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A functional approach to the boundedness of maximal function

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Abstract
We improve the boundedness of maximal function on the spaces defined by Choquet integrals associated to both Bessel and Riesz capacities. The capacities will be generalized to functionals as a means of proof.

Keywords Maximal functions · Choquet integrals · Capacities

Mathematics Subject Classification 31C15 · 42B25

1 Introduction

This paper addresses the following boundedness issues.

Theorem 1.1 *Let* $\alpha > 0$ *and* $s, t > 1$ *be such that* $\alpha s < n$, $1/s + 1/t = 1$. *For any* $q > t(n - \alpha s)/n$ *and measurable q.e. defined function* φ, *it follows that*

$$\int_{\mathbb{R}^n} (\mathcal{M}^{\mathrm{loc}}\varphi)^q dC \leq C(\alpha, s, n, q) \int_{\mathbb{R}^n} |\varphi|^q dC, \tag{1.1}$$

where $C(\alpha, s, n, q)$ *is a constant depending only on* α, s, n *and* q.

Theorem 1.2 *Let* $\alpha > 0$ *and* $s, t > 1$ *be such that* $\alpha s < n$, $1/s + 1/t = 1$. *For any* $q > t(n - \alpha s)/n$ *and measurable q.e. defined function* φ, *it follows that*

$$\int_{\mathbb{R}^n} (\mathcal{M}\varphi)^q dc \leq C(\alpha, s, n, q) \int_{\mathbb{R}^n} |\varphi|^q dc,$$

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where $C(\alpha, s, n, q)$ is a constant depending only on α, s, n and q.

Related notations are introduced as follows. Denote $\mathcal{M}^{\mathrm{loc}}(\cdot)$ and $\mathcal{M}$ the *local and global Hardy-Littlewood maximal functions* respectively by

$$\mathcal{M}^{\mathrm{loc}}(\varphi)(x) = \sup_{0<r\leq 1} \frac{1}{|B_r(x)|} \int_{B_r(x)} |\varphi(y)|dy, \quad x \in \mathbb{R}^n,$$
$$\mathcal{M}(\varphi)(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |\varphi(y)|dy, \quad x \in \mathbb{R}^n$$

for a locally integrable function φ on $\mathbb{R}^n$, the *Choquet integrals* $\int_{\mathbb{R}^n} |\cdot|dC$ of f that

$$\int_{\mathbb{R}^n} |f|dC = \int_0^\infty \mathrm{Cap}_{\alpha,s}(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})d\lambda,$$
$$\int_{\mathbb{R}^n} |f|dc = \int_0^\infty \mathrm{cap}_{\alpha,s}(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})d\lambda,$$

the Bessel and Riesz capacities $\mathrm{Cap}_{\alpha,s}(\cdot)$ and $\mathrm{cap}_{\alpha,s}(\cdot)$ respectively of a set $E \subseteq \mathbb{R}^n$ that

$$\mathrm{Cap}_{\alpha,s}(E) = \inf\{\|f\|_{L^s}^s : f \geq 0, G_\alpha * f \geq 1 \text{ on } E\},$$
$$G_\alpha(x) = \mathcal{F}^{-1}[(1 + |\cdot|^2)^{-\alpha/2}](x), \quad x \in \mathbb{R}^n,$$
$$\mathrm{cap}_{\alpha,s}(E) = \inf\{\|f\|_{L^s}^s : f \geq 0, I_\alpha * f \geq 1 \text{ on } E\},$$
$$I_\alpha(x) = \frac{1}{|x|^{n-\alpha}}, \quad x \in \mathbb{R}^n,$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform in $\mathbb{R}^n$. A property holds quasi-everywhere (q.e.) with respect to Bessel capacity if it holds except on a set E with $\mathrm{Cap}_{\alpha,s}(E) = 0$, a similar property with respect to Riesz capacity is defined canonically.

Note that the boundedness is valid for $q < 1$. A similar result is already obtained in [6] regarding the case for $q > (n-\alpha)/n$. Nevertheless, for certain values of α, s, t, n, one has $t(n-\alpha s)/n < (n-\alpha)/n$, so Theorem 1.1 can be viewed as an improvement to the boundedness obtained in [6] for certain cases. A similar issue regarding the boundedness of maximal function associated to Hausdorff content is addressed in [7].

2 A functional generalization of capacities

In what follows, we always assume that $\alpha > 0$, $s, t > 1$, $1/s + 1/t = 1$, $n \in \mathbb{N}$, and that $\alpha s < n$. Furthermore, we denote C_0 the class of compactly supported continuous functions on $\mathbb{R}^n$ and C_0^∞ is the subclass of infinitely differentiable functions of C_0. By $\mathcal{LSC}$ we mean the class of lower semi-continuous functions on $\mathbb{R}^n$. Let us write χ_E the characteristic function of a set $E \subseteq \mathbb{R}^n$. The notation $A \lesssim B$ will abbreviate the inequality that $A \leq CB$ for some constant $C > 0$ and $A \approx B$ simply means both $A \lesssim B$ and $B \lesssim A$.

Let $\mathbb{M}$ be a space equipped with a positive measure ν. By a *kernel* we mean a nonnegative g on $\mathbb{R}^n \times \mathbb{M}$ such that $g(\cdot, y)$ is lower semicontinuous on $\mathbb{R}^n$ for each $y \in \mathbb{M}$, and $g(x, \cdot)$ is measurable on $\mathbb{M}$ for each $x \in \mathbb{R}^n$. Suppose that μ is a positive measure on $\mathbb{R}^n$ and f is a nonnegative ν-measurable function, we define *potentials* $\mathcal{G}f$ and $\check{\mathcal{G}}\mu$ to be

$$\mathcal{G}f(x) = \int_{\mathbb{M}} g(x, y) f(y) d\nu(y), \quad x \in \mathbb{R}^n$$
$$\check{\mathcal{G}}\mu(y) = \int_{\mathbb{R}^n} g(x, y) d\mu(x), \quad y \in \mathbb{M}.$$

For any function φ on $\mathbb{R}^n$, we let

$$\Omega_\varphi = \{f : f \in L^s_+(\nu), \mathcal{G}f(x) \geq |\varphi(x)|^{1/s} \text{ for all } x \in \mathbb{R}^n\},$$

then we define $\mathcal{C}(\cdot) = \mathcal{C}_g(\cdot)$ to be

$$\mathcal{C}(\varphi) = \inf\{\|f\|^s_{L^s} : f \in \Omega_\varphi\}.$$

Therefore, $\mathcal{C}(\cdot)$ is a generalization of the capacity $C_{g,s}(\cdot)$ defined by

$$C_{g,s}(E) = \inf\{\|f\|^s_{L^s} : f \in L^s_+(\nu), \mathcal{G}f(x) \geq 1 \text{ on } E\}$$

for any set $E \subseteq \mathbb{R}^n$. We call $\mathcal{C}(\cdot)$ the *functional generalization* of $C_{g,s}(\cdot)$.

Suppose that $\mathbb{M} = \mathbb{R}^n$, g is said to be a *radially decreasing convolution kernel* if $g(x, y) = g_0(|x - y|)$, where $0 \leq g_0 \in \mathcal{LSC}$, g_0 is decreasing on $[0, \infty)$, and $\int_0^1 g_0(t)t^{n-1}dt < \infty$. For $f \in L^s_+(\mathbb{R}^n)$, we have $\mathcal{G}f = g * f$. The Bessel $G_\alpha(\cdot)$ and Riesz $I_\alpha(\cdot)$ are radially decreasing convolution kernels. Note that

$$G_\alpha(x) = (4\pi)^{\alpha/2}\Gamma(\alpha/2) \int_0^\infty t^{\frac{\alpha-n}{2}} e^{-\frac{\pi|x|^2}{t} - \frac{t}{4\pi}} \frac{dt}{t}, \quad x \in \mathbb{R}^n,$$

see [1, Section 1.2.4].

In the sequel, if a statement is stated with $\mathcal{C}(\cdot)$ without the subscript g as in $\mathcal{C}_g(\cdot)$, then the statement holds for any kernel g. We begin by introducing the following *subadditivity* property of $\mathcal{C}(\cdot)$.

Proposition 2.1 *$\mathcal{C}(\cdot)$ is subadditive.*

Proof Let $\mathcal{H}$ be the set of all nonnegative functions φ on $\mathbb{R}^n$ such that $\mathcal{G}f(x) \geq \varphi(x)^{1/s}$ for some function $f \in L^s_+(\nu)$ with $\|f\|_{L^s(\nu)} \leq 1$. We claim that $\mathcal{H}$ is convex. Suppose that $\varphi_1, \varphi_2 \in \mathcal{H}$ and $0 < c < 1$, then there are $f_1, f_2 \in L^s_+(\nu)$ such that $\mathcal{G}f_j(x) \geq \varphi_j(x)^{1/s}$ and $\|f_j\|_{L^s(\nu)} \leq 1$ for $j = 1, 2$. It follows by the reverse Minkowski's inequality that

$$\left(\mathcal{G}\big((cf_1^s + (1 - c)f_2^s)^{1/s}\big)(x)\right)^s$$

$$
\begin{aligned}
&= \left(\int_{\mathbb{M}} g(x,y)\left(cf_1(y)^s + (1-c)f_2(y)^s\right)^{1/s} d\nu(y)\right)^s \\
&\geq \left(\int_{\mathbb{M}} g(x,y)\left(cf_1(y)^s\right)^{1/s} d\nu(y)\right)^s + \left(\int_{\mathbb{M}} g(x,y)\left((1-c)f_2(y)^s\right)^{1/s} d\nu(y)\right)^s \\
&= c\left(\mathcal{G}f_1(x)\right)^s + (1-c)\left(\mathcal{G}f_2(x)\right)^s \\
&\geq (c\varphi_1 + (1-c)\varphi_2)(x).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\|(cf_1^s + (1-c)f_2^s)^{1/s}\|_{L^s(\nu)} &= \left(\int_{\mathbb{M}} \left(cf_1(y)^s + (1-c)f_2(y)^s\right) d\nu(y)\right)^{1/s} \\
&= \left(c\|f_1\|_{L^s(\nu)}^s + (1-c)\|f_2\|_{L^s(\nu)}^s\right)^{1/s} \\
&\leq (c + (1-c))^{1/s} \\
&= 1.
\end{aligned}
$$

