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Relative uniform convergence of difference double sequence of positive linear functions

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Abstract
In this article we introduce the notion of relative uniform convergence of difference double sequence of positive linear functions. We define the difference double sequence spaces $_2\ell_\infty(\Delta, ru)$, $_2c(\Delta, ru)$, $_2c_0(\Delta, ru)$,$_2\, {c_0}^B(\Delta, ru)$, $_2c^B(\Delta, ru)$, $_2c^R(\Delta, ru)$, $_2{c_0}^R(\Delta, ru)$ and study their topological properties.

Keywords Difference sequence space · Completeness · Relative uniform convergence · Solid space · Symmetric space · Monotone space

Mathematics Subject Classification 40A05 · 40C05 · 46A45 · 46B45

1 Introduction

Throughout the article $_2\omega$, $_2\ell_\infty(\Delta, ru)$, $_2c(\Delta, ru)$, $_2c_0(\Delta, ru)$, $_2c^R(\Delta, ru)$, $_2{c_0}^B(\Delta, ru)$, $_2{c_0}^R(\Delta, ru)$ and $_2c^B(\Delta, ru)$ denote all sequence space, relative uniform bounded, relative uniform convergence, relative uniform null, regular relative uniform convergence, relative uniform null bounded, regular relative uniform null , relative uniform bounded convergence difference double sequence space respectively.
A double sequence is a double infinite array of numbers by (x_{nk}). The notion of double sequence was introduced by Pringsheim [15]. Some earlier works on double sequence spaces are found in Bromwich [2]. Hardy [11] introduced the notion of regular convergence of double sequence. The double sequence has been investigated from different aspects by Basarir and Sonalcan [1], Das et al. [6], Datta and Tripathy [7,8], Tripathy and Sarma [19] and many others.

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The notion of uniform convergence of sequence of functions relative to a scale function was introduced by E. H. Moore. Chittenden [3–5] gave a formulation of the definition given by Moore as follows:

Definition 1.1 A sequence (f_n) of real, single-valued functions f_n of a real variable x, ranging over a compact subset D of real numbers, converges relatively uniformly on D in case there exist functions g and σ, defined on D, and for every $\varepsilon > 0$, there exists an integer n_o (dependent on ε) such that for every $n \geq n_o$, the inequality

$$\mid g(x) - f_n(x) \mid < \varepsilon \mid \sigma(x) \mid,$$

holds for every element x of D.

The function σ of the above definition is called a scale function.

The notion was further studied by many others [9,10,16].

Kizmaz [13] defined the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$, $c_0(\Delta)$ as follows:

$$Z(\Delta) = \{x = (x_k) : (\Delta x_k) \in Z\},$$

for $Z = \ell_\infty,\ c,\ c_0$ where $\Delta x_k = x_k - x_{k+1}, k \in N$.

These sequence spaces are Banach space under the norm

$$||(x_k)||_\Delta = \mid x_1 \mid + \sup_{k \in N} |\Delta x_k|.$$

Tripathy and Sarma [20,21] studied difference double sequence spaces and their topological properties.

The notion was further studied from different aspects by Tripathy [17], Tripathy and Goswami [18], Jena et al. [12], Paikray et al. [14] and many others.

A double sequence (x_{nk}) is said to be convergent in Pringshiem's sense if

$$\lim_{n,k \to \infty} x_{nk} = M, \text{ exists where } n, k \in N.$$

A double sequence (x_{nk}) is said to converge regularly if it is convergent in Pringsheim's sense to limit M and the following limit exists:

$$\lim_{n \to \infty} x_{nk} = P_k, \text{ exists for each } k \in N;$$
$$\lim_{k \to \infty} x_{nk} = L_n, \text{ exists for each } n \in N.$$

2 Preliminaries

Definition 2.1 A subset E of the set of all double sequence ${}_2w$ is said to be *solid* or *normal* if $(f_{nk}(x)) \in E \Rightarrow (\alpha_{nk} f_{nk}(x)) \in E$, for all (α_{nk}) of sequence of scalars with $\mid \alpha_{nk} \mid \leq 1$, for all $n, k \in N$.

