{\frac{\partial}{\partial z}}\frac{\partial}{\partial y} = -\frac{1}{z}\frac{\partial}{\partial y}, \qquad \nabla_{\frac{\partial}{\partial z}}\frac{\partial}{\partial z} = -\frac{1}{z}\frac{\partial}{\partial z}.
\end{gathered} \tag{14}
$$

Put $\nu = -\alpha z \frac{\partial}{\partial z}$. From (14), one easily verifies:

$$\nabla_{\partial_i} \nu = \alpha \left\{ \partial_i - \nu^\flat(\partial_i)\nu \right\}, \tag{15}$$

for all $1 \leq i \leq 3$. Thus, the vector field $\nu = -\alpha z \frac{\partial}{\partial z}$ is a concurrent-recurrent vector field. Also, we find the Ricci tensor as given below

$$(Ric_{ij}) = \begin{pmatrix} -\frac{2}{z^2} & 0 & 0 \\ 0 & -\frac{2}{z^2} & 0 \\ 0 & 0 & -\frac{2}{z^2} \end{pmatrix},$$

and so M is Einstein (being 3-dimensional it is of constant curvature).

Let $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$. Then, one may easily verify that

$$(\pounds_X g)(\partial_i, \partial_j) + 2Ric(\partial_i, \partial_j) + 4\alpha^2 g(\partial_i, \partial_j) = 0,$$

for all $1 \leq i, j \leq 3$. Hence, the metric g is a Ricci soliton having the potential field $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ and the soliton constant $\lambda = 2\alpha^2$.

2.2 Key lemmas

In this paper, we denote $Ric^\sharp$ for the Ricci operator defined by $g(Ric^\sharp Y, Z) = Ric(Y, Z)$ and s for the scalar curvature. From the Eq. (7), with straight forward computation we have:

Lemma 1 *In a Riemannian manifold admitting a concurrent-recurrent vector field, we have*

$$R(Y, Z)\nu = -\alpha^2\{\nu^\flat(Z)Y - \nu^\flat(Y)Z\}, \tag{16}$$

$$R(\nu, Y)Z = -\alpha^2\{g(Z, Y)\nu - \nu^\flat(Z)Y\}, \tag{17}$$

$$Ric(\cdot, \nu) = -(n-1)\alpha^2\nu^\flat(\cdot) \quad (\Rightarrow Ric^\sharp \nu = -(n-1)\alpha^2\nu), \tag{18}$$

$$(\nabla_Y \nu^\flat)Z = \alpha\{g(Z, Y) - \nu^\flat(Y)\nu^\flat(Z)\}. \tag{19}$$

Now, we prove:

Lemma 2 *A Riemannian manifold admitting a concurrent-recurrent vector field satisfies*

$$(\nabla_Y Ric^\sharp)\nu = -(n-1)\alpha^3 Y - \alpha Ric^\sharp Y, \tag{20}$$

$$(\nabla_\nu Ric^\sharp)Y = -2(n-1)\alpha^3 Y - 2\alpha Ric^\sharp Y, \tag{21}$$

$$\pounds_\nu Ric = 2\alpha^3(n-1)\{\nu^\flat \otimes \nu^\flat - g\}. \tag{22}$$

Proof First, we differentiate $Ric^\sharp \nu = -(n-1)\alpha^2 \nu$ along Y and avail (7) in order to deduce (20). From (16), we have

$$\sum_{i=1}^{n} g((\nabla_{e_i} R)(e_i, Z)\nu, W) = -(1-n)\alpha^3 g(Z, W) + \alpha Ric(Z, W)$$

where $\{e_i\}$ is an orthonormal frame. On the other hand, the second Bianchi identity enable us to obtain

$$\sum_{i=1}^{n} g((\nabla_{e_i} R)(W, \nu)Z, e_i) = g((\nabla_W Ric^\sharp)\nu, Z) - g((\nabla_\nu Ric^\sharp)W, Z).$$

We combine the above two equations and use (20) to find

$$g((\nabla_\nu Ric^\sharp)W, Z) = -2(n-1)\alpha^3 g(Z, W) - 2\alpha Ric(Z, W),$$

which proves (21). In light of (7) we see that

$$\pounds_\nu g = 2\alpha\{g - \nu^\flat \otimes \nu^\flat\}. \tag{23}$$

According to Yano [32], we know

$$(\pounds_X \nabla_Y g - \nabla_Y \pounds_X g - \nabla_{[X,Y]} g)(Z, W) = -g((\pounds_X \nabla)(Y, Z), W) - g((\pounds_X \nabla)(Y, W), Z).$$