As a result, the convexity of $\mathcal{H}$ is justified. Subsequently, we claim that

$$
\mathcal{C}(\varphi) = \inf\{c > 0 : \varphi \in c\mathcal{H}\}. \tag{2.1}
$$

Indeed, assume that $c > 0$ and $\varphi/c \in \mathcal{H}$, then some $f \in L_+^s(\nu)$ is such that $\|f\|_{L^s(\nu)} \leq 1$ and that $\mathcal{G}f \geq (\varphi/c)^{1/s}$. It follows that $\mathcal{G}(c^{1/s}f) \geq \varphi^{1/s}$ and $c \geq c\|f\|_{L^s}^s = \|c^{1/s}f\|_{L^s}^s \geq \mathcal{C}(\varphi)$. Consequently, $\inf\{c > 0 : \varphi \in c\mathcal{H}\} \geq \mathcal{C}(\varphi)$. On the other hand, assume that $f \in L_+^s(\nu)$ and $\mathcal{G}f \geq \varphi^{1/s}$. For any $\epsilon > 0$, we have $\mathcal{G}\left(f/(\|f\|_{L^s(\nu)} + \epsilon)\right) \geq \left(\varphi/(\|f\|_{L^s(\nu)} + \epsilon)^s\right)^{1/s}$ and hence $\varphi \in (\|f\|_{L^s(\nu)} + \epsilon)^s \cdot \mathcal{H}$. It follows that $(\|f\|_{L^s(\nu)} + \epsilon)^s \geq \inf\{c > 0 : \varphi \in c\mathcal{H}\}$ and the arbitrariness of $\epsilon > 0$ yields that $\mathcal{C}(\varphi) \geq \inf\{c > 0 : \varphi \in c\mathcal{H}\}$. Therefore, (2.1) is justified and the subadditivity of $\mathcal{C}(\cdot)$ then follows. □

Next, $\mathcal{C}(\cdot)$ possesses the outer regularity in the sense that

Proposition 2.2 *For any $\varphi \geq 0$, it holds $\mathcal{C}(\varphi) = \inf\{\mathcal{C}(\psi) : \psi \geq \varphi, \psi \in \mathcal{LSC}\}$.*

Proof We assume that $\mathcal{C}(\varphi) < \infty$. Let $\epsilon > 0$ be given. There is an $f \in L_+^s(\nu)$ such that $\mathcal{G}f \geq \varphi$ and that $\|f\|_{L^s(\nu)}^s < \mathcal{C}(\varphi) + \epsilon$. We write $((\mathcal{G}f)^s)^{1/s} = \mathcal{G}f$, then it follows by definition of $\mathcal{C}(\cdot)$ that $\mathcal{C}\left((\mathcal{G}f)^s\right) \leq \|f\|_{L^s(\nu)}^s$ and hence $\mathcal{C}\left((\mathcal{G}f)^s\right) < \mathcal{C}(\varphi) + \epsilon$. Now we note that $\mathcal{G}f \in \mathcal{LSC}$ (see [1, Proposition 2.3.2]). □

The aforementioned regularity resembles the property of capacity that

$$
\mathrm{Cap}_{\alpha,s}(E) = \inf\{\mathrm{Cap}_{\alpha,s}(G) : G \supseteq E, G \text{ open}\}
$$

for any set $E \subseteq \mathbb{R}^n$.

The next three propositions are the generalization of [1, Proposition 2.3.9], [1, Theorem 2.3.10], and [1, Proposition 2.3.12] respectively, which the proofs are almost identical and hence will be omitted. However, there are much to say about Proposition 2.4. In [1, Theorem 2.3.10], for any subset E of $\mathbb{R}^n$ with $C_{g,s}(E) < \infty$, there is a unique *capacitary function* f^E of E such that $f^E \in L^s_+(\nu)$ and $\mathcal{G}f^E \geq 1$ $(C_{g,s})$-q.e. on E, and

$$\int_{\mathbb{M}} (f^E)^s d\nu = C_{g,s}(E).$$

We also call $\mathcal{G}f^E$ as the *capacitary potential* of E. As a generalization, in the following Proposition 2.4, given any nonnegative φ, the term f_φ will serve as the *capacitary function* of φ, and $\mathcal{G}f_\varphi$ is then the *capacitary potential* of φ for which $\mathcal{G}f_\varphi(x) \geq \varphi(x)^{1/s}$ q.e. and

$$\int_{\mathbb{M}} (f_\varphi)^s d\nu = \mathcal{C}(\varphi).$$

Proposition 2.3 *For any $\varphi \geq 0$, by denoting $\overline{\Omega}_\varphi$ the closure of Ω_φ in $L^s(\nu)$, it follows that*

$$\overline{\Omega}_\varphi = \{f : f \in L^s_+(\nu), \mathcal{G}f(x) \geq \varphi(x)^{1/s} \text{ q.e. with respect to } C_{g,s}(\cdot)\}.$$

Proposition 2.4 *Let $\varphi \geq 0$ and $\mathcal{C}(\varphi) < \infty$, then there is a unique $f_\varphi \in L^s_+(\nu)$ such that $\mathcal{G}f_\varphi(x) \geq \varphi(x)^{1/s}$ q.e. with respect to $C_{g,s}$, and*

$$\int_{\mathbb{M}} (f_\varphi)^s d\nu = \mathcal{C}(\varphi).$$

Proposition 2.5 *If $\{\varphi_i\}_{i=1}^\infty$ is an increasing sequence of nonnegative functions with $\varphi = \sup \varphi_i$, then*

$$\mathcal{C}(\varphi) = \lim_{i\to\infty} \mathcal{C}(\varphi_i).$$

As a result of Proposition 2.5, we obtain the *Fatou property* of $\mathcal{C}(\cdot)$ that

$$\mathcal{C}\left(\liminf_{i\to\infty} \varphi_i\right) \leq \liminf_{i\to\infty} \mathcal{C}(\varphi_i), \quad \varphi_i \geq 0. \tag{2.2}$$

Proposition 2.5 also resembles the property of capacity that

$$\mathrm{Cap}_{\alpha,s}(E) = \lim_{i\to\infty} \mathrm{Cap}_{\alpha,s}(E_i)$$

for any increasing sequence $\{E_i\}$ of arbitrary subsets of $\mathbb{R}^n$ with union E.

The following Corollary 2.6 addresses the countably subadditivity of $\mathcal{C}(\cdot)$, which is a direct consequence of the Fatou property of $\mathcal{C}(\cdot)$.

Corollary 2.6 *Let* $\varphi_i \geq 0$, $i = 1, 2, \ldots$, *it follows that*

$$\mathcal{C}\left(\sum_{i=1}^{\infty}\varphi_i\right) \leq \sum_{i=1}^{\infty}\mathcal{C}(\varphi_i).$$

The Fatou property of $\mathcal{C}(\cdot)$ also entails the following corollary once we recall the fact that for any nonnegative $\varphi \in \mathcal{LSC}$, there is an increasing sequence $\{\psi_i\}$ of C_0 such that $\psi_i(x) \uparrow \varphi(x)$ pointwise everywhere.

Corollary 2.7 *For any* $0 \leq \varphi \in \mathcal{LSC}$, *it holds*

$$\mathcal{C}(\varphi) = \sup\{\mathcal{C}(\psi) : 0 \leq \psi \leq \varphi, \psi \in C_0\}.$$

The above corollary resembles the property of capacity that

$$\mathrm{Cap}_{\alpha,s}(G) \approx \sup\{\mathrm{Cap}_{\alpha,s}(K) : K \subseteq G, K \text{ compact}\}$$

for any open set G.

Theorem 2.8 *Assume that g is a radially decreasing kernel, then*

$$\mathcal{C}_g(\varphi) \approx \int_{\mathbb{R}^n}\varphi dC_{g,s}. \tag{2.3}$$

In particular, we have

$$\mathcal{C}_{G_\alpha}(\chi_E) \approx \mathrm{Cap}_{\alpha,s}(E),$$
$$\mathcal{C}_{I_\alpha}(\chi_E) \approx \mathrm{cap}_{\alpha,s}(E)$$

for any set $E \subseteq \mathbb{R}^n$.

Proof First of all, we have the *capacitary strong type inequality* that

$$\int_0^{\infty} C_{g,s}(\{x \in \mathbb{R}^n : (g * f(x))^s > \lambda\})d\lambda \lesssim \|f\|_{L^s(\mathbb{R}^n)}^s,$$

(see [1, Theorem 7.1.1]) which shows that

$$\int_{\mathbb{R}^n}\varphi dC_{g,s} = \int_0^{\infty} C_{g,s}(\{x \in \mathbb{R}^n : \varphi(x) > \lambda\})d\lambda \lesssim \mathcal{C}(\varphi).$$

Furthermore, let us write that

$$\varphi = \sum_{i=-\infty}^{\infty}\varphi\chi_{\{\lambda^i \leq \varphi < \lambda^{i+1}\}}.$$

By re-examining the proof of [1, Proposition 7.4.1] and combining Corollary 2.6, we deduce the estimate that

$$\mathcal{C}_g(\varphi) \leq \frac{c^2}{c-1}\int_0^\infty C_{g,s}(\{x \in \mathbb{R}^n : \varphi(x) > \lambda\})d\lambda \tag{2.4}$$

for any fixed constant $c > 1$. In fact, (2.4) holds regardless of whether g is a radially decreasing convolution kernel. Therefore, (2.3) is then established. □

The following proposition characterizes the zeros of $\mathcal{C}(\cdot)$.

Proposition 2.9 *Let $\varphi \geq 0$, then $\mathcal{C}(\varphi) = 0$ if and only if there is a $f \in L^s_+(\nu)$ such that*

$$\varphi^{-1}((0, +\infty]) \subseteq (\mathcal{G}f)^{-1}(\{+\infty\}).$$

Proof First we prove for sufficiency. We have

$$\mathcal{C}\left(\chi_{(\mathcal{G}f)^{-1}([\lambda,+\infty])}\right) \leq \frac{1}{\lambda^s}\|f\|^s_{L^s(\nu)}$$

for any $\lambda > 0$, it follows that $\mathcal{C}\left(\chi_{(\mathcal{G}f)^{-1}(\{+\infty\})}\right) = 0$ and hence

$$\mathcal{C}\left(\chi_{\varphi^{-1}((0,\infty])}\right) = 0.$$

We deduce from (2.4) that $\mathcal{C}(\varphi) = 0$. Note that $\mathcal{C}(\chi_E) = C_{g,s}(E)$ for any set $E \subseteq \mathbb{R}^n$ and (2.4) holds regardless of whether g is a radially decreasing convolution kernel.