Definition 2.2 Let

$$K = \Big\{(n_i, k_j) : i, j \in N; n_1 < n_2 < n_3 < \cdots \text{ and } k_1 < k_2 < k_3 < \cdots\Big\} \subseteq N \times N$$

and E be a subset of the set of all double sequence ${}_2w$. A K-step space of E is a sequence space

$$\lambda_K^E = \{(f_{n_i k_j}(x)) \in_2 \omega : (f_{nk}(x)) \in E\}.$$

A canonical pre-image of a sequence of functions $(f_{n_i k_j}(x)) \in E$ is a sequence of functions $(g_{nk}(x)) \in E$ defined by

$$g_{nk}(x) = \begin{cases} f_{nk}(x), & \text{if } (n, k) \in K; \\ \theta, & \text{otherwise.} \end{cases}$$

Definition 2.3 A double sequence space E is said to be *monotone* if it contains the canonical pre-images of all its step spaces.

Remark 2.1 From the above notions, it follows that if a sequence space E is solid then, E is monotone.

Definition 2.4 A double sequence space E is said to be *symmetric* if $(f_{nk}(x)) \in E \Rightarrow (f_{\pi(n,k)}(x)) \in E$, where π is a permutation of N.

Definition 2.5 A difference double sequence of functions $(\Delta f_{nk}(x))$ defined on a compact domain D is said to be *relatively uniformly convergent* if there exists a function $\sigma(x)$ defined on D and for every $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon)$ such that

$$| \Delta f_{nk}(x) - f(x) | < \varepsilon | \sigma(x) |,$$

for all $n, k \geq n_0$ holds for every element x of D. The difference operator Δ is defined by $\Delta f_{nk}(x) = \Delta f_{nk}(x) - \Delta f_{nk}(x) - \Delta f_{nk}(x)$, for all $n, k \in N$.

Remark 2.2 When $f = \theta$, the zero function, we get the definition of *null relative uniform* from the above definition.

Definition 2.6 A difference double sequence of functions $(\Delta f_{nk}(x))$ defined on a compact domain D is said to be *regular relative uniform convergent* if there exist functions $g(x), g_k(x), f_n(x), \sigma(x), \xi_n(x), \eta_k(x)$ defined on D, for every $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon)$ such that for all $x \in D$,

$$\begin{aligned} &| \Delta f_{nk}(x) - g(x) | < \varepsilon | \sigma(x) |, \text{ for all } n, k \geq n_0; \\ &| \Delta f_{nk}(x) - g_k(x) | < \varepsilon | \eta_k(x) |, \text{ for each } k \in N \text{ and for all } n \geq n_0; \\ &| \Delta f_{nk}(x) - f_n(x) | < \varepsilon | \xi_n(x) |, \text{ for each } n \in N \text{ and for all } k \geq n_0. \end{aligned}$$

Remark 2.3 When $g = g_k = f_n = \theta$, the zero function, we get the definition of *regular null relative uniform* from the above definition.

We introduce the following difference double sequence spaces defined over the normed space $(D, ||.||_{(\Delta,\sigma)})$.

$$Z(\Delta, ru) = \{(f_{nk}(x)) : (\Delta f_{nk}(x)) \in Z \text{ relative uniformly w.r.t. } \sigma(x)\},$$

where $Z ={}_2\ \ell_\infty,\ {}_2c_0{}^B,\ {}_2c^B,\ {}_2c^R\ {}_2c_0{}^R, {}_2c$ and ${}_2c_0$.
The above sequence spaces are normed by the norm defined by

$$||f(x)||_{(\Delta,\sigma)} = \sup_{n\geq 1} \sup_{||x||\leq 1} \frac{||f_{n1}(x)||\ ||\sigma(x)||}{||x||} + \sup_{k\geq 1} \sup_{||x||\leq 1} \frac{||f_{1k}(x)||\ ||\sigma(x)||}{||x||} + \sup_{n\geq 1; k\geq 1} \sup_{||x||\leq 1} \frac{||\Delta f_{nk}(x)||\ ||\sigma(x)||}{||x||}$$

3 Main Results

In this section we establish the results of this article.