Because of $\nabla g = 0$, it follows directly that

$$(\nabla_Y \pounds_X g)(Z, W) = g((\pounds_X \nabla)(Y, Z), W) + g((\pounds_X \nabla)(Y, W), Z).$$

Using the symmetric property of $\pounds_X \nabla$, from the above equation we have

$$2g((\pounds_X \nabla)(Y, Z), W) = (\nabla_Y \pounds_X g)(Z, W) + (\nabla_Z \pounds_X g)(W, Y) - (\nabla_W \pounds_X g)(Y, Z). \tag{24}$$

We differentiate (23) and use (19) and (24) to deduce

$$(\pounds_\nu \nabla)(Y, Z) = -2\alpha^2\{g(Y, Z)\nu - \nu^\flat(Y)\nu^\flat(Z)\nu\} \tag{25}$$

According to Yano [32], we know the following relation on any Riemannian manifold:

$$(\pounds_X R)(Y, Z)W = (\nabla_Y \pounds_X \nabla)(Z, W) - (\nabla_Z \pounds_X \nabla)(Y, W). \tag{26}$$

Now, we differentiate (25) along W and employ the identity (26) to find

$$(\pounds_\nu R)(Y, Z)W = 2\alpha^3\{g(Y, W)Z - g(Z, W)Y + \nu^\flat(Z)\nu^\flat(W)Y - \nu^\flat(Y)\nu^\flat(W)Z\}.$$

Contracting the above equation gives (22). □

As an outcome of Lemma 2, we have the following result which characterizes an Einstein manifold:

Theorem 4 *An n-dimensional connected Riemannian manifold (M, g) admits a concurrent-recurrent vector field ν is an Einstein manifold, if and only if, the Ricci operator satisfies*

$$(\nabla_Y Ric^\sharp)\nu = (\nabla_\nu Ric^\sharp)Y,$$

for any $Y \in \mathfrak{X}(M)$.

Proof If the given condition holds, then we have $Ric^\sharp Y = -(n-1)\alpha^2 Y$, that is, $Ric = -(n-1)\alpha^2 g$, and the converse is trivial. □

Lemma 3 *Let the metric of a Riemannian manifold M admitting a concurrent-recurrent vector field be a Ricci almost soliton. If $[X, \nu] \in L(\nu)$, where $L(\nu)$ is the linear span of ν, then M is Einstein.*

Proof First we take Lie derivative to the Ricci almost soliton equation (1) along ν and avail (22) and (23) to obtain

$$\pounds_\nu \pounds_X g = (4\alpha^3(n-1) - 4\lambda\alpha - 2\nu(\lambda))g - (4\alpha^3(n-1) - 4\lambda\alpha)\nu^\flat \otimes \nu^\flat. \quad (27)$$

Taking Lie derivative to (23) along the potential field X and using the Ricci almost soliton equation (1), we get

$$\begin{aligned}(\pounds_X \pounds_\nu g)(Y, Z) = &- 4\alpha\{Ric(Y, Z) + \lambda g(Y, Z)\} - 2\alpha\{(\pounds_X \nu^\flat)(Y)\nu^\flat(Z) \\ &+ (\pounds_X \nu^\flat)(Z)\nu^\flat(Y)\}. \end{aligned} \quad (28)$$

From O'Neill [26], we know the following fundamental identity on Lie derivative (see also Silva Filho [18, Proposition 1]):

$$\pounds_{[Y,Z]} g = \pounds_Y \pounds_Z g - \pounds_Z \pounds_Y g, \quad (29)$$

for all vector fields Y, Z. The Eqs. (27) and (28) and the above identity enables us to obtain

$$\begin{aligned}(\pounds_{[X,\nu]} g)(Y, Z) = &- 4\alpha Ric(Y, Z) - \{4\alpha^3(n-1) - 2\nu(\lambda)\}g(Y, Z) \\ &+ (4\alpha^3(n-1) - 4\lambda\alpha)\nu^\flat(Y)\nu^\flat(Z). \end{aligned} \quad (30)$$

Lie differentiating $g(\nu, \nu) = 1$ along the soliton field X and using (1) and (18), we find $g(\nu, \pounds_X \nu) = -((n-1)\alpha^2 - \lambda)$. So the hypothesis $[X, \nu] \in L(\nu)$ implies $[X, \nu] = (\lambda - (n-1)\alpha^2)\nu$. From here, it is not difficult to show

$$(\pounds_{[X,\nu]} g)(Y, Z) = Y(\lambda)\nu^\flat(Z) + Z(\lambda)\nu^\flat(Y) + 2\{\alpha\lambda - (n-1)\alpha^3\}\{g(Y, Z) - \nu^\flat(Y)\nu^\flat(Z)\}.$$