Now we prove for necessity. Assuming that $\mathcal{C}(\varphi) = 0$, then we can choose $f_i \in L^s_+(\nu)$ such that $\mathcal{G}f_i \geq \varphi^{1/s}$ and that $\|f_i\|^s_{L^s(\nu)} < 2^{-is}$ for $i = 1, 2, \ldots$ The function $f = \sum_i f_i$ satisfies that $\mathcal{G}f(x) \geq \sum_i \mathcal{G}f_i(x) = \infty$ for $\varphi(x) > 0$ and that $\|f\|_{L^s(\nu)} < 1$. □

Recall that we have the dual definition of the capacity that

$$C_{g,s}(K)^{1/s} = \sup\{\mu(K) : \mu \in \mathscr{M}^+(K), \|\check{\mathcal{G}}\mu\|_{L^t(\nu)} \leq 1\}$$

for any compact set $K \subseteq \mathbb{R}^n$, $1/s + 1/t = 1$, and $\mathscr{M}^+(K)$ is the space of positive measures on K (see [1, Theorem 2.5.1]). The following Theorem 2.10 corresponds to the dual definition of $\mathcal{C}(\cdot)$ under certain conditions imposed to the kernel g. Throughout Theorems 2.10 and 2.11 we will assume that $\mathbb{M}$ is locally compact and that

$$g(x, y) = h(\phi(x, y)), \quad x \in \mathbb{R}^n, \quad y \in \mathbb{M} \tag{2.5}$$

for some functions $\phi : \mathbb{R}^n \times \mathbb{M} \to [0, +\infty)$ and $h : [0, +\infty) \to (0, +\infty]$, where $\phi(F \times K)$ is bounded for any compact sets $F \subseteq \mathbb{R}^n$, $K \subseteq \mathbb{M}$, and h is decreasing. If $g(x) = g_0(|x|) > 0$ is a radially decreasing convolution kernel, then $\mathbb{M} = \mathbb{R}^n$ and we can simply take $h = g_0$ and $\phi(x, y) = |x - y|$.

Theorem 2.10 *Suppose that the kernel g satisfies the condition (2.5). For any function φ with compact support* $\mathrm{supp}(\varphi)$*, if*

$$\varphi|_{\mathrm{supp}(\varphi)}$$

is continuous with $\min_{\mathrm{supp}(\varphi)} \varphi > 0$*, then*

$$\mathcal{C}_g(\varphi) = \sup\left\{\left(\int_{\mathbb{R}^n} \varphi^{1/s} d\mu\right)^s : \mu \in \mathscr{M}^+(\mathrm{supp}(\varphi)), \|\check{\mathcal{G}}\mu\|_{L^t(\nu)} \leq 1\right\}.$$

Proof The following proof is modified from [2, Theorem 6.1]. For any $\psi \geq 0$, let

$$\begin{aligned}\mathcal{D}(\psi) &= \mathcal{C}_g(\psi^s),\\ \mathcal{M}(\psi) &= \sup\left\{\int_{\mathbb{R}^n} \psi d\mu : \mu \in \mathscr{M}^+(\mathrm{supp}(\psi)), \|\check{\mathcal{G}}\mu\|_{L^t(\nu)} \leq 1\right\},\end{aligned}$$

then by letting $\psi = \varphi^{1/s}$, we are to prove that

$$\mathcal{D}(\psi)^{1/s} = \mathcal{M}(\psi).$$

Let

$$\mathcal{M}_\psi = \left\{\mu \in \mathscr{M}^+(\mathrm{supp}(\psi)) : \int_{\mathbb{R}^n} \psi(x) d\mu = 1\right\},$$

and

$$\mathcal{F} = \{f \in L^s_+(\nu) : \|f\|_{L^s(\nu)} \leq 1\}.$$

We also let

$$\mathcal{D}_1(\psi) = \left(\sup_{\mathcal{F}} \inf_{\mathcal{M}_\psi} \int_{\mathbb{R}^n} \mathcal{G}f(x) d\mu\right)^{-s},$$

and

$$\mathcal{M}_1(\psi) = \left(\inf_{\mathcal{M}_\psi} \sup_{\mathcal{F}} \int_{\mathbb{R}^n} \mathcal{G}f(x) d\mu\right)^{-1}.$$

We claim that

$$\mathcal{D}_1(\psi)^{1/s} = \mathcal{M}_1(\psi). \tag{2.6}$$

The sets $\mathcal{M}_\psi$ and $\mathcal{F}$ are convex. The set $\mathcal{M}_\psi$ is vaguely compact by the observation that $\mu(\mathrm{supp}(\psi)) \leq (\min_{\mathrm{supp}(\psi)} \psi)^{-1}$ for $\mu \in \mathcal{M}_\psi$ and the Banach-Alaoglu Theorem.

The linearity of the maps

$$f \to \int_{\mathbb{R}^n} \mathcal{G} f(x) d\mu,$$
$$\mu \to \int_{\mathbb{R}^n} \mathcal{G} f(x) d\mu,$$

and the continuity of the second map allow us to invoke Fan's Minimax Theorem (see [1, Theorem 2.4.1]), and hence (2.6) follows by the minimax theorem. We are now to show that

$$\mathcal{D}(\psi) = \mathcal{D}_1(\psi), \tag{2.7}$$

and

$$\mathcal{M}(\psi) = \mathcal{M}_1(\psi). \tag{2.8}$$

We begin by showing that

$$\mathcal{D}_1(\psi) \leq \mathcal{D}(\psi). \tag{2.9}$$

We could assume that $\mathcal{D}(\psi) < \infty$. For any $\epsilon > 0$, there is an $f_\epsilon \in L^s_+(\nu)$ such that $\mathcal{G} f_\epsilon \geq \psi$ and

$$\| f_\epsilon \|^s_{L^s(\nu)} < \mathcal{D}(\psi) + \epsilon.$$

As a result,

$$\left\| \frac{f_\epsilon}{(\mathcal{D}(\psi) + \epsilon)^{1/s}} \right\|_{L^s(\nu)} \leq 1.$$

For any $\mu \in \mathcal{M}_\psi$, we have

$$\int_{\mathbb{R}^n} \mathcal{G} \left(\frac{f_\epsilon}{(\mathcal{D}(\psi) + \epsilon)^{1/s}} \right) (x) d\mu \geq \frac{1}{(\mathcal{D}(\psi) + \epsilon)^{1/s}}.$$

Thus,

$$\mathcal{D}(\psi) + \epsilon \geq \left(\int_{\mathbb{R}^n} \mathcal{G} \left(\frac{f_\epsilon}{(\mathcal{D}(\psi) + \epsilon)^{1/s}} \right) (x) d\mu \right)^{-s},$$

which implies that $\mathcal{D}(\psi) + \epsilon \geq \mathcal{D}_1(\psi)$, so (2.9) follows. Now we show that

$$\mathcal{D}(\psi) \leq \mathcal{D}_1(\psi). \tag{2.10}$$

We assume that $\mathcal{D}_1(\psi) < \infty$. For any $\epsilon > 0$, there is a $f_\epsilon \in \mathcal{F}$ such that

$$\left(\inf_{\mu \in \mathcal{M}_\psi} \int_{\mathbb{R}^n} \mathcal{G} f_\epsilon(x) d\mu\right)^{-s} < \mathcal{D}_1(\psi) + \epsilon.$$

Thus,

$$1 \leq \inf_{\mu \in \mathcal{M}_\psi} \int_{\mathbb{R}^n} \mathcal{G}(f_\epsilon \cdot (\mathcal{D}_1(\psi) + \epsilon)^{1/s})(x) d\mu.$$

Fix an $x \in \operatorname{supp}(\psi)$, and let $d\mu = (\psi(x))^{-1} d\delta_x$, where $d\delta_x$ is the point mass measure at x, then $\int_{\mathbb{R}^n} \psi(x) d\mu = 1$ and hence

$$\begin{aligned} 1 &\leq \mathcal{G}(f_\epsilon \cdot (\mathcal{D}_1(\psi) + \epsilon)^{1/s})(x) \cdot \frac{1}{\psi(x)} \\ \psi(x) &\leq \mathcal{G}(f_\epsilon \cdot (\mathcal{D}_1(\psi) + \epsilon)^{1/s})(x). \end{aligned}$$

Since $\|f_\epsilon\|_{L^s(\nu)} \leq 1$, we get

$$\mathcal{D}(\psi) \leq \|f_\epsilon \cdot (\mathcal{D}_1(\psi) + \epsilon)^{1/s}\|_{L^s(\nu)}^s \leq \mathcal{D}_1(\psi) + \epsilon,$$

so (2.10) follows and hence (2.7). We are now to show (2.8). As before, we will separate the cases to

$$\mathcal{M}_1(\psi) \leq \mathcal{M}(\psi), \tag{2.11}$$

and

$$\mathcal{M}(\psi) \leq \mathcal{M}_1(\psi). \tag{2.12}$$

We note that $\mathcal{M}_1(\psi) \geq 0$ since $0 \in \mathcal{F}$. Assume at the moment that

$$\mathcal{M}_1(\psi) < \infty, \tag{2.13}$$

we invoke the dual pair $(L^t(\nu), L^s(\nu))$, then for every $\epsilon > 0$, there is a measure $\mu \in \mathcal{M}_\psi$ satisfying

$$\begin{aligned} \mathcal{M}_1(\psi) &< \left(\sup_{f \in \mathcal{F}} \int_{\mathbb{R}^n} \mathcal{G} f(x) d\mu\right)^{-1} + \epsilon \\ &= \left(\sup_{f \in \mathcal{F}} \int_{\mathbb{M}} f(y)(\check{\mathcal{G}}\mu)(y) d\nu\right)^{-1} + \epsilon \\ &= \|\check{\mathcal{G}}\mu\|_{L^t(\nu)}^{-1} + \epsilon. \end{aligned}$$

Set $\sigma = \|\check{\mathcal{G}}\mu\|_{L^t(\nu)}^{-1}\mu$, we get

$$\mathcal{M}_1(\psi) - \epsilon < \|\check{\mathcal{G}}\mu\|_{L^t(\nu)}^{-1} = \int_{\mathbb{R}^n} \psi(x)d\sigma \leq \mathcal{M}(\psi),$$

so (2.11) follows. Now we justify (2.12). For any $\mu \in \mathscr{M}^+(\text{supp}(\psi))$ with $\|\check{\mathcal{G}}\mu\|_{L^s(\nu)} \leq 1$, and $f \in \mathcal{F}$, set $\sigma = \left(\int_{\mathbb{R}^n} \psi(x)d\mu\right)^{-1}\mu$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{G}f(x)d\sigma &= \int_{\mathbb{M}} (\check{\mathcal{G}}\sigma)(y)f(y)d\nu \\ &= \left(\int_{\mathbb{R}^n} \psi(x)d\mu\right)^{-1} \int_{\mathbb{M}} (\check{\mathcal{G}}\mu)(y)f(y)d\nu \\ &\leq \left(\int_{\mathbb{R}^n} \psi(x)d\mu\right)^{-1},\end{aligned}$$

by the dual pair $(L^t(\nu), L^s(\nu))$. Therefore,

$$\int_{\mathbb{R}^n} \psi(x)d\nu \leq \mathcal{M}_1(\psi)$$

which implies (2.12), so (2.8) is established as well.