Theorem 3.1 *The sequence spaces $Z(\Delta, ru)$ where, $Z ={}_2\ \ell_\infty,\ {}_2c_0{}^B,\ {}_2c^B,\ {}_2c^R$ and ${}_2c_0{}^R$ are normed linear spaces.*

Proof We established the theorem for the case of ${}_2\ell_\infty(\Delta, ru)$.
Let α and β be the scalars and $(f_{nk}(x)) \in_2 \ell_\infty(\Delta, ru)$, $(g_{nk}(x)) \in_2 \ell_\infty(\Delta, ru)$.
Then, we have

$$||(\Delta f_{nk}(x)\sigma(x))|| \leq M_1||x||, \quad \text{for all } x \in D \text{ and}$$
$$||(\Delta g_{nk}(x)\sigma(x))|| \leq M_2||x||, \quad \text{for all } x \in D.$$

Without the loss of generality, we can consider the same scale function for the sequence of functions $(f_{nk}(x))$ and $(g_{nk}(x))$.
We have,

$$\begin{aligned} ||(\Delta(\alpha f_{nk}(x)) + \beta g_{nk}(x))\sigma(x))|| &\leq |\alpha|\ ||(\Delta f_{nk}(x)\sigma(x))|| \\ &\quad + |\beta|\ ||(\Delta g_{nk}(x)\sigma(x))|| \\ &\leq |\alpha|M_1||x|| + |\beta|M_2||x|| \\ &\leq (|\alpha|M_1 + |\beta|M_2)||x||,\ \alpha, \beta \text{ are absolutely scalable.} \end{aligned}$$

Hence, ${}_2\ell_\infty(\Delta, ru)$ is a linear space.
Next we prove that ${}_2\ell_\infty(\Delta, ru)$ is a normed space:
(i) Clearly $||f(x)||_{(\Delta,\sigma)} \geq 0$, for all $x \in D$.

$$||f(x)||_{(\Delta,\sigma)} = \left\{ \sup_{n\geq 1} \sup_{||x||\leq 1} \frac{||(f_{n1}(x)\sigma(x))||}{||x||} + \sup_{k\geq 1} \sup_{||x||\leq 1} \frac{||(f_{1k}(x)\sigma(x))||}{||x||} + \sup_{n\geq 1; k\geq 1} \sup_{||x||\leq 1} \frac{||(\Delta f_{nk}(x)\sigma(x))||}{||x||} \right\} = 0$$

$\Rightarrow f_{n1}(x)\sigma(x) = 0$, $f_{1k}(x)\sigma(x) = 0$ and $\Delta f_{nk}(x)\sigma(x) = 0$.

Therefore, $f(x) = (f_{nk}(x)) = 0$, since $\sigma(x) \neq 0$, for $x \in D$.

Conversely, let $f(x) = 0$.

Then, we have $||f(x)||_{(\Delta,ru)} = 0$.

(ii)

$$||\alpha f(x)||_{(\Delta,\sigma)} = \left\{ \sup_{n\geq 1} \sup_{||x||\leq 1} \frac{||(\alpha f_{n1}(x)\sigma(x))||}{||x||} + \sup_{k\geq 1} \sup_{||x||\leq 1} \frac{||(\alpha f_{1k}(x)\sigma(x))||}{||x||} + \sup_{n\geq 1; k\geq 1} \sup_{||x||\leq 1} \frac{||(\alpha \Delta f_{nk}(x)\sigma(x))||}{||x||} \right\}$$

$\Rightarrow ||\alpha f(x)||_{(\Delta,\sigma)} = | \alpha | \, ||f(x)||_{(\Delta,\sigma)}$.

(iii) It can be easily verified that

$$||f(x) + g(x)||_{(\Delta,\sigma)} \leq ||f(x)||_{(\Delta,\sigma)} + ||g(x)||_{(\Delta,\sigma)}.$$

Since the three conditions of norms are satisfied, ${}_2\ell_\infty(\Delta, ru)$ is a normed space. Similarly, we can show that the other sequence spaces are also normed linear space. □

Theorem 3.2 *Let $(D, ||.||_{(\Delta,\sigma)})$ be a complete normed space. The sequence spaces $Z(\Delta, ru)$ where, $Z ={}_2 \ell_\infty$, ${}_2c_0{}^B$, ${}_2c^B$, ${}_2c^R$ and ${}_2c_0{}^R$ are complete.*

Proof Let $(f^i(x)) = (f^i_{nk}(x))$ be a Cauchy sequence in ${}_2\ell_\infty(\Delta, ru)$.
For a given $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $x \in D$,

$||f^i(x) - f^j(x)||_{(\Delta,\sigma)} \leq \frac{\varepsilon}{3}$, for all $i, j \geq n_0$.