Comparing the above equation with (30) gives

$$Y(\lambda)\nu^\flat(Z)+Z(\lambda)\nu^\flat(Y)+2\{\alpha\lambda-(n-1)\alpha^3\}\{g(Y,Z)-\nu^\flat(Y)\nu^\flat(Z)\}+4\alpha Ric(Y,Z)$$
$$+\{4\alpha^3(n-1)-2\nu(\lambda)\}g(Y,Z)-(4\alpha^3(n-1)-4\lambda\alpha)\nu^\flat(Y)\nu^\flat(Z)=0. \qquad (31)$$

Finally, we plug $Y=Z=\nu$ in the above equation to obtain $\lambda=(n-1)\alpha^2$, and we substitute this into (31) to claim $Ric=-\alpha^2(n-1)g$. Hence, M is Einstein. □

3 Proof of the main results

3.1 Proof of Theorem 1

Proof First, we operate ∇_W to the Ricci soliton equation (1) to deduce

$$(\nabla_W \pounds_X g)(Y,Z)=-2(\nabla_W Ric)(Y,Z).$$

We fetch the above equation into the Eq. (24) to infer

$$g((\pounds_X\nabla)(Y,Z),U)=(\nabla_U Ric)(Y,Z)-(\nabla_Y Ric)(Z,U)-(\nabla_Z Ric)(U,Y).$$

Now, we set $Z=\nu$ in the above equation and use the Eqs. (20) and (21) to yield

$$(\pounds_X\nabla)(Y,\nu)=2(n-1)\alpha^3 Y+2aRic^\sharp Y. \qquad (32)$$

Now, we differentiate the above equation along Z and use the identity (7) to find

$$(\nabla_Z\pounds_X\nabla)(Y,\nu)=-\alpha(\pounds_X\nabla)(Y,Z)+2a^2\nu^\flat(Z)\{(n-1)\alpha^2 Y+Ric^\sharp Y\}+2a(\nabla_Z Ric^\sharp)Y.$$

We feed the above equation into the identity (26) and then use the symmetric property of $\pounds_X\nabla$ to arrive

$$(\pounds_X R)(Y,Z)\nu=-2a^2\nu^\flat(Z)\{(n-1)\alpha^2 Y+Ric^\sharp Y\}+2a^2\nu^\flat(Y)\{(n-1)\alpha^2 Z$$
$$+Ric^\sharp Z\}+2a\{(\nabla_Y Ric^\sharp)Z-(\nabla_Z Ric^\sharp)Y\}. \qquad (33)$$

We set $Z=\nu$ in the above equation to derive

$$(\pounds_X R)(Y,\nu)\nu=0. \qquad (34)$$

Now Lie-differentiating $R(Y,\nu)\nu=-\alpha^2\{Y-\nu^\flat(Y)\nu\}$ (which can be obtained by (16)) along X gives

$$(\pounds_X R)(Y,\nu)\nu=-\alpha^2 g(Y,\pounds_X\nu)\nu+2a^2\nu^\flat(\pounds_X\nu)Y+\alpha^2(\pounds_X\nu^\flat)(Y)\nu.$$

Combining the above two equations and using the fact that $\alpha \neq 0$ is constant, we have

$$(\pounds_X \nu^\flat)(Y)\nu = g(Y, \pounds_X \nu)\nu - 2\nu^\flat(\pounds_X \nu)Y. \tag{35}$$

Now, with the help of (18), Eq. (1) reads

$$(\pounds_X g)(Y, \nu) = 2(n-1)\alpha^2 \nu^\flat(Y) - 2\lambda \nu^\flat(Y). \tag{36}$$

Substituting $Y = \nu$ in the above equation, we get

$$\nu^\flat(\pounds_X \nu) = -(n-1)\alpha^2 + \lambda. \tag{37}$$

Now Lie-differentiating $\nu^\flat(Y) = g(Y, \nu)$ yields

$$(\pounds_X \nu^\flat)(Y) = (\pounds_X g)(Y, \nu) + g(Y, \pounds_X \nu). \tag{38}$$

We use the above equation and (37) in (35) and then consider the fact that $\alpha \neq 0$ to arrive

$$0 = -\alpha[-\alpha^2(n-1) + \lambda]Y - [(n-1)\alpha^3 - \lambda\alpha]\nu^\flat(Y)\nu. \tag{39}$$