We now justify (2.13). Assume the contrary that a sequence $\{\mu_j\} \subseteq \mathcal{M}_\psi$ is such that

$$\sup_{f\in\mathcal{F}} \int_{\mathbb{R}^n} \mathcal{G}f(x)d\mu_j \to 0.$$

By the dual pair $(L^t(\nu), L^s(\nu))$, we get immediately that

$$\|\check{\mathcal{G}}\mu_j\|_{L^t(\nu)} \to 0. \tag{2.14}$$

Denote $F = \text{supp}(\psi)$ and choose a compact set $K \subseteq \mathbb{M}$ such that $\nu(K) > 0$. Let $\eta_0 > 0$ be such that $\phi(x, y) < \eta_0$ for all $x \in F$ and $y \in K$, see the notations in (2.5). We have

$$\check{\mathcal{G}}\mu_j(y) = \int_{\mathbb{R}^n} h(\phi(x, y))d\mu_j \geq h(\eta_0)\mu_j(F), \quad y \in K,$$

then $\mu_j(F) \leq \|\check{\mathcal{G}}\mu_j\|_{L^t(\nu)}(h(\eta_0)\nu(K))^{-1}$, and hence the sequence $\{\mu_j(F)\}$ is bounded. By Banach-Alaoglu Theorem, there exists a subnet $\{\mu_{j_k}\}$ converging vaguely to a measure μ and hence $\int_{\mathbb{R}^n} \psi(x)d\mu = 1$. On the other hand, we have $\mu_{j_k}(F) \to 0$, so $\int_{\mathbb{R}^n} \psi(x)d\mu = 0$, we get a contradiction, and (2.13) follows. □

By the dual definition of capacity, one obtains the *capacitary measure* $\mu^K \in \mathcal{M}^+(K)$ for K such that $f^K = (\check{\mathcal{G}}\mu^K)^{t-1}$, and

$$\mu^K(K) = \int_{\mathbb{M}} (\check{\mathcal{G}}\mu^K)^t d\nu = \int_{\mathbb{R}^n} \mathcal{G} f^K d\mu^K = C_{g,s}(K),$$

(see [1, Theorem 2.5.3]). The following theorem is a counterpart for the case of $\mathcal{C}(\cdot)$, and we call μ_φ the *capacitary measure* for φ.

Theorem 2.11 *Suppose that the kernel g satisfies the condition (2.5). For any function φ with compact support* $\operatorname{supp}(\varphi)$*, if*

$$\varphi|_{\operatorname{supp}(\varphi)}$$

is continuous with $\min_{\operatorname{supp}(\varphi)} \varphi > 0$*, then there is a* $\mu_\varphi \in \mathscr{M}^+(\operatorname{supp}(\varphi))$ *such that* $f_\varphi = (\check{G}\mu_\varphi)^{1/(s-1)}$*, and*

$$\int_{\mathbb{R}^n} \varphi^{1/s} d\mu_\varphi = \int_{\mathbb{M}} (\check{G}\mu_\varphi)^t d\nu = \int_{\mathbb{R}^n} \mathcal{G} f_\varphi d\mu_\varphi = \mathcal{C}_g(\varphi). \tag{2.15}$$

Proof Let $\{\mu_i\}$ be a sequence in $\mathscr{M}^+(\operatorname{supp}(\varphi))$ such that $\|\check{\mathcal{G}}\mu_i\|_{L^t(\nu)} = 1$ and

$$\lim_{i\to\infty} \left(\int_{\mathbb{R}^n} \varphi^{1/s} d\mu_i\right)^s = \mathcal{C}_g(\varphi).$$

Since $\mu_i(\operatorname{supp}(\varphi)) \leq (\min_{\operatorname{supp}(\varphi)} \varphi)^{-1}$ for all $i = 1, 2, \ldots$, we can assume that $\{\mu_i\}$ converges vaguely to a measure $\mu \in \mathscr{M}^+(\operatorname{supp}(\varphi))$, and hence

$$\left(\int_{\mathbb{R}^n} \varphi^{1/s} d\mu\right)^s = \mathcal{C}_g(\varphi).$$

By [1, Proposition 2.3.2], $\check{\mathcal{G}}\mu(y) \in \mathcal{LSC}$ on $\mathscr{M}^+(\operatorname{supp}(\varphi))$ for each $y \in \mathbb{M}$, it follows that $\|\check{\mathcal{G}}\mu\|_{L^t(\nu)} \leq 1$, and thus, $\|\check{\mathcal{G}}\mu\|_{L^t(\nu)} = 1$ by Theorem 2.10.

Now we let

$$\mu_\varphi = \left(\int \varphi^{1/s} d\mu\right)^{s/t} \mu,$$

then

$$\int_{\mathbb{M}} (\check{\mathcal{G}}\mu_\varphi)^t d\nu = \|\check{\mathcal{G}}\mu\|_{L^t(\nu)}^t \left(\int \varphi^{1/s} d\mu\right)^s = \mathcal{C}_g(\varphi).$$

On the other hand,

$$\int_{\mathbb{R}^n} \varphi^{1/s} d\mu_\varphi = \left(\int_{\mathbb{R}^n} \varphi^{1/s} d\mu\right)^{s/t} \left(\int_{\mathbb{R}^n} \varphi^{1/s} d\mu\right) = \mathcal{C}_g(\varphi).$$

Let f_φ be the capacitary function for φ in Theorem 2.4, so that $\mathcal{G} f_\varphi(x) \geq \varphi(x)^{1/s}$ q.e. with respect to $C_{g,s}$. Let

$$S = \{x \in \mathrm{supp}(\varphi) : \mathcal{G} f_\varphi(x) < \varphi(x)^{1/s}\}$$

and $K \subseteq S$ be a compact set, then $C_{g,s}(K) = 0$. Applying Theorem 2.10 to the function χ_K, we have

$$C_{g,s}(K) \geq \left(\frac{\mu_\varphi(K)}{\mathcal{C}(\varphi)^{1/t}}\right)^s,$$

and hence $\mu_\varphi(K) = 0$. As S is a Borel set, it follows that $\mu_\varphi(S) = 0$ and hence $\mathcal{G} f_\varphi(x) \geq \varphi(x)^{1/s}$ a.e. with respect to μ_φ. By Fubini's theorem and Hölder's inequality, we have

$$\begin{aligned}
\mathcal{C}_g(\varphi) &= \int_{\mathbb{R}^n} \varphi^{1/s} d\mu_\varphi \\
&\leq \int_{\mathbb{R}^n} \mathcal{G} f_\varphi d\mu_\varphi \\
&= \int_{\mathbb{M}} \check{\mathcal{G}}\mu_\varphi f_\varphi d\nu \\
&\leq \|\check{\mathcal{G}}\mu_\varphi\|_{L^t(\nu)} \|f_\varphi\|_{L^s(\nu)} \\
&= \mathcal{C}_g(\varphi)^{1/t} \mathcal{C}_g(\varphi)^{1/s} \\
&= \mathcal{C}_g(\varphi).
\end{aligned}$$

The equality in Hölder's inequality implies that $(f_\varphi)^s = (\check{G}\mu_\varphi)^t$. □

On the other hand, one may suspect whether we can weaken the q.e. condition in Proposition 2.3. In fact, for any set $E \subseteq \mathbb{R}^n$, one has

$$\mathrm{Cap}_{\alpha,s}(E) = \inf \|f\|^s_{L^s(\mathbb{R}^n)},$$

where the infimum is taken for all $f \in L^s_+(\mathbb{R}^n)$ such that $G_\alpha * f(x) \geq 1$ a.e. on some neighborhood of E (see [1, Corollary 2.6.8]). Before looking at the general case for $\mathcal{C}(\cdot)$, we need the following lemma, which is a simple generalization of [1, Proposition 2.6.7].

Lemma 2.12 *Suppose that $g(x) = g_0(|x|)$ is a radially decreasing convolution kernel, continuous on $\mathbb{R}^n \setminus \{0\}$, and such that $\int_{|x|>1} g(x)^t dx < \infty$. Assume that there is an L and a $\delta > 0$ such that g_0 satisfies*

$$g_0(r) \leq L g_0(2r), \quad 0 < r \leq \delta. \tag{2.16}$$

Let $f \in L^s_+(\mathbb{R}^n)$, $0 \leq \varphi \in \mathcal{LSC}$, and suppose that $g * f(x) \geq \varphi(x)$ a.e. on an open set U, then $g * f(x) \geq \varphi(x)$ everywhere on U.

Proof Without loss of generality, we can assume that $g * f(x) \geq \varphi(x)$ a.e. on a neighborhood of 0, and prove that $g * f(0) \geq \varphi(0)$. We can also assume that

$$g * f(0) = \int_{\mathbb{R}^n} g(x) f(x) dx < \infty.$$

Let $0 < a < b$ and define a weight function $\eta_{a,b}$ by

$$\eta_{a,b}(x) = \frac{g(x)}{\int_{|y|<|x|} g(y)dy}, \quad a < |x| < b,$$

and $\eta_{a,b}(x) = 0$ otherwise. We set $G(r) = \int_{|y|<r} g(y)dy$, it follows that $\int_{\mathbb{R}^n} \eta_{a,b}(x)dx = \log G(b) - \log G(a)$. Since $\int_0^1 g_0(t)t^{n-1}dt < \infty$, we have $\lim_{a\to 0} G(a) = 0$. For any $b > 0$, we can choose an $a > 0$ such that $\int_{\mathbb{R}^n} \eta_{a,b}(x)dx = 1$.