Then,

$$\left\{ \sup_{n\geq 1} \sup_{||x||\leq 1} \frac{||(f^i_{n1}(x) - f^j_{n1}(x))\sigma(x)||}{||x||} + \sup_{k\geq 1} \sup_{||x||\leq 1} \frac{||(f^i_{1k}(x) - f^j_{1k}(x))\sigma(x)||}{||x||} + \sup_{n\geq 1; k\geq 1} \sup_{||x||\leq 1} \frac{||(\Delta f^i_{nk}(x) - \Delta f^j_{nk}(x))\sigma(x)||}{||x||} \right\} \leq \frac{\varepsilon}{3}, \tag{1}$$

for all $i, j \geq n_0$.

$\Rightarrow (f^i_{n1}(x))$ is a is relative uniform Cauchy in D w.r.t. $\sigma(x)$, for all $x \in D$ and for each $n \in N$.

$\Rightarrow (f^i_{n1}(x))$ converges relatively uniformly in D w.r.t. $\sigma(x)$, for all $x \in D$ and for each $n \in N$.

Let

$$\lim_{i\to\infty} f^i_{n1}(x) = f_{n1}(x), x \in D \tag{2}$$

Similarly,

$$\lim_{i\to\infty} f^i_{1k}(x) = f_{1k}(x), x \in D \tag{3}$$

$$\lim_{i\to\infty} \Delta f^i_{nk}(x) = \Delta f_{nk}(x), x \in D \tag{4}$$

From the above three equations we get,

$$\lim_{i\to\infty} f^i_{nk}(x) = f_{nk}(x), \text{ for all } x \in D \tag{5}$$

From Eq. (1), we have for all $i, j \geq n_0$ and for all $x \in D$,

$$\lim_{j\to\infty} \frac{||(f^i_{n1}(x) - f^j_{n1}(x))\sigma(x)||}{||x||} = \frac{||(f^i_{n1}(x) - f_{n1}(x))\sigma(x)||}{||x||} \leq \frac{\varepsilon}{3},$$

for each $n \in N$.
Similarly,

$$\lim_{j\to\infty} \frac{||(f^i_{1k}(x) - f^j_{1k}(x))\sigma(x)||}{||x||} = \frac{||(f^i_{1k}(x) - f_{1k}(x))\sigma(x)||}{||x||} \leq \frac{\varepsilon}{3},$$

for each $k \in N$ and

$$\lim_{j\to\infty} \frac{||(\Delta f^i_{nk}(x) - \Delta f^j_{nk}(x))\sigma(x)||}{||x||} = \frac{||(\Delta f^i_{nk}(x) - \Delta f_{nk}(x))\sigma(x)||}{||x||} \leq \frac{\varepsilon}{3},$$

for all $n, k \in N$.

Since $\frac{\varepsilon}{3}$ is not dependent on n and k we have,

$$\sup_{n\geq 1} \sup_{||x||\leq 1} \frac{||(f^i_{n1}(x) - f_{n1}(x))\sigma(x)||}{||x||} \leq \frac{\varepsilon}{3};$$
$$\sup_{k\geq 1} \sup_{||x||\leq 1} \frac{||(f^i_{1k}(x) - f_{1k}(x))\sigma(x)||}{||x||} \leq \frac{\varepsilon}{3};$$
$$\sup_{n\geq 1; k\geq 1} \sup_{||x||\leq 1} \frac{||(\Delta f^i_{nk}(x) - \Delta f_{nk}(x))\sigma(x)||}{||x||} \leq \frac{\varepsilon}{3}.$$

Evidently, $||f^i_{nk}(x) - f_{nk}(x)||_{(\Delta,\sigma)} = \Bigg\{ \sup_{n\geq 1} \sup_{||x||\leq 1} \frac{||(f^i_{n1}(x) - f_{n1}(x))\sigma(x)||}{||x||}$

$$+\sup_{k\geq 1}\sup_{||x||\leq 1}\frac{||(f^i_{1k}(x)-f_{1k}(x))\sigma(x)||}{||x||}$$
$$\left.+\sup_{n\geq 1;k\geq 1}\sup_{||x||\leq 1}\frac{||(\Delta f^i_{nk}(x)-\Delta f_{nk}(x))\sigma(x)||}{||x||}\right\}\leq\varepsilon.$$

$\Rightarrow ||f^i(x)-f(x)||_{(\Delta,\sigma)}\leq\varepsilon$, for all $i\geq n_0$, for all $x\in D$.