Then the above equation leads to

$$[(n-1)\alpha^3 - \lambda\alpha](Y - \nu^\flat(Y)\nu) = 0,$$

and contracting this over Y shows that $\lambda = (n-1)\alpha^2$. Hence, the soliton is expanding with soliton constant $\lambda = (n-1)\alpha^2$. □

3.2 Proof of Corollary 1

Proof In dimension three, the Riemann curvature tenor is given by

$$\begin{aligned} R(Y, Z)W =& g(Z, W)Ric^\sharp Y - g(Y, W)Ric^\sharp Z + g(Ric^\sharp Z, W)Y - g(Ric^\sharp Y, W)Z \\ &- \frac{s}{2}\{g(Z, W)Y - g(Y, W)Z\}. \end{aligned} \tag{40}$$

Setting $Z = W = \nu$ in the above equation and using (18), we get

$$Ric^\sharp Y = \left(\frac{s}{2} + \alpha^2\right) Y + \left(-3\alpha^2 - \frac{s}{2}\right) \nu^\flat(Y)\nu. \tag{41}$$

Since $\lambda = (n-1)\alpha^2$ (which follows from above theorem), the Eqs. (36)–(38) gives $(\pounds_X g)(Y, \nu) = 0$, $\nu^\flat(\pounds_X \nu) = 0$ and $(\pounds_X \nu^\flat)(Y) = g(Y, \pounds_X \nu)$. Using these and (41) in the Lie-derivative of $Ric(Y, \nu) = -2\alpha^2 \nu^\flat(Y)$ gives

$$\left(\frac{s}{2} + 3\alpha^2\right) g(Y, \pounds_X \nu) = 0.$$

Suppose if $s = -6\alpha^2$, then (41) shows that $Ric^\sharp Y = -2\alpha^2 Y$ and using this in (40) implies the manifold is of constant curvature $-\alpha^2$.

So that we assume $s \neq -6\alpha^2$ on some open set $\mathcal{U}$ of M. So that $\pounds_X \nu = 0$ on $\mathcal{U}$, and this together with the identity (7) implies

$$\nabla_\nu X = \alpha\{X - \nu^\flat(X)\nu\}. \tag{42}$$

Making use of $(\pounds_X g)(Y, \nu) = 0$, the above equation and (42) gives

$$g(\nabla_Y X, \nu) = -g(\nabla_\nu X, Y) = -ag(Y, X) + \alpha\nu^\flat(Y)\nu^\flat(X). \tag{43}$$

According to Duggal and Sharma [17], we write

$$(\pounds_X \nabla)(Y, Z) = \nabla_Y \nabla_Z - \nabla_{\nabla_Y Z} X + R(X, Y)Z.$$

Now, we set $Z = \nu$ in the above equation and use the relations (7), (16), (42) and (43) to find

$$(\pounds_X \nabla)(Y, \nu) = 0.$$

Thus (32) implies that $Ric^\sharp Y = -2\alpha^2 Y$ and contracting this we find $s = -6\alpha^2$. This leads to a contradiction as $s \neq -6\alpha^2$ on $\mathcal{U}$ and completes the proof. □

3.3 Proof of Theorem 2

Proof In the light of (2), we infer

$$\nabla_Y \text{grad}\, \gamma = -Ric^\sharp Y - \lambda Y. \tag{44}$$

We operate the above equation by ∇_Z to obtain

$$\nabla_Z \nabla_Y \text{grad}\, \gamma = -(\nabla_Z Ric^\sharp)Y - Ric^\sharp(\nabla_Z Y) - Z(\lambda)Y - \lambda \nabla_Z Y.$$

Using this in the definition of curvature tensor, we obtain

$$R(Y, Z)\text{grad}\, \gamma = (\nabla_Z Ric^\sharp)Y - (\nabla_Y Ric^\sharp)Z + Z(\lambda)Y - Y(\lambda)Z. \tag{45}$$

Now, we take scalar product of the above equation with ν and utilize the Eq. (20) to deduce

$$g(R(Y, Z)\text{grad}\, \gamma, \nu) = Z(\lambda)\nu^\flat(Y) - Y(\lambda)\nu^\flat(Z).$$

From (16), we see that

$$g(R(Y, Z)\nu, \text{grad}\, \gamma) = \alpha^2\{Z(\gamma)\nu^\flat(Y) - Y(\gamma)\nu^\flat(Z)\}.$$