For small enough $b > 0$,

$$\inf_{B_b(0)} \varphi \leq \int_{\mathbb{R}^n} \eta_{a,b}(x)(g * f)(x)dx = \int_{\mathbb{R}^n} (\eta_{a,b} * g)(y) f(y)dy. \tag{2.17}$$

Fix a ρ such that $0 < \rho \leq \delta$. Then,

$$\lim_{a,b\to 0} \eta_{a,b} * g(y) = \lim_{a,b\to 0} \int_{\mathbb{R}^n} \eta_{a,b}(x)g(y - x)dx = g(y)$$

uniformly for $|y| \geq \rho$, and also

$$\int_{\mathbb{R}^n} \eta_{a,b}(x)g(y - x)dx \leq g_0(|y| - b).$$

Thus, for any $0 < R < \infty$ and $\epsilon > 0$,

$$\lim_{a,b\to 0} \int_{\rho\leq|y|\leq R} (\eta_{a,b} * g)(y) f(y)dy = \int_{\rho\leq|y|\leq R} g(y) f(y)dy,$$

and

$$\int_{|y|\geq R} (\eta_{a,b} * g)(y) f(y)dy \leq \|f\|_{L^s(\mathbb{R}^n)} \left(\int_{|y|\geq R-b} g(y)^t dy\right)^{1/t} < \epsilon,$$

if $R > 0$ is large enough.

To estimate $\eta_{a,b} * g(y)$ for $|y| \leq \rho$, we observe that if $|x - y| \leq 2^{-1}|y|$, then $|x| \geq 2^{-1}|y|$, and $|x| \geq |x - y|$. Thus,

$$\int_{|x-y|\leq 2^{-1}|y|} \eta_{a,b}(x)g(y - x)dx = \int_{|x-y|\leq 2^{-1}|y|} \frac{g(x)g(y - x)}{\int_{|t|<|x|} g(t)dt} dx$$

$$\leq g(2^{-1}y)\int_{|x-y|\leq 2^{-1}|y|}\frac{g(y-x)}{\int_{|t|<|x-y|}g(t)dt}dx$$
$$\leq g(2^{-1}y)\int_{\mathbb{R}^n}\eta_{a,b}(x)dx$$
$$= g(2^{-1}y).$$

On the other hand,

$$\int_{|x-y|\geq 2^{-1}|y|}\eta_{a,b}(x)g(y-x)dx \leq g(2^{-1}y)\int_{\mathbb{R}^n}\eta_{a,b}(x)dx = g(2^{-1}y).$$

The assumption (2.16) gives that $\eta_{a,b} * g(y) \leq 2Lg(y)$ for $|y| \leq 2\delta$, and thus

$$\int_{|y|\leq\rho}(\eta_{a,b} * g)(y)f(y)dy \leq 2L\int_{|y|\leq\rho}g(y)f(y)dy < \epsilon,$$

if $\rho > 0$ is small enough.

We obtain from (2.17) and the lower semi-continuity of φ that

$$\varphi(0) \leq \liminf_{b\to 0}\left(\inf_{B_b(0)}\varphi\right) \leq \int_{\rho\leq|y|\leq R}g(y)f(y)dy + 2\epsilon.$$

The proposition follows if $\rho \to 0$ and $R \to \infty$. □

As a corollary, we have the following result.

Corollary 2.13 *Let g be a kernel satisfying the conditions in Lemma 2.12. For any $0 \leq \varphi \in \mathcal{LSC}$, it follows that*

$$\mathcal{C}_g(\varphi) = \inf \|f\|_{L^s}^s,$$

where the infimum is taken for all $f \in L^s_+(\mathbb{R}^n)$ such that $g * f(x) \geq \varphi(x)^{1/s}$ a.e.

The proof of Theorem 1.1 will be given independently in Sect. 3 as an application of the functional generalization of *local Riesz capacities*. However, if we seek for the proof of Theorem 1.1 under the range that $s > 2-\alpha/n$, one may apply the functional generalization of Bessel capacities without the technicalities as in Sect. 3 (see Sect. 4 for the remarks). The proof of Theorem 1.2 will be put in the last section.

3 Local Riesz capacities and proof of Theorem 1.1

In this section we will review some background materials about the local Riesz capacities. The following approach is adopted from [8, Section 3.6.1], which is equivalent to [1, Section 4.4]. As usual, we denote $\alpha > 0$, $s, t > 1$, $1/s + 1/t = 1$, $n \in \mathbb{N}$, and that $\alpha s < n$.

Let $\chi = \chi_{B_1(0)}$, $\mathbb{M} = \mathbb{R}^n \times \mathbb{R}$, and $0 < \rho < \infty$. Define the kernel g_ρ on $\mathbb{R}^n \times \mathbb{M}$ by $g_\rho(x, (y, z)) = z^{-(n-\alpha)}\chi_{(0,\rho)}(z)\chi((x-y)/z)$. We also define the measure ν_ρ on $\mathbb{M}$ by

$$d\nu_\rho(y, z) = \chi_{(0,\rho)}(z)dy \times \frac{dz}{z}.$$

If f is a function on $\mathbb{M}$, we see that

$$\mathcal{G}_\rho f(x) = \int_0^\rho \left(\int_{|x-y|<z} \frac{f(y, z)}{z^{n-\alpha}} dy \right) \frac{dz}{z}.$$

Besides that, for $\mu \in \mathscr{M}^+(\mathbb{R}^n)$, it follows that

$$\check{\mathcal{G}}_\rho \mu(y, z) = \frac{1}{z^{n-\alpha}} \chi_{(0,\rho)}(z)\mu(B_z(y)).$$

We define the local Riesz capacity $\mathcal{R}_{\alpha,s,\rho}(\cdot)$ by

$$\mathcal{R}_{\alpha,s,\rho}(E) = \inf\{\|f\|_{L^s(\nu_\rho)}^s : f \in L_+^s(\nu_\rho), \mathcal{G}_\rho f \geq 1 \text{ on } E\}$$

for any set $E \subseteq \mathbb{R}^n$, and the corresponding functional $\mathcal{C}_{\mathcal{R},\rho}(\cdot)$ is defined by

$$\mathcal{C}_{\mathcal{R},\rho}(\varphi) = \inf\{\|f\|_{L^s(\nu_\rho)}^s : f \in L_+^s(\nu_\rho), \mathcal{G}_\rho f \geq \varphi^{1/s}\}$$

for any $\varphi \geq 0$. Subsequently, the *nonlinear potential* $\mathcal{V}_\rho^\mu$ is defined by $\mathcal{V}_\rho^\mu = \mathcal{G}_\rho((\check{\mathcal{G}}_\rho \mu)^{1/(s-1)})$ and hence

$$\mathcal{V}_\rho^\mu(x) = \int_0^\rho \left(\int_{|x-y|<z} \left(\frac{\mu(B_z(y))}{z^{n-\alpha}} \right)^{\frac{1}{s-1}} \frac{1}{z^{n-\alpha}} dy \right) \frac{dz}{z}, \quad x \in \mathbb{R}^n.$$

For any compact set $K \subseteq \mathbb{R}^n$, the capacitary measure μ^K has the properties that

$$\begin{aligned} \mathcal{V}_\rho^{\mu^K}(x) &\geq 1 \quad \text{q.e. with respect to } \mathcal{R}_{\alpha,s,\rho}(\cdot) \text{ on } K, \\ \mathcal{V}_\rho^{\mu^K}(x) &\leq 1 \quad \text{for every } x \in \operatorname{supp}(\mu^K), \end{aligned}$$

and such that

$$\mu^K(K) = \int_{\mathbb{R}^n} \mathcal{V}_\rho^{\mu^K} d\mu^K = \mathcal{R}_{\alpha,s,\rho}(K).$$

On the other hand, we also define the *Wolff potential* W_ρ^μ by

$$W_\rho^\mu(x) = \int_0^\rho \left(\frac{\mu(B_z(x))}{z^{n-\alpha s}} \right)^{\frac{1}{s-1}} \frac{dz}{z}, \quad x \in \mathbb{R}^n.$$

We observe that $B_z(y) \subseteq B_{2z}(x)$ for $|x-y|<z$, and hence

$$\mathcal{V}_\rho^\mu(x) \lesssim \int_0^\rho \mu(B_{2z}(x))^{\frac{1}{s-1}} \frac{1}{z^{\frac{n-\alpha}{s-1}-\alpha}} dy \frac{dz}{z} \approx W_{2\rho}^\mu(x).$$

Besides that, $B_z(x) \subseteq B_{2z}(y)$ for $|x-y|<z$ by symmetry, then

$$\begin{aligned}\mathcal{V}_\rho^\mu(x) &\approx \int_0^{\rho/2} \left(\int_{|x-y|<2z} \left(\frac{\mu(B_{2z}(y))}{z^{n-\alpha}} \right)^{\frac{1}{s-1}} \frac{1}{z^{n-\alpha}} dy \right) \frac{dz}{z} \\ &\geq \int_0^{\rho/2} \left(\int_{|x-y|<z} \left(\frac{\mu(B_{2z}(y))}{z^{n-\alpha}} \right)^{\frac{1}{s-1}} \frac{1}{z^{n-\alpha}} dy \right) \frac{dz}{z} \\ &\geq \int_0^{\rho/2} \mu(B_z(x))^{\frac{1}{s-1}} \frac{1}{z^{\frac{n-\alpha}{s-1}-\alpha}} dy \frac{dz}{z} \\ &= W_{\rho/2}^\mu(x).\end{aligned}$$

Therefore, we obtain the estimate that

$$W_{\rho/2}^\mu \lesssim \mathcal{V}_\rho^\mu \lesssim W_{2\rho}^\mu. \tag{3.1}$$