Therefore, $(f^i(x)-f(x))\in_2\ell_\infty(\Delta,ru)$, for all $i\geq n_0$, for all $x\in D$.

Then, $f(x)=f^i(x)-(f^i(x)-f(x))\in_2\ell_\infty(\Delta,ru)$ since, ${}_2\ell_\infty(\Delta,ru)$ is a linear space.

Hence, ${}_2\ell_\infty(\Delta,ru)$ is complete.

We can show that the other sequence spaces are also complete in the same process.□

Result 3.1 The sequence spaces $Z(\Delta,ru)$ where, $Z={}_2\ell_\infty,\ {}_2{c_0}^B,\ {}_2c^B,\ {}_2c^R,\ {}_2{c_0}^R, {}_2c$ and ${}_2c_0$ are not monotone.

Proof The proof follows from the following example:

Example 3.1 Consider the sequence of functions $(f_{nk}(x))$, $f_{nk}:[0,1]\rightarrow R$ defined by

$$f_{nk}(x)=\begin{cases}\frac{1}{x}, & \text{when } x\in(0,1];\\ 0, & \text{when } x=0.\end{cases}$$
$$\Delta f_{nk}(x)=f_{nk}(x)-f_{n+1,k}(x)-f_{n,k+1}(x)+f_{n+1,k+1}(x)=0.$$

$(\Delta f_{nk}(x))$ converge uniformly to zero function on [0, 1] w.r.t. the constant scale function $\sigma(x)=1$.

Then, $(\Delta f_{nk}(x))\in Z(\Delta,ru)$ where, $Z={}_2\ell_\infty,\ {}_2{c_0}^B,\ {}_2c^B,\ {}_2c^R,\ {}_2{c_0}^R, {}_2c$ and ${}_2c_0$.

Let us consider the sequence of functions $(g_{nk}(x))$, the pre-image of the sequence of functions $(f_{nk}(x))$ defined by

$$g_{nk}(x)=\begin{cases}f_{nk}(x), & \text{if } n \text{ and } k \text{ both are odd};\\ 0, & \text{otherwise}.\end{cases}$$

The difference sequence of functions $(\Delta g_{nk}(x))$ of the sequence of functions $(g_{nk}(x))$ is given by

$$\Delta g_{nk}(x)=\begin{cases}\frac{1}{x}, & \text{when } n+k \text{ is even, for all } x\in(0,1];\\ -\frac{1}{x}, & \text{when } n+k \text{ is odd, for all } x\in(0,1].\end{cases}$$

$(\Delta g_{nk}(x))$ is not null uniformly w.r.t. any scale function $\sigma(x)$.

Then, $(\Delta g_{nk}(x))\notin_2 c_0(\Delta,ru)$.

Hence, the spaces ${}_2\ell_\infty(\Delta,ru),\ {}_2{c_0}^B(\Delta,ru),\ {}_2c^B(\Delta,ru),\ {}_2c^R(\Delta,ru),\ {}_2c(\Delta,ru),\ {}_2c_0(\Delta,ru),\ {}_2{c_0}^R(\Delta,ru)$ are not monotone.

Remark 3.1 Since soild implies monotone and from the Result 3.1. it follows that the spaces ${}_2\ell_\infty(\Delta, ru)$, ${}_2{c_0}^B(\Delta, ru)$, ${}_2c^B(\Delta, ru)$, ${}_2c^R(\Delta, ru)$, ${}_2c(\Delta, ru)$, ${}_2c_0(\Delta, ru)$, ${}_2{c_0}^R(\Delta, ru)$ are not solid in general.