We combine above two equations and then plug $Z = \nu$ to yield

$$Y(\lambda + \alpha^2\gamma) = \nu(\lambda + \alpha^2\gamma)\nu^\flat(Y),$$

and this is equivalent to

$$d(\lambda + \alpha^2\gamma) = \nu(\lambda + \alpha^2\gamma)\nu^\flat. \tag{46}$$

Now, we put $Y = \nu$ in the Eq. (45) and then take scalar product with W to infer

$$g(R(\nu, Z)\text{grad}\,\gamma, W) = \alpha Ric(Y, W) + ((n-1)\alpha^3 - \nu(\lambda))g(Z, W) + Z(\lambda)\nu^\flat(W).$$

On the other hand, from (16) we easily find

$$g(R(\nu, Y)W, \text{grad}\,\gamma) = \alpha^2\{-g(Y, W)\nu(\gamma) + \nu^\flat(W)Y(\gamma)\}.$$

Now, we combine the above two equations to obtain

$$\alpha Ric(Y, W) = ((1-n)\alpha^3 + \nu(\lambda + \alpha^2\gamma))g(Y, W) - \nu(\lambda + \alpha^2\gamma)\nu^\flat(Y)\nu^\flat(W). \tag{47}$$

We contract the above equation to get

$$\nu(\lambda + \alpha^2\gamma) = \frac{\alpha s}{n-1} + n\alpha^3. \tag{48}$$

Using the above equation in (47), one can easily have

$$Ric = \left(\frac{s}{n-1} + \alpha^2\right) g - \left(\frac{s}{n-1} + n\alpha^2\right) \nu^\flat \otimes \nu^\flat, \tag{49}$$

Now, we contract Eq. (45) over Y to find

$$Ric(Z, \text{grad}\,\gamma) = \frac{1}{2}Z(s) + (n-1)Z(\lambda).$$

Comparing the above equation with (49), we get

$$\frac{n-1}{2}Z(s) + (n-1)^2 Z(\lambda) = (s + (n-1)\alpha^2)Z(\gamma) - (s + n(n-1)\alpha^2)\nu(\gamma)\nu^\flat(Z), \tag{50}$$

for all $Z \in \mathfrak{X}(M)$. Now, we plug $Z = \nu$ in the above equation and use (48) to infer

$$ds(\nu) = -2(n-1)\left\{\frac{\alpha s}{n-1} + n\alpha^3\right\}. \tag{51}$$

From $d\nu^\flat = \frac{1}{2}\{Y\nu^\flat(Z) - Z\nu^\flat(Y) - \nu^\flat([Y, Z])\}$ and (19), we see that the 1-form $\nu^\flat$ is closed, that is, $d\nu^\flat = 0$. Now,operating (46) by the exterior derivative d, and since $d^2 = 0$ and $d\nu^\flat = 0$, we obtain $\frac{\alpha}{n-1} ds \wedge \nu^\flat = 0$, and since $\alpha \neq 0$, we have

$$ds(W)\nu^\flat(U) - ds(U)\nu^\flat(W) = 0,$$

for any vector fields W, U. Putting ν in place of U in the above equation and utilizing (51), we see that $ds(W) = ds(\nu)\nu^\flat(W)$, which means

$$\text{grad } s = \nu(s)\nu. \tag{52}$$

We use the above equation in (50) and substitute $Z = Z - \nu^\flat(Y)\nu$ to deduce

$$(s + n(n-1)\alpha^2)(\text{grad } \gamma - \nu(\gamma)\nu) = 0.$$

If $s = -\alpha^2 n(n-1)$, then it follows from (49) that $Ric = -(n-1)\alpha^2 g$, so that M is Einstein. Now, suppose that $s \neq -\alpha^2 n(n-1)$ on some open set $\mathcal{U}$ of M. Then, the above equation shows that $grad\gamma = \nu(\gamma)\nu$. From here, it is not hard to see

$$[X, \nu] = [grad\gamma, \nu] = -\nabla_\nu grad\gamma = (\lambda - (n-1)\alpha^2)\nu,$$

where we used (7), (44) and the second identity of (18). Then, it follows from Lemma 3 that $Ric = \alpha^2(n-1)g$ on $\mathcal{U}$, and contracting this shows $s = -\alpha^2 n(n-1)$. This is a contradiction and consequently proves our result. □

3.4 Proof of Corollary 2

Proof Invoking Theorem 2, we see that M is Einstein, that is, $Ric^\sharp Y = -(n-1)\alpha^2 Y$. We feed this in (40) to conclude the result. □

3.5 Proof of Corollary 3

Proof The result follows from the Theorem 3, Corollary 1 and Corollary 2. □

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Declarations

Conflict of interest The author declares that he have no conflict of interest.

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