On the other hand, let us recall the *inhomogeneous Riesz kernel* $\mathcal{I}_{\alpha,\rho}(\cdot)$ and *non-linear inhomogeneous Riesz potential* V_ρ^μ defined by

$$\mathcal{I}_{\alpha,\rho}(x) = |x|^{-(n-\alpha)} \chi_{|x|<\rho}$$

and $V_\rho^\mu = \mathcal{I}_{\alpha,\rho}((\mathcal{I}_{\alpha,\rho}\mu)^{1/(s-1)})$ respectively. A *Wolff type inequality* says that

$$\int_{\mathbb{R}^n} V_\rho^\mu d\mu \approx \int_{\mathbb{R}^n} W_\rho^\mu d\mu, \tag{3.2}$$

see [8, Theorem 3.6.6]. By an application of Fubini's theorem, one has

$$\int_{\mathbb{R}^n} V_\rho^\mu d\mu = \int_{\mathbb{R}^n} (\mathcal{I}_{\alpha,\rho}\mu)^t dx, \tag{3.3}$$

then we deduce from (3.3) that

$$\int_{\mathbb{R}^n} V_\rho^\mu d\mu \approx \int_{\mathbb{R}^n} V_{\rho/2}^\mu d\mu, \tag{3.4}$$

see the proof of [8, Lemma 3.3.8]. As a consequence, we combine (3.1), (3.2), and (3.4) to obtain the estimate that

$$\int_{\mathbb{R}^n} V_\rho^\mu d\mu \approx \int_{\mathbb{R}^n} V_{\rho/2}^\mu d\mu$$

$$
\begin{aligned}
&\approx \int_{\mathbb{R}^n} W^{\mu}_{\rho/2} d\mu \\
&\lesssim \int_{\mathbb{R}^n} \mathcal{V}^{\mu}_{\rho} d\mu \\
&\lesssim \int_{\mathbb{R}^n} W^{\mu}_{2\rho} d\mu \\
&\approx \int_{\mathbb{R}^n} V^{\mu}_{2\rho} d\mu \\
&\approx \int_{\mathbb{R}^n} V^{\mu}_{\rho} d\mu,
\end{aligned}
$$

and hence

$$\int_{\mathbb{R}^n} V^{\mu}_{\rho} d\mu \approx \int_{\mathbb{R}^n} \mathcal{V}^{\mu}_{\rho} d\mu. \tag{3.5}$$

Following the similar reasoning given in the proof of [8, Theorem 3.3.7], we conclude by the equivalence in (3.5) that

$$\mathcal{R}_{\alpha,s,\rho}(E) \approx \mathrm{Cap}_{\alpha,s}(E) \tag{3.6}$$

for any $E \subseteq \mathbb{R}^n$.

The equivalence in (3.6) suggests the following fact.

Proposition 3.1 *For any $\varphi \geq 0$, it holds $\mathcal{C}_{\mathcal{R},\rho}(\varphi) \approx \mathcal{C}_{G_\alpha}(\varphi)$.*

The proof needs several technical lemmas. First of all, let us address the *quasi-subadditivty* of $\mathcal{C}_{\mathcal{R},\rho}(\cdot)$ as the following shown.

Lemma 3.2 *Let $\{\mathcal{B}_j\}_{j\geq 0}$ be a covering of $\mathbb{R}^n$ by balls with unit diameter. Let this covering have a finite multiplicity, depending only on n. It follows that*

$$\mathcal{C}_{\mathcal{R},\rho}(\varphi) \approx \sum_{j\geq 0} \mathcal{C}_{\mathcal{R},\rho}(\varphi\chi_{\mathcal{B}_j})$$

for any $\varphi \geq 0$.

Proof By subadditivity of $\mathcal{C}_{\mathcal{R},\rho}(\cdot)$, we just need to address on

$$\sum_{j\geq 0} \mathcal{C}_{\mathcal{R},\rho}(\varphi\chi_{\mathcal{B}_j}) \lesssim \mathcal{C}_{\mathcal{R},\rho}(\varphi).$$

We start with an observation that

$$
\begin{aligned}
&\mathcal{C}_{\mathcal{R},\rho}(\varphi) \\
&\quad = \inf\{\|f\|^s_{L^s(\nu_\rho)} : f \in L^s(\nu_\rho), \mathcal{G}_\rho f \geq \varphi^{1/s} \text{ q.e. with respect to } \mathcal{R}_{\alpha,s,\rho}(\cdot)\}.
\end{aligned} \tag{3.7}
$$

We note that by Proposition 2.9 (or simply [1, Proposition 2.3.7]), the term $\mathcal{G}_\rho f(x) = \mathcal{G}_\rho f^+(x) - \mathcal{G}_\rho f^-(x)$ is defined and finite q.e. with respect to $\mathcal{R}_{\alpha,s,\rho}(\cdot)$. Suppose that $\mathcal{G}_\rho f(x) \geq \varphi(x)^{1/s}$ for such an x, then $\mathcal{G}_\rho f^+(x) \geq \varphi(x)^{1/s}$. We also have $\|f^+\|_{L^s(\nu_\rho)} \leq \|f\|_{L^s(\nu_\rho)}$, then (3.7) follows by Propositions 2.3 and 2.4.

Now we pick an $f \in L^s(\nu)$ such that $G_\rho f \geq \varphi^{1/s}$ q.e. as in (3.7). Let O_j be the center of $\mathcal{B}_j$, $O_0 = 0$, and $\eta_j = \eta(x - O_j)$, where $\eta \in C_0^\infty(2\mathcal{B}_0)$ and $\eta = 1$ on $\mathcal{B}_0$, then $(\mathcal{G}_\rho f)\eta_j \geq \varphi^{1/s}\chi_{\mathcal{B}_j}$ q.e. and hence

$$\sum_{j\geq 0} \mathcal{C}_{\mathcal{R},\rho}(\varphi\chi_{\mathcal{B}_j}) \leq \sum_{j\geq 0} \|(\mathcal{G}_\rho f)\eta_j\|_{W^{\alpha,s}}^s.$$

Now we appeal to Strichartz formula that

$$\sum_{j\geq 0} \|u\eta_j\|_{W^{\alpha,s}}^s \approx \|u\|_{W^{\alpha,s}}^s,$$

(see [4, Theorem 3.1.2]), we obtain

$$\sum_{j\geq 0} \mathcal{C}_{\mathcal{R},\rho}(\varphi\chi_{\mathcal{B}_j}) \lesssim \|\mathcal{G}_\rho f\|_{W^{\alpha,s}}^s = \|f\|_{L^s(\nu_\rho)}^s,$$

then the result follows by (3.7). □

Lemma 3.3 *For any function φ with compact support* supp(φ) *and*

$$\mathrm{diam}(\mathrm{supp}(\varphi)) < 1,$$

if $\varphi|_{\mathrm{supp}(\varphi)}$ is continuous with $\min_{\mathrm{supp}(\varphi)} \varphi > 0$, then

$$\mathcal{C}_{\mathcal{R},\rho}(\varphi) = \sup\left\{\left(\int_{\mathbb{R}^n} \varphi^{1/s} d\mu\right)^s : \mu \in \mathscr{M}^+(\mathrm{supp}(\varphi)), \|\check{\mathcal{G}}_\rho\mu\|_{L^t(\nu_\rho)} \leq 1\right\}.$$

Proof This lemma does not follow immediately by Theorem 2.10 since it is not clear if the defining kernel satisfies (2.5). We let $F = \mathrm{supp}(\phi)$, by re-examining the proof of Theorem 2.10, one only needs to deduce from (2.14) that the sequence $\{\mu_j(F)\}$ is bounded.

Let $\{B_{\rho/4}(O_k)\}_{k\geq 0}$ be a covering for $\mathbb{R}^n$ with finite multiplicity and denote $\mathcal{B}_k = B_{\rho/4}(O_k)$. Then there is a dimensional constant c_n such that the set $\mathcal{N} = \{k : \mathcal{B}_k \cap F \neq \emptyset\}$ has elements no more than c_n. As a consequence,

$$\begin{aligned}\|\check{\mathcal{G}}_\rho\mu_j\|_{L^s(\nu_\rho)}^s &= \int_0^\rho \int_{\mathbb{R}^n} \left(\frac{\mu_j(B_z(y))}{z^{n-\alpha}}\right)^s dy\frac{dz}{z}\\ &\geq C_n \sum_{k\geq 0} \int_{\rho/2}^\rho \int_{\mathcal{B}_k} \left(\frac{\mu_j(B_z(y))}{z^{n-\alpha}}\right)^s dy\frac{dz}{z}\end{aligned}$$

$$\geq C_n \sum_{k\geq 0} \int_{\rho/2}^{\rho} \int_{\mathcal{B}_k} \left(\frac{\mu_j(\mathcal{B}_k)}{z^{n-\alpha}} \right)^s dy \frac{dz}{z}$$

$$= C_{n,\rho} \sum_{k\geq 0} \mu_j(\mathcal{B}_k \cap F)^s, \tag{3.8}$$

where the dimensional constant C_n in (3.8) depends on the multiplicity of $\{\mathcal{B}_j\}_{j\geq 0}$. Finally, we note that

$$\mu_j(F)^s \leq c_n^{s-1} \sum_{k\geq 0} \mu_j(\mathcal{B}_k \cap F)^s \leq C'_{n,\rho} \|\check{\mathcal{G}}_\rho \mu_j\|^s_{L^s(\nu_\rho)},$$

so the sequence $\{\mu_j(F)\}$ is bounded. □

Proof of Proposition 3.1 Suppose that φ is given as in the assumption of Lemma 3.3 and $\mu \in \mathscr{M}^+(\mathrm{supp}(\varphi))$. We have

$$\int_{\mathbb{R}^n} (\mathcal{I}_{\alpha,\rho}\mu)^t dx \approx \int_{\mathbb{R}^n} (G_\alpha * \mu)^t dx,$$

see [1, Theorem 3.6.2], then we deduce from (3.3) and (3.5) that

$$\int_{\mathbb{R}^n} (G_\alpha * \mu)^t dx \approx \int_{\mathbb{R}^n} \mathcal{V}_\rho^\mu d\mu.$$

Keep in mind that $\mathcal{V}_\rho^\mu = \mathcal{G}_\rho((\check{\mathcal{G}}_\rho \mu)^{1/(s-1)})$ and

$$\|\check{\mathcal{G}}_\rho \mu\|^t_{L^t(\nu_\rho)} = \int_{\mathbb{R}^n} \mathcal{V}_\rho^\mu d\mu,$$

then $\mathcal{C}_{\mathcal{R},\rho}(\varphi) \approx \mathcal{C}_{G_\alpha}(\varphi)$ follows by Theorem 2.10 and Lemma 3.3.