Result 3.2 The sequence spaces $Z(\Delta, ru)$ where, $Z ={}_2 \ell_\infty$, ${}_2{c_0}^B$, ${}_2c^B$, ${}_2c^R$, ${}_2{c_0}^R$,${}_2 c$ and ${}_2c_0$ are not symmetric. □

Proof The proof of the result follows from the following example:

Example 3.2 Consider the sequence of functions $(f_{nk}(x))$, $f_{nk} : [0, 1] \to R$ defined by

$$f_{nk}(x) = \begin{cases} x, & \text{when } n = 1, k \in N; \\ 0 & \text{otherwise.} \end{cases}$$

The difference sequence of functions $(\Delta f_{nk}(x))$ of the sequence of functions $(f_{nk}(x))$ is given by

$$\Delta f_{nk}(x) = f_{nk}(x) - f_{n,k+1}(x) - f_{n+1,k}(x) + f_{n+1,k+1}(x) = 0.$$

$(\Delta f_{nk}(x))$ is relative uniform convergent w.r.t. the constant scale function 1.

Then, $(\Delta f_{nk}(x)) \in Z(\Delta, ru)$ where $Z ={}_2 \ell_\infty$, ${}_2{c_0}^B$, ${}_2c^B$, ${}_2c^R$, ${}_2{c_0}^R$,${}_2 c$ and ${}_2c_0$.

Let $(g_{nk}(x))$ be the rearranged sequence of functions of $(f_{nk}(x))$ defined by

$$g_{nk}(x) = \begin{cases} x, & \text{when } n = k = i^2, i \in N; \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\Delta g_{nk}(x) = \begin{cases} x, & \text{for } n = k = i^2 \text{ and } n = k = i^2 - 1; i \in N; \\ -x, & \text{for } n = i^2 - 1, k = i^2 \text{ and } n = i^2, k = i^2 - 1; i \in N; \\ 0, & \text{otherwise.} \end{cases}$$

One cannot get a scale function such that the sequence of functions $(\Delta g_{nk}(x))$, a null relative uniform.

Then, $(\Delta g_{nk}(x)) \notin_2 c_o(\Delta, ru)$.

Hence, the sequence spaces ${}_2\ell_\infty(\Delta, ru)$, ${}_2{c_0}^B(\Delta, ru)$, ${}_2c^B(\Delta, ru)$, ${}_2c^R(\Delta, ru)$, ${}_2c(\Delta, ru)$, ${}_2c_0(\Delta, ru)$ and ${}_2{c_0}^R(\Delta, ru)$ are not symmetric. □

Theorem 3.3 *(i) $Z(ru) \subset Z(\Delta, ru)$, for $Z ={}_2 \ell_\infty$, ${}_2{c_0}^B$, ${}_2c^B$, ${}_2c^R$, ${}_2{c_0}^R$, ${}_2c$ and ${}_2c_0$ and the inclusions are strict.*
(ii) $Z_0(\Delta, ru) \subset Z(\Delta, ru)$, for $Z ={}_2 \ell_\infty$, ${}_2{c_0}^B$, ${}_2c^B$, ${}_2c^R$, ${}_2{c_0}^R$, ${}_2c$ and ${}_2c_0$ and the inclusions are strict.

Proof (i)

$$\text{Let } (f_{nk}(x)) \in_2 c(ru) \tag{6}$$

Then, for all $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon)$ such that

$$| f_{nk}(x) - f(x) | < \frac{\varepsilon}{4} | \sigma(x) |,$$

for all $n \geq n_0$ and $k \geq n_0$.
For all $n \geq n_0$ and $k \geq n_0$ we have,

$$\begin{aligned}
| \Delta f_{nk}(x) | &\leq | f_{nk}(x) - f(x) | + | f_{n,k+1}(x) - f(x) | + | f_{n,k+1}(x) - f(x) | \\
&\quad + | f_{n+1,k+1}(x) - f(x) | \\
&\leq \frac{\varepsilon}{4} | \sigma(x) | + \frac{\varepsilon}{4} | \sigma(x) | + \frac{\varepsilon}{4} | \sigma(x) | + \frac{\varepsilon}{4} | \sigma(x) | + \frac{\varepsilon}{4} | \sigma(x) | \\
&\leq \varepsilon | \sigma(x) | \\
| \Delta f_{nk}(x) | &\leq \varepsilon | \sigma(x) | . \\
(f_{nk}(x)) &\in_2 c_0(\Delta, ru)
\end{aligned} \tag{7}$$

From Eqs. (6) and (7) we get, ${}_2c(ru) \subset_2 c_0(\Delta, ru)$.
Similarly, we can prove for the other sequence spaces.
The following example shows that the inclusions are strict. □

Example 3.3 Consider the sequence of functions $(f_{nk}(x))$, $f_{nk}(x) : [0, 1] \to R$ defined by

$$f_{nk}(x) = \begin{cases} x, & \text{when } n \text{ is odd, } k \in N; \\ 0 & \text{otherwise.} \end{cases}$$

The difference sequence of functions $(\Delta f_{nk}(x))$ of the sequence of functions $(f_{nk}(x))$ is given by

$$\Delta f_{nk}(x) = f_{nk}(x) - f_{n,k+1}(x) - f_{n+1,k}(x) + f_{n+1,k+1}(x) = 0.$$

One cannot get a scale function such that the ordinary sequence of functions $(f_{nk}(x))$, a null relative uniform. However its difference sequence of functions, $(\Delta f_{nk}(x))$ is null relative uniform w.r.t. the constant scale function 1.

Hence, the inclusion are strict.
(ii) It is obvious that $Z_0(\Delta, ru) \subset Z(\Delta, ru)$, for $Z =_2 \ell_\infty,\ {}_2{c_0}^B,\ {}_2c^B,\ {}_2c^R,\ {}_2{c_0}^R, {}_2c$ and ${}_2c_0$, hence we omitted the prove.
The following example shows that inclusions are strict.

Example 3.4 Consider the sequence of functions $(f_{nk}(x))$, $f_{nk}(x) : [0, 1] \to R$ defined by

$$f_{nk}(x) = \begin{cases} (n+k)x, & \text{for } n = 1, k \in N; \\ nkx & \text{otherwise.} \end{cases}$$

We have $\Delta f_{nk}(x) = x$.
$(\Delta f_{nk}(x))$ is relative uniform convergent to 1 on [0,1] w.r.t. the scale function $\sigma(x)$ defined by

$$\sigma(x) = \begin{cases} \frac{1}{x}, & \text{for } x \in (0, 1]; \\ 0 & \text{for } x = 0. \end{cases}$$

Hence the inclusions are strict.

Result 3.3 A diference double sequence of functions $(\Delta f_{nk}(x))$ is regular relative uniform convergent over a compact subset D w.r.t. a scale function $\sigma(x)$ then, $(\Delta f_{nk}(x))$ is also relative uniform convergent w.r.t. a scale function $\sigma(x)$ but not conversely.

Example 3.5 Consider the sequence of functions $(f_{nk}(x))$, $f_{nk}(x) : [0, 1] \to R$ defined by

$$f_{nk}(x) = \begin{cases} x, & \text{when } n = 1 \text{ and } k \text{ is odd}, k \in N; \\ 0 & \text{otherwise.} \end{cases}$$

$(\Delta f_{nk}(x))$ is given by

$$\Delta f_{nk}(x) = \begin{cases} x, & \text{when } n = 1;\ k \text{ is odd}, k \in N; \\ -x, & \text{when } n = 1;\ k \text{ is even}, k \in N; \\ 0 & \text{otherwise.} \end{cases}$$

We have seen that $(\Delta f_{nk}(x))$ is relative uniform convergent to the zero function θ on [0, 1] w.r.t. the constant scale function 1. However, in case of regular relative uniform convergence, the first row of the sequence of functions $(\Delta f_{nk}(x))$ fails to converge relatively uniformly w.r.t. a scale function $\sigma(x)$.

Hence, the above result is justified.

Remark 3.2 All the results for the case of regularly convergent sequences will be same as that of the Pringsheim's sense convergence.

4 Conclusions

In this article we have studied difference double sequences of positive linear operators from the point of view of relative uniform convergence. This is the first article on this topic and it is expected that it will attract researcher for further investigation and applications.

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Conflict of interest The authors declare that they have no conflict of interest.

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