Now we drop the assumption that $\mathrm{diam}(\mathrm{supp}(\varphi)) < 1$. Lemma 3.2 implies

$$\mathcal{C}_{\mathcal{R},\rho}(\varphi) \approx \sum_{j\geq 0} \mathcal{C}_{\mathcal{R},\rho}(\varphi \chi_{\mathcal{B}_j}) \approx \sum_{j\geq 0} \mathcal{C}_{G_\alpha}(\varphi \chi_{\mathcal{B}_j}) \approx \mathcal{C}_{G_\alpha}(\varphi).$$

Indeed, the last $\approx$ follows by the quasi-additivity of $\mathcal{C}_{G_\alpha}(\cdot)$ which the proof is exactly the same as in Lemma 3.2.

Now we assume that $\varphi \in C_0$. Let $\varphi_n = (\varphi + n^{-1})\chi_{\mathrm{supp}(\varphi)}$, we get

$$\mathcal{C}_{\mathcal{R},\rho}(\varphi) \leq \mathcal{C}_{\mathcal{R},\rho}(\varphi_n) \approx \mathcal{C}_{G_\alpha}(\varphi_n) \lesssim \mathcal{C}_{G_\alpha}(\varphi) + \frac{1}{n} \cdot \mathrm{Cap}_{\alpha,s}(\mathrm{supp}(\varphi)).$$

Taking $n \to \infty$, we have $\mathcal{C}_{\mathcal{R},\rho}(\varphi) \lesssim \mathcal{C}_{G_\alpha}(\varphi)$. By symmetry and (3.6), one obtains $\mathcal{C}_{\mathcal{R},\rho}(\varphi) \approx \mathcal{C}_{G_\alpha}(\varphi)$.

Now we assume that $\varphi \in \mathcal{LSC}$ and pick a $0 \leq \psi \in C_0$, so

$$\mathcal{C}_{\mathcal{R},\rho}(\psi) \approx \mathcal{C}_{G_\alpha}(\varphi) \leq \mathcal{C}_{G_\alpha}(\varphi).$$

By Corollary 2.7, one has $\mathcal{C}_{\mathcal{R},\rho}(\varphi) \lesssim \mathcal{C}_{G_\alpha}(\varphi)$, the other direction follows by symmetry.

One can argue by Proposition 2.2 for the general case that $\varphi \geq 0$, the result now follows. □

In the sequel, for any $\mu \in \mathscr{M}^+(\mathbb{R}^n)$ and $\rho > 0$, we denote

$$\mathcal{M}_\rho(\mu)(x) = \sup_{0<r\leq\rho} \frac{\mu(B_r(x))}{r^n}, \quad x \in \mathbb{R}^n.$$

Lemma 3.4 *Let $\rho > 0$ and $x \in \mathbb{R}^n$. For any compactly supported positive measure μ, it follows that*

$$\int_0^\rho \left(\frac{\mu(B_z(x))}{z^{n-\alpha s}}\right)^{\frac{1}{s-1}} \frac{dz}{z} \lesssim \|\mu\|^{\frac{\alpha s}{n(s-1)}} \mathcal{M}_\rho(\mu)(x)^{\frac{n-\alpha s}{n(s-1)}}. \tag{3.9}$$

In other words,

$$W_\rho^\mu \lesssim \|\mu\|^{\frac{\alpha s}{n(s-1)}} \mathcal{M}_\rho(\mu)^{\frac{n-\alpha s}{n(s-1)}}.$$

Proof Let $0 < \delta \leq \rho$ be determined later. We write

$$\begin{aligned}
&\int_0^\rho \left(\frac{\mu(B_z(x))}{z^{n-\alpha s}}\right)^{\frac{1}{s-1}} \frac{dz}{z} \\
&= \int_0^\delta \left(\frac{\mu(B_z(x))}{z^{n-\alpha s}}\right)^{\frac{1}{s-1}} \frac{dz}{z} + \int_\delta^\rho \left(\frac{\mu(B_z(x))}{z^{n-\alpha s}}\right)^{\frac{1}{s-1}} \frac{dz}{z} \\
&= I_1 + I_2.
\end{aligned}$$

We have

$$\begin{aligned}
I_1 &= \int_0^\delta z^{\frac{\alpha s}{s-1}} \left(\frac{\mu(B_z(x))}{z^n}\right)^{\frac{1}{s-1}} \frac{dz}{z} \\
&\leq \mathcal{M}_\rho(\mu)(x)^{\frac{1}{s-1}} \int_0^\delta z^{\frac{\alpha s}{s-1}} \frac{dz}{z} \\
&= \frac{s-1}{\alpha s} \cdot \mathcal{M}_\rho(\mu)(x)^{\frac{1}{s-1}} \delta^{\frac{\alpha s}{s-1}}.
\end{aligned} \tag{3.10}$$

On the other hand,

$$I_2 \leq \|\mu\|^{\frac{1}{s-1}} \int_\delta^\infty \frac{1}{z^{\frac{n-\alpha s}{s-1}}} \frac{dz}{z} = \frac{s-1}{n-\alpha s} \|\mu\|^{\frac{1}{s-1}} \frac{1}{\delta^{\frac{n-\alpha s}{s-1}}}. \tag{3.11}$$

In view of (3.9), we can actually assume that μ is supported in $B_\rho(x)$, then

$$\mathcal{M}_\rho(\mu)(x) \geq \frac{\mu(B_\rho(x))}{\rho^n} = \frac{\|\mu\|}{\rho^n}.$$

If we choose

$$\delta = \left(\frac{\|\mu\|}{\mathcal{M}_\rho(\mu)(x)}\right)^{\frac{1}{n}},$$

it satisfies that $0 < \delta \leq \rho$ and

$$\mathcal{M}_\rho(\mu)(x)^{\frac{1}{s-1}}\delta^{\frac{\alpha s}{s-1}} = \|\mu\|^{\frac{1}{s-1}}\frac{1}{\delta^{\frac{n-\alpha s}{s-1}}},$$

then (3.9) follows by routine simplification of (3.10) and (3.11). □

The following proposition addresses the fact that W_ρ^μ satisfies the almost *Muckenhoupt local A_1 condition.*

Proposition 3.5 *For $0 < \delta < n(s-1)/(n-\alpha s)$, and $x_0 \in \mathbb{R}^n$, it follows that*

$$\mathcal{M}^{\text{loc}}\left(\left(\int_0^2\left(\frac{\mu(B_z(\cdot))}{z^{n-\alpha s}}\right)^{\frac{1}{s-1}}\frac{dz}{z}\right)^\delta\right)(x_0) \lesssim \left(\int_0^4\left(\frac{\mu(B_z(x_0))}{z^{n-\alpha s}}\right)^{\frac{1}{s-1}}\frac{dz}{z}\right)^\delta. \tag{3.12}$$

In other words,

$$\mathcal{M}^{\text{loc}}(W_2^\mu) \lesssim W_4^\mu.$$

Proof Let $0 < r < 1$ be fixed. We are to estimate

$$I = \frac{1}{r^n}\int_{B_r(x_0)}\left(\int_0^2\left(\frac{\mu(B_z(x))}{z^{n-\alpha s}}\right)^{\frac{1}{s-1}}\frac{dz}{z}\right)^\delta dx \lesssim I_1 + I_2,$$

where

$$I_1 = \frac{1}{r^n}\int_{B_r(x_0)}\left(\int_0^r\left(\frac{\mu(B_z(x))}{z^{n-\alpha s}}\right)^{\frac{1}{s-1}}\frac{dz}{z}\right)^\delta dx,$$

$$I_2 = \frac{1}{r^n}\int_{B_r(x_0)}\left(\int_r^2\left(\frac{\mu(B_z(x))}{z^{n-\alpha s}}\right)^{\frac{1}{s-1}}\frac{dz}{z}\right)^\delta dx.$$

For the integral I_1, we can assume that μ is supported in $B_{2r}(x_0)$. By Lemma 3.4, it suffices to estimate

$$\int_{B_r(x_0)} \mathcal{M}_r(\mu)(x)^{\frac{\delta(n-\alpha s)}{n(s-1)}} dx.$$

Let $p = \delta(n - \alpha s)/(n(s - 1)) < 1$, we appeal to the inequality that

$$\int_E |F|^p dx \leq \frac{1}{1-p} |E|^{1-p} \|F\|_{L^{1,\infty}(E)}^p$$

for any measurable set $E \subseteq \mathbb{R}^n$ with $|E| < \infty$ and $F \in L^{1,\infty}(E)$, here $L^{1,\infty}(E)$ is the weak Lebesgue space on E (see [3, Exercise 1.1.11]). As a consequence,

$$\begin{aligned} I_1 &\lesssim \|\mu\|^{\frac{\delta\alpha s}{n(s-1)}} r^{-n} |B_r(x_0)|^{1-p} \|\mathcal{M}_r \mu\|_{L^{1,\infty}}^p \\ &\lesssim r^{-\frac{\delta(n-\alpha s)}{s-1}} \|\mu\|^{\frac{\delta\alpha s}{n(s-1)}+p} \qquad (3.13)\\ &= \left(r^{-\frac{n-\alpha s}{s-1}} \mu(B_{2r}(x_0))^{\frac{1}{s-1}} \right)^\delta \\ &\lesssim \left(\int_{2r}^{3r} \left(\frac{\mu(B_z(x_0))}{z^{n-\alpha s}} \right)^{\frac{1}{s-1}} \frac{dz}{z} \right)^\delta \\ &\leq \left(\int_0^3 \left(\frac{\mu(B_z(x_0))}{z^{n-\alpha s}} \right)^{\frac{1}{s-1}} \frac{dz}{z} \right)^\delta, \qquad (3.14) \end{aligned}$$

where we have used the weak type (1, 1) boundedness of $\mathcal{M}(\cdot)$ in (3.13), where

$$\mathcal{M}(\mu)(\cdot) = \sup_{r>0} \frac{\mu(B_r(\cdot))}{r^n},$$

and note that $\mathcal{M}_r(\cdot) \leq \mathcal{M}(\cdot)$ for all $r > 0$.

For the integral I_2, we observe that $B_z(x) \subseteq B_{2z}(x_0)$ for $x \in B_r(x_0)$ and $z > r$, this gives

$$\begin{aligned} I_2 &\lesssim \left(\int_r^2 \left(\frac{\mu(B_{2z}(x_0))}{z^{n-\alpha s}} \right)^{\frac{1}{s-1}} \frac{dz}{z} \right)^\delta \\ &\approx \left(\int_{2r}^4 \left(\frac{\mu(B_z(x_0))}{z^{n-\alpha s}} \right)^{\frac{1}{s-1}} \frac{dz}{z} \right)^\delta \\ &\leq \left(\int_0^4 \left(\frac{\mu(B_z(x_0))}{z^{n-\alpha s}} \right)^{\frac{1}{s-1}} \frac{dz}{z} \right)^\delta, \qquad (3.15) \end{aligned}$$

then (3.12) follows by combining (3.14) and (3.15). □

Proof of Theorem 1.1 The result in [6] has tackled the case for $q \geq 1$. Therefore, we assume that $t(n-\alpha s)/n < q < 1$. Suppose that φ is given as in the assumption of Lemma 3.3. It suffices to prove that

$$\mathcal{C}_{G_\alpha}((\mathcal{M}^{\mathrm{loc}}\varphi)^q) \lesssim \mathcal{C}_{G_\alpha}(\varphi^q),$$

which in turn is equivalent to

$$\mathcal{C}_{\mathcal{R},8}((\mathcal{M}^{\mathrm{loc}}\varphi)^q) \lesssim \mathcal{C}_{\mathcal{R},1}(\varphi^q) \tag{3.16}$$

by Proposition 3.1. Let $\mu = \mu_\varphi$ be the capacitary measure for φ such that $\mathcal{V}_1^\mu = \mathcal{G}_1((\check{\mathcal{G}}_1\mu)^{1/(s-1)}) \geq \varphi^{q/s}$ and that

$$\mathcal{C}_{\mathcal{R},1}(\varphi^q) = \int_{\mathbb{R}^n} \mathcal{V}_1^\mu d\mu.$$

By (3.1), we see that $\varphi^{q/s} \lesssim W_2^\mu$ and hence $\mathcal{M}^{\mathrm{loc}}\varphi \lesssim \mathcal{M}^{\mathrm{loc}}((W_2^\mu)^{s/q})$. Using Proposition 3.5, one obtains $(\mathcal{M}^{\mathrm{loc}}\varphi)^{q/s} \lesssim W_4^\mu$ provided that

$$\frac{s}{q} < \frac{n(s-1)}{n-\alpha s},$$

which holds by the assumption that $q > t(n-\alpha s)/n$. As a consequence, $(\mathcal{M}^{\mathrm{loc}}\varphi)^{q/s} \lesssim \mathcal{V}_8^\mu$ by (3.1) again. By the definition of $\mathcal{C}_{\mathcal{R},8}(\cdot)$ and Wolff's inequality, we have

$$\mathcal{C}_{\mathcal{R},8}((\mathcal{M}^{\mathrm{loc}}\varphi)^q) \leq \|(\check{\mathcal{G}}_8\mu)^{1/(s-1)}\|_{L^s(\nu_\rho)}^s = \int_{\mathbb{R}^n} \mathcal{V}_8^\mu d\mu \approx \int_{\mathbb{R}^n} \mathcal{V}_1^\mu d\mu,$$

so (3.16) follows.

We are now to extend the validity of (3.16) for general $\varphi \geq 0$. If we drop that $\mathrm{diam}(\mathrm{supp}(\varphi)) < 1$, we can argue by Lemma 3.2 that

$$\varphi \leq \sum_{j\geq 0} \varphi\chi_{\mathrm{supp}(\varphi)\cap\mathcal{B}_j},$$

$$\mathcal{M}^{\mathrm{loc}}(\varphi^q) \leq \sum_{j\geq 0} \mathcal{M}^{\mathrm{loc}}(\varphi^q\chi_{\mathrm{supp}(\varphi)\cap\mathcal{B}_j}),$$

where we have used the elementary inequality that $(\sum_j a_j)^q \leq \sum_j a_j^q$ for $a_j \geq 0$ and $0 < q < 1$. Taking $\mathcal{C}_{\mathcal{R},8}(\cdot)$ both sides,

$$\begin{aligned}\mathcal{C}_{\mathcal{R},8}(\mathcal{M}^{\mathrm{loc}}(\varphi^q)) &\leq \sum_{j\geq 0} \mathcal{C}_{\mathcal{R},8}(\mathcal{M}^{\mathrm{loc}}(\varphi^q\chi_{\mathrm{supp}(\varphi)\cap\mathcal{B}_j}))\\ &\lesssim \sum_{j\geq 0} \mathcal{C}_{\mathcal{R},1}(\varphi^q\chi_{\mathrm{supp}(\varphi)\cap\mathcal{B}_j}),\end{aligned} \tag{3.17}$$

so (3.16) follows by the finite multiplicity of $\{\mathcal{B}_j\}$.

For the case that $\varphi \in C_0$, one can argue as in Proposition 3.1. Now assuming that $\varphi \in \mathcal{LSC}$, find a sequence $\{\varphi_n\}$ of C_0 such that $\varphi_n(x) \uparrow \varphi(x)$, then (3.16) follows by the Fatou property of $\mathcal{C}_{\mathcal{R},\rho}(\cdot)$. The general case that $\varphi \geq 0$ can be argued by the outer regularity of $\mathcal{C}_{\mathcal{R},\rho}(\cdot)$, the proof is now complete. □

4 Concluding remarks and proof of Theorem 1.2

As a maneuver for the proof of Theorem 1.1, we replace the Choquet integrals $\int_{\mathbb{R}^n} |\cdot| dC$ with the functionals $\mathcal{C}(\cdot)$. In proving Theorem 1.1, the countably subadditivity in (3.17) is crucial. However, the Choquet integral is generally not subadditive but

$$\int_{\mathbb{R}^n} (f+g) dC \leq 2\int_{\mathbb{R}^n} f dC + 2\int_{\mathbb{R}^n} g dC$$

for $f, g \geq 0$, so it fails to be countably subadditive generally. We also use approximation by outer regularity of $\mathcal{C}(\cdot)$ which the Choquet integral seems to be lacking. As in Proposition 2.11, we are able to switch the functional $\mathcal{C}(\cdot)$ to the regular integral associated to the nonlinear potential and Proposition 3.5 comes into play subsequently, it is unknown if the Choquet integral can be switched to the form of (2.15).

On the other hand, one has the exact Muckenhoupt local A_1 condition for $G_\alpha((G_\alpha * \mu)^{1/(s-1)})$ that

$$\mathcal{M}^{\mathrm{loc}}\left((G_\alpha * ((G_\alpha * \mu)^{1/(s-1)}))^\delta\right) \lesssim (G_\alpha * ((G_\alpha * \mu)^{1/(s-1)}))^\delta \tag{4.1}$$

for $0 < \delta < n(s-1)/(n-\alpha s)$ and $s > 2-\alpha/n$. Therefore, for the case that $s > 2-\alpha/n$, one may prove Theorem 1.1 without going through the materials regarding local Riesz capacities, it suffices to replace (3.12) with (4.1) in Proposition 3.5. However, for the case that $s \leq 2-\alpha/n$, (4.1) holds only for $0 < \delta < n/(n-\alpha)$ and the bound $n/(n-\alpha)$ is simply too crude for our purpose. Meanwhile, Proposition 3.5 holds for any $s > 1$ with the expected bound. There is no need to obtain the exact Muckenhoupt local A_1 condition in proving Theorem 1.1 since all the functionals $\mathcal{C}_{\mathcal{R},\rho}(\cdot)$ are equivalent for any $\rho > 0$.

A way of proving Theorem 1.2 can be described in few lines. As contrast to the machinery in Sect. 3, we start with

$$\begin{aligned}
\mathbb{M} &= \mathbb{R}^n \times (0,\infty),\\
d\nu(y,z) &= dy \times \frac{dz}{z},\\
\mathcal{G}f(x) &= \int_0^\infty \left(\int_{|x-y|<z} \frac{f(y,z)}{z^{n-\alpha}} dy\right)\frac{dz}{z}.
\end{aligned}$$

One has

$$\mathcal{R}_{\alpha,s}(E)=\inf\{\|f\|_{L^s(\nu)}^s : f\in L_+^s(\nu), \mathcal{G}f\geq 1 \text{ on } E\}\approx \mathrm{cap}_{\alpha,s}(E).$$

The most crucial proposition as contrast to that of (3.12) is the following.

$$\mathcal{M}\left(\left(\int_0^\infty\left(\frac{\mu(B_z(\cdot))}{z^{n-\alpha s}}\right)^{\frac{1}{s-1}}\frac{dz}{z}\right)^\delta\right)(x_0)\lesssim\left(\int_0^\infty\left(\frac{\mu(B_z(x_0))}{z^{n-\alpha s}}\right)^{\frac{1}{s-1}}\frac{dz}{z}\right)^\delta$$

for $0<\delta<n(s-1)/(n-\alpha s)$, and $x_0\in\mathbb{R}^n$. The proof now follows in the same fashion as Theorem 1.1.

We end this section by giving another remarks about $\mathcal{C}(\cdot)$. In [6] we have showed that

$$\mathcal{I}(\varphi)=\inf\{\|f\|_{\mathcal{Z}'} : f\in L_+^s(\mathbb{R}^n), G_\alpha * f\geq\varphi \text{ q.e.}\}\approx\int_{\mathbb{R}^n}\varphi dC,\quad \varphi\geq 0,$$

where $\mathcal{Z}=M_t^{\alpha,s}$, $1/s+1/t=1$, is the Sobolev Multiplier space, and $\mathcal{Z}'$ is the Köthe dual of $\mathcal{Z}$, see [5] for details. There are some reasons that we do not take $\mathcal{I}(\cdot)$ as a functional generalization of the capacity. The subadditivity of $\mathcal{I}(\cdot)$ holds apparently, but it is unclear that $\mathcal{I}(\cdot)$ has the Fatou property as in (2.2). It is also unclear that $\mathcal{I}(\cdot)$ satisfies the inner or outer regularity. Since $\mathcal{Z}'$ is not reflexive nor uniformly convex in general, Proposition 2.4 fails to hold for $\mathcal{I}(\cdot)$, neither will do for Theorem 2.11. In some sense, it is the reflexivity and uniformly convexity of the space $L^s(\mathbb{R}^n)$ that makes $\mathcal{C}(\cdot)$ having those specific properties, which $\mathcal{I}(\cdot)$ is lacking. As a consequence, we will take $\mathcal{C}(\cdot)$ rather than $\mathcal{I}(\cdot)$ as the functional generalization of the capacity.

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